REGULATORS OF CANONICAL EXTENSIONS ARE TORSION: THE SMOOTH DIVISOR CASE

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Abstract. In this paper, we prove a generalization of Reznikov's theorem which says that the Chern-Simons classes and in particular the Deligne Chern classes (in degrees > 1) are torsion, of a flat vector bundle on a smooth complex projective variety. We consider the case of a smooth quasi–projective variety with an irreducible smooth divisor at infinity. We define the Chern-Simons classes of the Deligne's *canonical extension* of a flat vector bundle with unipotent monodromy at infinity, which lift the Deligne Chern classes and prove that these classes are torsion.

Contents

1. Introduction 1
2. Idea for the construction of secondary classes 4
3. The $\mathcal{C}^\infty$-trivialization of canonical extensions 10
4. Patched connection on the canonical extension 13
5. Compatibility with the Deligne Chern class 20
6. Rigidity of the secondary classes 22
7. A deformational variant of the patching construction in K-theory 25
8. Hermitian K-theory and variations of Hodge structure 31
9. The generalization of Reznikov's theorem 37
References 39

1. Introduction

Chern, Simons [Cn-Sm] and Cheeger [Ch-Sm] introduced a theory of differential cohomology on smooth manifolds. For complex vector bundles with connection, they defined classes or the secondary invariants in the ring of differential characters. These classes lift the closed Chern form defined by the curvature of the given connection. In particular when the connection is flat, the secondary invariants yield classes in the cohomology with $\mathbb{R}/\mathbb{Z}$-coefficients. These are the Chern-Simons classes of flat connections.

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The following question was raised in [Ch-Sm, p.70-71] (see also [Bl, p.104]) by Cheeger and Simons:

Question 1.1. *Suppose X is a smooth manifold and (E,∇) is a flat connection on X. Are the Chern-Simons classes $\widehat{c_p}(E,\nabla)$ of (E,∇) torsion in $H^{2p-1}(X,\mathbb{R}/\mathbb{Z})$, for $p\geq 2$?*

Suppose X is a smooth projective variety defined over the complex numbers. Let (E,∇) be a vector bundle with a flat connection ∇. S. Bloch [Bl] showed that for a unitary connection the Chern-Simons classes are mapped to the Chern classes of E in the Deligne cohomology. The above Question 1.1 together with his observation led him to conjecture that the Chern classes of flat bundles are torsion in the Deligne cohomology of X, in degrees at least two.

A. Beilinson defined universal secondary classes and H. Esnault [Es] constructed secondary classes using a modified splitting principle in the $\mathbb{C}/\mathbb{Z}$-cohomology. These classes are shown to be lifting of the Chern classes in the Deligne cohomology. These classes also have an interpretation in terms of differential characters, and the original $\mathbb{R}/\mathbb{Z}$ classes of Chern-Simons are obtained by the projection $\mathbb{C}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$. The imaginary parts of the $\mathbb{C}/\mathbb{Z}$ classes are Borel's volume regulators $Vol_{2p-1}(E,\nabla)\in H^{2p-1}(X,\mathbb{R})$. All the constructions give the same class in odd degrees, called as the secondary classes on X (see [DHZ], [Es3] for a discussion on this).

Reznikov [Re], [Re2] showed that the secondary classes of (E,∇) are torsion in the cohomology $H^{2p-1}(X,\mathbb{C}/\mathbb{Z})$ of X, when $p\geq 2$. In particular, he proved the above mentioned conjecture of Bloch.

Our aim here is to extend this result when X is smooth and quasi–projective with an irreducible smooth divisor D at infinity. We consider a flat bundle on X which has unipotent monodromy around the divisor D. We define secondary classes on X which extend the classes on $X-D$ of the flat connection and which lift the Deligne Chern classes of the canonical extension [De] on X.

Our main theorem is

Theorem 1.2. *Suppose X is a smooth quasi–projective variety defined over $\mathbb{C}$. Let (E,∇) be a flat connection on $U:=X-D$ associated to a representation $\rho:\pi_1(U)\to GL_r(\mathbb{C})$. Assume that D is a smooth and irreducible divisor and $(\overline{E},\overline{\nabla})$ be the Deligne canonical extension on X with unipotent monodromy around D. Then the secondary classes*

$$\widehat{c_p}(\rho/X)\in H^{2p-1}(X,\mathbb{C}/\mathbb{Z})$$

of $(\overline{E},\overline{\nabla})$ are torsion, for $p>1$. If, furthermore, X is projective then the Chern classes of $\overline{E}$ are torsion in the Deligne cohomology of X, in degrees >1.

What we do here can easily be generalized to the case when D is smooth and has several disjoint irreducible components. On the other hand, the generalization to a normal crossings divisor presents significant new difficulties which we don't yet know how to handle, so this will be left for the future.

The main constructions in this paper are as follows. We will consider the following situation. Suppose X is a smooth manifold, and $D \subset X$ is a connected smooth closed subset of real codimension 2. Let $U := X - D$ and suppose we can choose a reasonable tubular neighborhood B of D. Let $B^* := B \cap U = B - D$. It follows that $\pi_1(B^*) \to \pi_1(B)$ is surjective. The diagram

$$\begin{array}{ccc} B^* & \rightarrow & B \\ \downarrow & & \downarrow \\ U & \rightarrow & X \end{array} \tag{1}$$

is a homotopy pushout diagram. Note also that B retracts to D, and B^* has a tubular structure:

$$B^* \cong S \times (0,1)$$

where $S \cong \partial B$ is a circle bundle over D.

We say that (X, D) is *complex algebraic* if X is a smooth complex quasiprojective variety and D an irreducible smooth divisor.

Suppose we are given a representation $\rho : \pi_1(U) \to GL_r(\mathbb{C})$, corresponding to a local system L over U, or equivalently to a vector bundle with flat connection (E, ∇). Let γ be a loop going out from the basepoint to a point near D, once around, and back. Then $\pi_1(B)$ is obtained from $\pi_1(B^*)$ by adding the relation $\gamma \sim 1$. We assume that *the monodromy of ρ at infinity is unipotent*, by which we mean that $\rho(\gamma)$ should be unipotent. The logarithm is a nilpotent transformation

$$N := \log \rho(\gamma) := (\rho(\gamma) - I) - \frac{1}{2}(\rho(\gamma) - I)^2 + \frac{1}{3}(\rho(\gamma) - I)^3 - ...,$$

where the series stops after a finite number of terms.

In this situation, there is a canonical and natural way to extend the bundle E to a bundle $\overline{E}$ over X, known as the *Deligne canonical extension* [De]. The connection ∇ extends to a meromorphic connection $\overline{\nabla}$ whose polar terms involved look locally like $\frac{1}{2\pi i} N . \frac{dz}{z}$, for z a local equation for D. In an appropriate frame the singularities of $\overline{\nabla}$ are only in the strict upper triangular region of the connection matrix. In the complex algebraic case, (E, ∇) are holomorphic, and indeed algebraic with algebraic structure uniquely determined by the requirement that ∇ have regular singularities. The extended bundle $\overline{E}$ is algebraic on X and $\overline{\nabla}$ becomes a logarithmic connection [De].

We will define *extended regulator classes*

$$\widehat{c}_p(\rho/X) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$$

which restrict to the usual regulator classes on U. Their imaginary parts define *extended volume regulators* which we write as $Vol_{2p-1}(\rho/X) \in H^{2p-1}(X, \mathbb{R})$.

The technique for defining the extended regulator classes is to construct a *patched connection* $\nabla^{\#}$ over X. This will be a smooth connection, however it is not flat. Still, the curvature comes from the singularities of $\overline{\nabla}$ which have been smoothed out, so the curvature is upper-triangular. In particular, the Chern forms for $\nabla^{\#}$ are still identically zero. The Cheeger-Simons theory of differential characters provides a class of $\nabla^{\#}$ in the group of differential characters, mapping to the group of closed forms. Since the image, which is the Chern form, vanishes, the differential character lies in the kernel of this map which is exactly $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$ [Ch-Sm, Cor. 2.4]. This is the construction of the regulator class.

The proof of Dupont-Hain-Zucker that the regulator class lifts the Deligne Chern class, goes through word for word here to show that this extended regulator class lifts the Deligne Chern class of the canonical extension $\overline{E}$ in the complex algebraic case. For this part, we need X projective.

We also give a different construction of the regulator classes, using the deformation theorem in K-theory. The filtration which we will use to define the patched connection, also leads to a polynomial deformation on B^* between the representation ρ and its associated-graded. Then, using the fact that $BGL(F[t])^+$ is homotopy-equivalent to $BGL(F)^+$ and the fact that the square (1) is a homotopy pushout, this allows us to construct a map from X to $BGL(F)^+$ and hence pull back the universal regulator classes. Corollary 7.5 below says that these are the same as the extended regulators defined by the patched connection. On the other hand, the counterpart of the deformation construction in hermitian K-theory allows us to conclude that the extended volume regulator is zero whenever ρ underlies a complex variation of Hodge structure in the complex algebraic case.

A rigidity statement for the patched connections is discussed and proved in more generality in §6. All of the ingredients of Reznikov's original proof [Re2] are now present for the extended classes, including Mochizuki's theorem that any representation can be deformed to a complex variation of Hodge structure [Mo]. Thus we show the generalization of Reznikov's result.

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2. Idea for the construction of secondary classes

We begin by recalling the differential cohomology introduced by Chern, Cheeger and Simons [Ch-Sm],[Cn-Sm]. Since we want to look at logarithmic connections, we consider these cohomologies on complex analytic varieties and on their smooth compactifications. Our aim is to define secondary classes in the $\mathbb{C}/\mathbb{Z}$-cohomology for logarithmic connections which have unipotent monodromy along a smooth boundary divisor. A gluing construction was suggested by Deligne, which uses gluing of secondary classes on the open variety and on a tubular neighborhood of the boundary divisor. In §4 this will be made precise using a patched connection.

Let X be a nonsingular variety defined over the complex numbers. In the following discussion we will interchangeably use the notation X for the algebraic variety or the underlying complex analytic space.

Conventions. Denote by $\mathbb{Z}(p)$ the subgroup of $\mathbb{C}$ generated by $(2\pi i)^p$. For each subgroup L of $\mathbb{C}$, set

$$L(p) = L \otimes_{\mathbb{Z}} \mathbb{Z}(p).$$

The isomorphism $\mathbb{Z} \to \mathbb{Z}(p)$ that takes 1 to $(2\pi i)^p$ induces a canonical isomorphism

$$H^\bullet(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z}(p)). \tag{2}$$

The Chern classes of complex vector bundles lie in $H^\bullet(X, \mathbb{Z}(p))$ which is the image of the usual topological Chern classes, under the isomorphism given by (2).

2.1. **Analytic differential characters on** X [Ch-Sm].

Let $S_k(X)$ denote the group of k-dimensional smooth singular chains on X, with integer coefficients. Let $Z_k(X)$ denote the subgroup of cycles. Let us denote

$$S^\bullet(X, \mathbb{Z}) := \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathbb{Z})$$

the complex of $\mathbb{Z}$ -valued smooth singular cochains, whose boundary operator is denoted by δ. The group of smooth differential k-forms on X with complex coefficients is denoted by $A^k(X)$ and the subgroup of closed forms by $A^k_{cl}(X)$. Then $A^\bullet(X)$ is canonically embedded in $S^\bullet(X)$, by integrating forms against the smooth singular chains. In fact, we have an embedding

$$i_{\mathbb{Z}} : A^\bullet(X) \hookrightarrow S^\bullet(X, \mathbb{C}/\mathbb{Z}).$$

The group of differential characters of degree k is defined as

$$\widehat{H^k}(X, \mathbb{C}/\mathbb{Z}) := \{(f, \alpha) \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(X), \mathbb{C}/\mathbb{Z}) \oplus A^k(X) : \delta(f) = i_{\mathbb{Z}}(\alpha) \text{ and } d\alpha = 0\}.$$

There is a canonical and functorial exact sequence:

$$0 \longrightarrow H^{k-1}(X, \mathbb{C}/\mathbb{Z}) \longrightarrow \widehat{H^k}(X, \mathbb{C}/\mathbb{Z}) \longrightarrow A^k_{cl}(X, \mathbb{Z}) \longrightarrow 0. \tag{3}$$

Here $A^k_{cl}(X, \mathbb{Z}) := \ker(A^k_{cl}(X) \longrightarrow H^k(X, \mathbb{C}/\mathbb{Z}))$. Similarly, one defines the group of differential characters $\widehat{H^k}(X, \mathbb{R}/\mathbb{Z})$ with $\mathbb{R}/\mathbb{Z}$-coefficients.

For the study of infinitesimal variations of differential characters, we have the following remark about the tangent space.

Lemma 2.1. *The group of differential characters has the structure of infinite dimensional abelian Lie group. Its tangent space at the origin (or by translation, at any point) is* *naturally identified as*

$$T_0\left(\widehat{H^k}(X,\mathbb{C}/\mathbb{Z})\right) = \frac{A^{k-1}(X,\mathbb{C})}{dA^{k-2}(X,\mathbb{C})}.$$

Proof. A tangent vector corresponds to a path (f_t, α_t). An element $\beta \in A^{k-1}(X,\mathbb{C})$ maps to the path given by $f_t(z) := t\int_z \beta$ and $\alpha_t := td(\beta)$. Looking at the above exact sequence (3), we see that this map induces an isomorphism from $A^{k-1}(X,\mathbb{C})/dA^{k-2}(X,\mathbb{C})$ to the tangent space of $\widehat{H^k}(X,\mathbb{C}/\mathbb{Z})$. $\square$

2.2. **Secondary classes and the Cheeger-Chern-Simons classes.** Suppose (E,∇) is a vector bundle with a connection on X. Then the Chern forms

$$c_k(E,\nabla) \in A^{2k}_{cl}(X,\mathbb{Z})$$

for $0 \le k \le$ rank (E), are defined using the universal Weil homomorphism [Cn-Sm]. There is an GL_r-invariant, symmetric, homogeneous and multilinear polynomial $\mathcal{P}_k$ of degree k in k variables on the Lie algebra $\mathbf{gl}_r$ such that if Ω is the curvature of ∇ then $c_k(E,\nabla) = \mathcal{P}_k(\frac{-1}{2\pi i}\Omega,\ldots,\frac{-1}{2\pi i}\Omega)$. Here $\mathcal{P}_k$ are defined as follows;

$$\det(I_r + X) = 1 + \mathcal{P}_1(X) + \mathcal{P}_2(X) + ... + \mathcal{P}_r(X),\ \ X \in \mathbf{gl}_r(\mathbb{C}).$$

When $X_i = X$ for each i, then $\mathcal{P}_k(X,...,X) = \mathrm{trace}(\wedge^k X)$ (see [Gri-Ha, p.403]), however the wedge product here is taken in the variable $\mathbb{C}^r$, not the wedge of forms on the base. If X is a diagonal matrix with eigenvalues $\lambda_1,\ldots,\lambda_r$ then $\mathcal{P}_k(X,...,X) = \sum_I \lambda_{i_1}\cdots\lambda_{i_k}$. We can also express $\mathcal{P}_k$ in terms of the traces of products of matrices. In this expression, the highest order term of $\mathcal{P}_k$ is the symmetrization of $Tr(X_1\cdots X_k)$ multiplied by a constant, the lower order terms are symmetrization's of $Tr(X_1\cdots X_{i_1})Tr(\cdots)\cdots Tr(X_{i_a+1}\cdots X_k)$, with suitable constant coefficients.

The characteristic classes

$$\widehat{c_k}(E,\nabla) \in \widehat{H^{2k}}(X,\mathbb{C}/\mathbb{Z})$$

are defined in [Ch-Sm] using a factorization of the universal Weil homomorphism and looking at the universal connections [Na-Ra]. These classes are functorial lifting of $c_k(E,\nabla)$.

One of the key properties of these classes is the variational formula in case of a family of connections. If $\{\nabla_t\}$ is a $\mathcal{C}^\infty$ family of connections on E, then—referring to Lemma 2.1 for the tangent space of the space of differential characters—we have the formula

$$\frac{d}{dt}\widehat{c_k}(E,\nabla_t) = k\mathcal{P}(\frac{d}{dt}\nabla_t,\Omega_t,\ldots,\Omega_t), \tag{4}$$

see [Ch-Sm, Proposition 2.9].

If E is topologically trivial, then any connection is connected by a path to the trivial connection for which the characteristic class is defined to be zero. The variational formula thus serves to characterize $\widehat{c_k}(E,\nabla_t)$ for all t.

Remark 2.2. *If the form $c_k(E,\nabla)$ is zero, then the class $\widehat{c_k}(E,\nabla)$ lies in $H^{2k-1}(X,\mathbb{C}/\mathbb{Z})$. If (E,∇) is a flat bundle, then $c_k(E,\nabla)=0$ and the classes $\widehat{c_k}(E,\nabla)$ are called the* secondary classes *or* regulators *of (E,∇). Notice that the class depends on the choice of ∇. We will also refer to these classes as the Chern-Simons classes in $\mathbb{C}/\mathbb{Z}$-cohomology.*

In the case of a flat bundle, after going to a finite cover the bundle is topologically trivial by the result of Deligne-Sullivan which will be discussed in §3 below. Thus, at least the pullback to the finite cover of $\widehat{c_k}(E,\nabla)$ can be understood using the variational methods described above.

Beilinson's theory of universal secondary classes yield classes for a flat connection (E,∇),

$$\widehat{c_k}(E,\nabla)\in H^{2k-1}(X,\mathbb{C}/\mathbb{Z}),\ k\geq 1 \tag{5}$$

which are functorial and additive over exact sequences. Furthermore, Esnault [Es] using a modified splitting principle, Karoubi [Ka2] using K-theory have defined secondary classes. These classes are functorial and additive. These classes then agree with the universally defined class in (5) (see [Es, p.323]).

When X is a smooth projective variety, Dupont-Hain-Zucker [Zu], [DHZ] and Brylinski [Br] have shown that the Chern–Simons classes are lifting of the Deligne Chern class $c_k^{\mathcal{D}}(E)$ under the map obtained by dividing out by the Hodge filtered piece F^k,

$$H^{2k-1}(X,\mathbb{C}/\mathbb{Z})\longrightarrow H^{2k-1}(X,\mathbb{C}/\mathbb{Z}(k))\longrightarrow H^{2k}_{\mathcal{D}}(X,\mathbb{Z}(k)).$$

By functoriality and additive properties, the classes in (5) lift the Chern-Simons classes defined above using differential characters, via the projection

$$\mathbb{C}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}.$$

In fact, Cheeger-Simons explicitly took the real part in their formula at the start of [Ch-Sm, §4]. See also [Bl] for unitary connections, [So], [Gi-So] when X is smooth and projective; for a discussion on this see [Es3].

2.3. **Secondary classes of logarithmic connections.** Suppose X is a nonsingular variety and $D\subset X$ an irreducible smooth divisor. Let $U:=X-D$. Choose a tubular neighborhood B of D and let $B^*:=B\cap U=B-D$.

Let (E,∇) be a complex analytic vector bundle on U with an integrable connection ∇. Consider a logarithmic extension $(\overline{E},\overline{\nabla})$ (see [De]) on X of the connection (E,∇). Assuming that the residues are nilpotent, we want to show that the classes $\widehat{c_k}(E,\nabla)\in H^{2i-1}(U,\mathbb{C}/\mathbb{Z})$ extend on X to give classes in the cohomology with $\mathbb{C}/\mathbb{Z}$-coefficients which map to the Deligne Chern class of $\overline{E}$.

We want to use the Mayer-Vietoris sequence (a suggestion from Deligne) to motivate a construction of secondary classes in this situation. The precise construction will be carried out in §4.

Consider the residue transformation

$$\eta : \overline{E} \longrightarrow \overline{E} \otimes \Omega_X(\log D) \xrightarrow{res} \overline{E} \otimes \mathcal{O}_D.$$

By assumption η is nilpotent and let r be the order of η.

Consider the *Kernel filtration* of $\overline{E}_D$ induced by the kernels of the operator η:

$$0 = W_{0,D} \subset W_{1,D} \subset W_{2,D} \subset ... \subset W_{r,D} = \overline{E}_D.$$

Here

$$W_{j,D} := \mathrm{kernel}(\eta^{\circ\, j} : \overline{E}_D \longrightarrow \overline{E}_D).$$

Denote the graded pieces

$$\mathbf{Gr}_j(\overline{E}_D) := W_{j,D}/W_{j+1,D}$$

and the associated graded

$$\mathbf{Gr}(\overline{E}_D) := \oplus_{j=0}^{r-1} \mathbf{Gr}_j(\overline{E}_D).$$

Lemma 2.3. *Each graded piece $\boldsymbol{Gr}^j(\overline{E}_D)$ (for $0 \leq j < r$) is endowed with a flat connection along D. Furthermore, the filtration of $\overline{E}_D$ by $W_{j,D}$ extends to a filtration of $\overline{E}$ by holomorphic subbundles W_r defined in a tubular neighborhood B of the divisor D. On B^* these subbundles are preserved by the connection ∇, and ∇ induces on each graded piece $\boldsymbol{Gr}_j(\overline{E}_{B^*})$ a connection which extends to a flat connection over B, and induces the connection mentioned in the first phrase, on $\boldsymbol{Gr}^j(\overline{E}_D)$.*

Proof. Suppose n is the dimension of the variety X. Consider a product of n-open disks Δ^n with coordinates $(t_1, t_2, ..., t_n)$ around a point of the divisor D so that D is locally defined by $t_1 = 0$. Let γ be the generator of the fundamental group of the punctured disk $\Delta^n - \{t_1 = 0\}$. Then γ is the monodromy operator acting on a fiber E_t, for $t \in \Delta^n - \{t_1 = 0\}$. The operator

$$N = \log\, \gamma = (\gamma - I) - \frac{1}{2}(\gamma - I)^2 + \frac{1}{3}(\gamma - I)^3 - ...$$

is nilpotent since by assumption the local monodromy γ is unipotent. Further, the order of unipotency of γ coincides with the order of nilpotency of N. Consider the filtration on the fiber E_t induced by the operator N:

$$0 = W^0(t) \subset W^1(t) \subset ... \subset W^r(t) = E_t.$$

such that

$$W^j(t) := \mathrm{kernel}(N^j : E_t \longrightarrow E_t).$$

Denote the graded pieces

$$\mathbf{gr}^j_t := W^j(t)/W^{j+1}(t).$$

Then we notice that the operator N acts trivially on the graded pieces $\mathbf{gr}^j_t$. This means that γ acts as identity on $\mathbf{gr}^j_t$. In other words, $\mathbf{gr}^j_t$ (for $t \in \Delta^n$) forms a local system on Δ^n and extends as a local system $\mathbf{gr}^j$ in a tubular neighborhood B of D in X.

The operation of γ around D can be extended to the boundary (see [De] or [Es-Vi, c) Proposition]). More precisely, the operation γ (resp. N) extends to the sheaf $\overline{E}$ and defines an endomorphism $\overline{\gamma}$ (resp.$\overline{N}$) of $\overline{E}_D$ such that

$$\exp(-2\pi i.\eta) = \overline{\gamma}_D.$$

This implies that the kernels defined by the residue transformation η and $\overline{N}$ are the same over D. The graded piece $\mathbf{Gr}^j$ is the bundle associated to the local system $\mathbf{gr}^j$ in a tubular neighborhood B of D in X. □

Corollary 2.4. *If (E_B, ∇_B) denotes the restriction of $(\overline{E}, \overline{\nabla})$ on the tubular neighborhood B, then in the K_0-group $K_{an}(B)$ of analytic vector bundles, we have the equality*

$$E_B = \boldsymbol{Gr}(E_B) = \oplus_j \boldsymbol{Gr}^j(E_B).$$

□

Corollary 2.5. *We can define the secondary classes of the restriction (E_B, ∇_B) to be*

$$\widehat{c}_i(E_B, \nabla_B) := \widehat{c}_i(\boldsymbol{Gr}(E_B))$$

in $H^{2i-1}(B, \mathbb{C}/\mathbb{Z})$.

□

For the above construction, we could have replaced the kernel filtration by Deligne's *monodromy weight filtration*

$$0 = W_{-r-1} \subset \ldots \subset W_r = E$$

or indeed by any filtration of the flat bundle (E_{B^*}, ∇_{B^*}) satisfying the following condition: we say that $W.$ is *graded-extendable* if it is a filtration by flat subbundles or equivalently by sub-local systems, and if each associated-graded piece Gr_j^W corresponds to a local system which extends from B^* to B.

Consider a tubular neighborhood B of D, as obtained in Lemma 2.3, and $B^* := B \cap U = B - D$. Associate the Mayer-Vietoris sequence for the pair (U, B):

$$H^{2i-2}(B^*, \mathbb{C}/\mathbb{Z}) \to H^{2i-1}(X, \mathbb{C}/\mathbb{Z}) \to H^{2i-1}(B, \mathbb{C}/\mathbb{Z}) \oplus H^{2i-1}(U, \mathbb{C}/\mathbb{Z})$$
$$\to H^{2i-1}(B^*, \mathbb{C}/\mathbb{Z}) \to .$$

Consider the restrictions (E_B, ∇_B) on B and (E, ∇) on U. Then we have the secondary classes, defined in Corollary 2.5,

$$\widehat{c}_i(E_B, \nabla_B) \in H^{2i-1}(B, \mathbb{C}/\mathbb{Z}) \tag{6}$$

and

$$\widehat{c}_i(E, \nabla) \in H^{2i-1}(U, \mathbb{C}/\mathbb{Z}) \tag{7}$$

such that

$$\widehat{c}_i(E_B, \nabla_B)|_{B^*} = \widehat{c}_i(E, \nabla)|_{B^*} \in H^{2i-1}(B^*, \mathbb{C}/\mathbb{Z}).$$

The above Mayer-Vietoris sequence yields a class

$$\widehat{c}_i(\overline{E},\overline{\nabla}) \in H^{2i-1}(X,\mathbb{C}/\mathbb{Z}) \tag{8}$$

which is obtained by gluing the classes in (6) and (7).

As such, the Mayer-Vietoris sequence doesn't uniquely determine the class: there is a possible indeterminacy by the image of $H^{2i-2}(B^*,\mathbb{C}/\mathbb{Z})$ under the connecting map. Nonetheless, we will show in §4, using a patched connection, that there is a canonically determined class $\widehat{c}_i(\overline{E},\overline{\nabla})$ as above which is functorial and additive (§6) and moreover it lifts the Deligne Chern class (§5).

3. The $\mathcal{C}^\infty$-trivialization of canonical extensions

To further motivate the construction of regulator classes, we digress for a moment to give a generalization of the result of Deligne and Sullivan on topological triviality of flat bundles, to the case of the canonical extension. The topological model of the canonical extension we obtain in this section, on an idea communicated to us by Deligne [De3], motivates the construction of a filtration triple in §7.3 which is required to define regulator classes using K-theory.

Suppose X is a proper $\mathcal{C}^\infty$-manifold of dimension d. Let E be a complex vector bundle of rank n. It is well-known that if $N \geq \frac{d}{2}$, then the Grassmanian manifold $\mathrm{Grass}(n,\mathbb{C}^{n+N})$ of n-dimensional subspaces of $\mathbb{C}^{n+N}$, classifies complex vector bundles of rank n on manifolds of dimension $\leq d$. In other words, given a complex vector bundle E on X, there exists a morphism

$$f : X \longrightarrow \mathrm{Grass}(n,\mathbb{C}^{n+N})$$

such that the pullback $f^*\mathcal{U}$ of the tautological bundle $\mathcal{U}$ on $\mathrm{Grass}(n,\mathbb{C}^{n+N})$ is E. If the morphism f is homotopic to a constant map then E is trivial as a $\mathcal{C}^\infty$-bundle. This observation is used to obtain an upper bound for the order of torsion of Betti Chern classes of flat bundles.

3.1. $\mathcal{C}^\infty$-trivialization of flat bundles.

Suppose E is equipped with a flat connection ∇. Then the Chern-Weil theory implies that the Betti Chern classes $c_i^B(E) \in H^{2i}(X,\mathbb{Z})$ are torsion. An upper bound for the order of torsion was given by Grothendieck [Gk]. An explanation of the torsion-property is given by the following theorem due to Deligne and Sullivan:

Theorem 3.1. [De-Su] *Let V be a complex local system of dimension n on a compact polyhedron X and $\mathcal{V} = V \otimes \mathcal{O}_X$ be the corresponding flat vector bundle. There exists a finite surjective covering $\pi : \tilde{X} \longrightarrow X$ of X such that the pullback vector bundle $\pi^*\mathcal{V}$ is trivial as a $\mathcal{C}^\infty$-bundle.*

An upper bound for the order of torsion is also prescribed in their proof which depends on the field of definition of the monodromy representation.

3.2. **$\mathcal{C}^\infty$-trivialization of canonical extensions.** Suppose X is a complex analytic variety $D \subset X$ a smooth irreducible divisor, and put $U := X - D$. Consider a flat vector bundle (E, ∇) on U and its canonical extension $(\overline{E}, \overline{\nabla})$ on X. Assume that the residues of $\overline{\nabla}$ are nilpotent. Then a computation of the de Rham Chern classes by Esnault [Es-Vi, Appendix B] shows that these classes are zero. This implies that the Betti Chern classes of $\overline{E}$ are torsion. We want to extend the Deligne-Sullivan theorem in this case, reflecting the torsion property of the Betti Chern classes.

Proposition 3.2. *Let E be a flat vector bundle on $U = X - D$, with unipotent monodromy around D. There is a finite covering $\tilde{U} \longrightarrow U$ such that if $\tilde{X}$ is the normalization of X in $\tilde{U}$, then the canonical extension of π^*E to $\tilde{X}$ is trivial as a $\mathcal{C}^\infty$-bundle.*

Note, in this statement, that the normalization $\tilde{X}$ is smooth, and the ramification of the map $\tilde{X} \to X$ is topologically constant along D.

The following proof of this proposition is due to Deligne and we reproduce it from [De3].

Given a flat connection (E, ∇) on U with unipotent monodromy along D, by Lemma 2.3, there is a vector bundle $\mathcal{F}^r$ with a filtration on a tubular neighborhood B of D:

$$(0) = \mathcal{F}^0 \subset \mathcal{F}^1 \subset ... \subset \mathcal{F}^r = \overline{E}|_B$$

such that the graded pieces are flat connections associated to local systems V_i.

Suppose the monodromy representation of (E, ∇) is given by

$$\rho : \pi_1(X) \longrightarrow GL(A)$$

where $A \subset \mathbb{C}$ is of finite type over $\mathbb{Z}$. The filtration of the previous paragraph is also a filtration of local systems of A-modules over B^*. Then the canonical extension itself should be trivial as soon as for two maximal ideals q_1, q_2 of A having distinct residue field characteristic, ρ is trivial mod q_1 and q_2. Consider a finite étale cover

$$\pi' : U' \longrightarrow U \tag{9}$$

corresponding to the subgroup of $\pi_1(U, u)$ formed of elements g such that $\rho(g) \equiv 1$, mod q_1 and mod q_2. The index of this subgroup divides the order of $GL_r(A/q_1) \times GL_r(A/q_2)$ (see [De-Su]). Construct a further cover

$$\pi : \tilde{U} \longrightarrow U' \longrightarrow U$$

such that the filtration and local systems V_i are constant mod q_1 and mod q_2.

The proof of Proposition 3.2 now follows from a topological result which we formulate as follows. Suppose a polytope X is the union of polytopes U and B, intersecting along B^*. Suppose we are given:

(1) On U, there is a flat vector bundle $\mathcal{V}$ coming from a local system V_A of free A-modules of rank n.

(2) a filtration F of V_A on B^* such that the graded piece $\mathbf{gr}^i_F$ is a local system of free

A-modules of rank n_i.
(3) local systems V_A^i on B extending the $\mathbf{gr}_F^i$ on B^*.

Suppose these data are trivial mod q_1, q_2, i.e., we have constant V_A, constant filtration and constant extensions.

From (V_A, F, V_A^i) we get using the embedding $A \subset \mathbb{C}$ a flat vector bundle $\mathcal{V}$, a filtration F and extensions $\mathcal{V}^i$. One can use these to construct a vector bundle on X (no longer flat), unique up to non-unique isomorphisms as follows: on B^* pick a vector bundle splitting of the filtration and use it to glue to form a vector bundle $\overline{\mathcal{V}}$ on X. This should be the topological translation of "canonical extension".

Lemma 3.3. *In the above situation, the vector bundle $\overline{\mathcal{V}}$ is trivial.*

Proof. As in [De-Su], one constructs algebraic varieties

$$U_1 \cap B_1 \hookrightarrow U_1,\, U_1 \cap B_1 \hookrightarrow B_1$$

over Spec $(\mathbb{Z})$, which are unions of affine spaces, with the homotopy of

$$U \cap B \hookrightarrow U,\, U \cap B \hookrightarrow B.$$

In $(U_1 \cup B_1) \times \mathbb{A}^1$, let us take the closed subscheme

$$(U_1 \times \{1\}) \cup ((U_1 \cap B_1) \times \mathbb{A}^1) \cup (B_1 \times \{0\}).$$

This is a scheme over Spec $\mathbb{Z}$.

Over Spec A, our data gives a vector bundle $\widetilde{\mathcal{V}}$: on U_1, given by V_A, on B_1 by $\oplus V_A^i$, on $(U_1 \cap B_1) \times \mathbb{A}^1$ by an interpolation of them: given by the subcoherent sheaves $\sum t^i.F^i$ of the pullback of V_A (deformation of a filtration to a grading). More precisely, on $(U_1 \cap B_1) \times \mathbb{A}^1$, we consider the coherent subsheaf

$$\sum_i t^i \cdot F_i \subset A[t] \otimes V_A.$$

It is locally free over $(U_1 \cap B_1) \times \mathbb{A}^1$, so it corresponds to a vector bundle. When $t = 1$, on $U_1 \times \{1\}$, this yields the vector bundle given by V_A. When $t = 0$, on $B_1 \times \{0\}$, we get the associated graded vector bundle of the filtration F on $B_1 \times \{0\}$.

If we extend scalars to $\mathbb{C}$, we obtain yet another model $\widetilde{\mathcal{V}}_{\mathbb{C}}$ of the canonical extension. Now mod q_1, q_2, we obtain a trivial bundle and the arguments in [De-Su] apply. Indeed, consider the classifying map

$$f : X \to \mathrm{Grass}(n, \mathbb{C}^{n+N})$$

such that the universal bundle on the Grassmanian pulls back to the vector bundle $\widetilde{\mathcal{V}}$ on X. Here $\dim X = d$ and $N \geq \frac{d}{2}$. Consider the fiber space $X' \to X$ whose fiber at $x \in X$ is the space of linear embeddings of the vector space $\widetilde{\mathcal{V}}_x$ in $\mathbb{C}^{n+N}$. The problem is reduced to showing that the classifying map $f' : X' \to \mathrm{Grass}(n, \mathbb{C}^{n+N})$ composed with the projection to the d-th coskeleton of the Grassmanian is homotopically trivial. Since the Grassmanian is simply connected, by Hasse principle for morphisms [Su], it follows that the above composed map is homotopically trivial if and only if for all l the l-adic completions

$$f'_{\hat{l}} : X'_{\hat{l}} \to \mathrm{cosq}_d(\mathrm{Grass}(n, \mathbb{C}^{n+N}))_{\hat{l}} = \mathrm{cosq}_d((\mathrm{Grass}(n, \mathbb{C}^{n+N})_{\hat{l}})$$

are homotopically trivial. Since, there is a maximal ideal q of A whose residue field characteristic is different from l and such that ρ and the local systems V_i and filtrations are trivial mod q, the bundle $\widetilde{\mathcal{V}}$ is trivial mod q. The lemma from [De-Su, Lemme] applies directly to conclude that $f'_{\hat{l}}$ is homotopically trivial. This concludes the lemma. □

4. Patched connection on the canonical extension

The basic idea for making canonical the lifting in (8) is to patch together connections sharing the same block-diagonal part, then apply the Chern-Simons construction to obtain a class in the group of differential characters. The projection to closed forms is zero because the Chern forms of a connection with strictly upper-triangular curvature are zero. Then, the resulting secondary class is in the kernel in the exact sequence (3).

In this section we will consider a somewhat general open covering situation. However, much of this generality is not really used in our main construction of §4.4 where X will be covered by only two open sets and the filtration is trivial on one of them. We hope that the more general formalism, or something similar, will be useful for the normal-crossings case in the future.

4.1. **Locally nil-flat connections.** Suppose we have a manifold X and a bundle E over X, provided with the following data of local filtrations and connections: we are given a covering of X by open sets V_i, and for each i an increasing filtration W^i of the restricted bundle $E|_{V_i}$ by strict subbundles; and furthermore on the associated-graded bundles $Gr^{W^i}(E|_{V_i})$ we are given flat connections $\nabla_{i,\mathrm{Gr}}$. We don't for the moment assume any compatibility between these for different neighborhoods. Call $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$ a *pre-patching collection.*

We say that a connection ∇ on E is *compatible with the pre-patching collection* if on each V_i, ∇ preserves the filtration W^i and induces the flat connection $\nabla_{i,\mathrm{Gr}}$ on the associated-graded $Gr^{W^i}(E|_{V_i})$.

Proposition 4.1. *Suppose (X, E, ∇) is a connection compatible with a pre-patching collection $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$. Then:*
(a) The curvature form Ω of ∇ is strictly upper triangular with respect to the filtration W^i over each neighborhood V_i;
(b) In particular if $\mathcal{P}$ is any invariant polynomial of degree k then $\mathcal{P}(\Omega, \ldots, \Omega) = 0$, for example $Tr(\Omega \wedge \cdots \wedge \Omega) = 0$; and
(b) The Chern-Simons class of ∇ defines a class $\widehat{c_p}(E, \nabla) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$.

Proof. (a): On V_i the connection preserves W_i and induces a flat connection on the graded pieces. This implies exactly that Ω is strictly upper-triangular with respect to W_i, that is to say that as an $End(E)$-valued 2-form we have $\Omega : W^i_k \to A^2(X, W^i_{k-1})$.
(b): It follows immediately that $Tr(\Omega \wedge \cdots \wedge \Omega) = 0$, and the other invariant polynomials are deduced from these by polynomial operations so they vanish too.
(c): The Chern-Simons class of ∇ projects to zero in $A^k_{cl}(X, \mathbb{Z})$ by (b), so by the basic exact sequence (3) it defines a class in $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$. □

A fundamental observation about this construction is that the class $\widehat{c_p}(E, \nabla)$ depends only on the pre-patching collection and not on the choice of ∇.

Lemma 4.2. *Suppose* $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$ *is a given pre-patching collection, and suppose* ∇_0 *and* ∇_1 *are connections compatible with this collection. Then the Chern-Simons classes are equal:*

$$\widehat{c_p}(E, \nabla_0) = \widehat{c_p}(E, \nabla_1) \ \ \text{in} \ \ H^{2p-1}(X, \mathbb{C}/\mathbb{Z}).$$

Proof. Choose any affine path ∇_t of compatible connections between ∇_0 and ∇_1. For $t = 0, 1$ this coincides with the previous ones. Let Ω_t denote the curvature form of ∇_t and let ∇'_t denote the derivative with respect to t.

By Lemma 2.1, note that the tangent space to the group of differential characters (at any point) is given by

$$T(\widehat{H^{2p}}(X, \mathbb{C}/\mathbb{Z})) = \frac{A^{2p-1}(X)}{dA^{2p-2}}.$$

With respect to this description of the tangent spaces, the derivative of the Chern-Simons class is given by

$$p\mathcal{P}(\nabla'_t, \Omega_t, \ldots, \Omega_t) = pTr(\nabla'_t \wedge \Omega_t^{p-1}) + \ldots.$$

See §2.2, also [Ch-Sm, Proposition 2.9].

On any local neighborhood V_i, note that ∇_t preserves the filtration W^i, and induces the original flat connection on Gr^{W_i}; hence for all t, Ω_t and ∇'_t are strictly upper triangular. It follows that $Tr(\nabla'_t \wedge \Omega_t^{a-1}) = 0$ and $Tr(\Omega_t^b) = 0$ so all the terms in $p\mathcal{P}(\nabla'_t, \Omega_t, \ldots, \Omega_t)$ vanish (see [Gri-Ha, p.403] for the explicit formula of $\mathcal{P}$). By the variational formula (4), the class in $\widehat{H^{2p}}(X, \mathbb{C}/\mathbb{Z})$ defined by ∇_t is independent of t. In other words, the ∇_t all define the same class in $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$. □

Say that a bundle with connection (X, E, ∇) is *locally nil-flat* if there exists a pre-patching collection for which ∇ is compatible. On the other hand, say that a pre-patching collection $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$ is a *patching collection* if there exists at least one compatible connection. Any compatible connection will be called a *patched connection*.

The above Proposition 4.1 and Lemma 4.2 say that if (X, E, ∇) is a locally nil-flat connection, then we get a Chern-Simons class $\widehat{c_p}(E, \nabla)$, and similarly given a patching collection we get a class defined as the class associated to any compatible connection; and these classes are all the same so they only depend on the patching collection so they could be denoted by

$$\widehat{c}_p(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\}) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}).$$

4.2. **Refinements.** If we are given a filtration W_k of a bundle E by strict subbundles, a *refinement* W'_m is another filtration by strict subbundles such that for any k there is an $m(k)$ such that $W_k = W'_{m(k)}$. In this case, W' induces a filtration $Gr^W(W')$ on $Gr^W(E)$. It will be useful to have a criterion for when two filtrations admit a common refinement.

Lemma 4.3. *Suppose E is a $\mathcal{C}^\infty$ vector bundle over a manifold, and $\{U_k\}_{k\in K}$ is a finite collection of strict subbundles containing 0 and E. Then it is the collection of bundles in a filtration of E, if and only if the following criterion is satisfied: for all $j, k \in K$ either $U_k \subset U_j$ or $U_j \subset U_k$. Suppose $\{W_i\}_{i\in I}$ and $\{U_k\}_{k\in K}$ are two filtrations of E. Then they admit a common refinement if and only if the following criterion is satisfied: for any $i \in I$ and any $k \in K$, either $W_i \subset U_k$ or else $U_k \subset W_i$.*

Proof. We prove the first part. If the collection corresponds to a filtration then it obviously satisfies the criterion. Suppose given a collection of strict subbundles satisfying the criterion. The relation $i \le j \Leftrightarrow U_i \subset U_j$ induces a total order on K, and with respect to this total order the collection is a filtration.

Now the second part of the lemma follows immediately from the first: the two filtrations admit a common refinement if and only if the union of the two collections satisfies the criterion of the first part. Given that $\{W_i\}_{i\in I}$ and $\{U_k\}_{k\in K}$ are already supposed to be filtrations, they already satisfy the criterion separately. The only other case is when $i \in I$ and $k \in K$ which is precisely the criterion of this part. □

Corollary 4.4. *Suppose E is a bundle with N filtrations, every two of which admit a common refinement. Then the N filtrations admit a common refinement. Furthermore there exists a common refinement in which each component bundle comes from at least one of the original filtrations.*

Proof. The union of the three collections satisfies the criterion of the first part of Lemma 4.3, since that criterion only makes reference to two indices at at time. This union satisfies the condition in the last sentence. □

A *refinement* of a pre-patching collection is a refinement $\tilde{V}_j$ of the open covering, with index set J mapping to the original index set I by a map denoted $j \mapsto i(j)$, and open subsets $\tilde{V}_j \subset V_{i(j)}$ such that the $\tilde{V}_j$ still cover X. Plus, on each $\tilde{V}_j$ a filtration $\tilde{W}^j$ of $E|_{\tilde{V}_j}$ which is a refinement of the restriction of $W^{i(j)}$ to $\tilde{V}_j$. Finally we assume that over $\tilde{V}_j$ the connection $\nabla_{i(j),\mathrm{Gr}}$ on $\mathrm{Gr}^{W^i}(E)$ preserves the induced filtration $\mathrm{Gr}^{W^{i(j)}}(\tilde{W}^j_\cdot)$ and the refined connection $\tilde{\nabla}_{j,\mathrm{Gr}}$ is the connection which is induced by $\nabla_{i(j),\mathrm{Gr}}$ on the associated-graded $\mathrm{Gr}^{\tilde{W}_j}(E)$.

Lemma 4.5. *Suppose ∇ is a patched connection compatible with a pre-patching collection $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$, and suppose $(X, E, \{(\tilde{V}_j, \tilde{W}^j, \tilde{\nabla}_{j,\mathrm{Gr}})\})$ is a refinement for $j \mapsto i(j)$. Then ∇ is also a patched connection compatible with $(X, E, \{(\tilde{V}_j, \tilde{W}^j, \tilde{\nabla}_{j,\mathrm{Gr}})\})$.*

Proof. The connection ∇ induces on $\mathrm{Gr}^{W^i}(E)$ the given connection $\nabla_{i,\mathrm{Gr}}$. By the definition of refinement, this connection in turn preserves the induced filtration $\mathrm{Gr}^{W^{i(j)}}(\tilde{W}^j)$. It follows that ∇ preserves $\tilde{W}^j$. Furthermore, $\nabla_{i,\mathrm{Gr}}$ induces on $\mathrm{Gr}^{\tilde{W}_j}(E)$ the connection $\tilde{\nabla}_{j,\mathrm{Gr}}$ in the data of the refinement, and since ∇ induced $\nabla_{i,\mathrm{Gr}}$ it follows that ∇ induces $\tilde{\nabla}_{j,\mathrm{Gr}}$ on $\mathrm{Gr}^{\tilde{W}_j}(E)$. □

Corollary 4.6. *If two patching collections*

$$(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\}) \text{ and } (X, E, \{(\tilde{V}_j, \tilde{W}^j, \tilde{\nabla}_{j,\mathrm{Gr}})\})$$

admit a common refinement, then

$$\widehat{c_p}(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\}) = \widehat{c_p}(X, E, \{(\tilde{V}_j, \tilde{W}^j, \tilde{\nabla}_{j,\mathrm{Gr}})\}).$$

Proof. Let ∇ and $\tilde{\nabla}$ denote compatible connections for the two patching collections. By the previous lemma, they are both compatible with the common refined patching collection. By Lemma 4.2 applied to the refinement, $\widehat{c_p}(E, \nabla) = \widehat{c_p}(E, \tilde{\nabla})$. But $\widehat{c_p}(E, \nabla)$ and $\widehat{c_p}(E, \tilde{\nabla})$ are respectively ways of calculating $\widehat{c_p}(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$ and $\widehat{c_p}(X, E, \{(\tilde{V}_j, \tilde{W}^j, \tilde{\nabla}_{j,\mathrm{Gr}})\})$, so these last two are equal. □

4.3. **Construction of a patched connection.** Suppose we have a pre-patching collection $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$. In order to construct a compatible connection, we need the following compatibility condition on the intersections $V_i \cap V_j$.

Condition 4.7. *We say that the pre-patching collection satisfies the* patching compatibility condition *if for any point $x \in V_i \cap V_j$ there is a neighborhood V'_x of x and a common refinement $\tilde{W}^x$ of both filtrations W^i and W^j on V'_x, consisting of bundles coming from these filtrations, such that the connections $\nabla_{i,\mathrm{Gr}}$ and $\nabla_{j,\mathrm{Gr}}$ both preserve the filtrations induced by $\tilde{W}^x$ on the respective associated graded bundles $Gr^{W^i}(E|_{V'_x})$ and $Gr^{W^j}(E|_{V'_x})$. Furthermore we require that the induced connections on $Gr^{\tilde{W}^x}(E|_{V'_x})$ be the same.*

Lemma 4.8. *Suppose $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$ is a pre-patching collection which satisfies the criterion 4.7. Then for any point x lying in several open sets $V_{i_1}, \ldots, V_{i_N}$, there is a smaller neighborhood $x \in V''_x \subset V_{i_1} \cap \cdots \cap V_{i_N}$ and a common refinement U^x of all of the filtrations W^{i_j}, $j = 1, \ldots, N$ on $E|_{V''_x}$, such that the induced filtrations on any of the associated graded pieces $\mathrm{Gr}^{W^{i_j}}(E|_{V''_x})$ are preserved by the connections $\nabla_{i_j,\mathrm{Gr}}$, and the connections all induce the same connection on the associated graded of the common refined filtration U^x.*

Proof. Fix $x \in V_{i_1} \cap \cdots \cap V_{i_N}$. Choose any neighborhood of x contained in the intersection. The filtrations W^{i_j}, $j = 1, \ldots, N$ admit pairwise common refinements by Condition 4.7. Therefore by Corollary 4.4, they admit a single refinement U^x common to all, and furthermore the component bundles U^x_a are taken from among the component bundles of the different W^{i_j}.

Now, on an associated-graded piece $\mathrm{Gr}^{W^{i_j}}(E|_{V''_x})$ consider one of the bundles in the induced filtration $\mathrm{Gr}^{W^{i_j}}(U^x_a)$. This comes from another filtration, so it is equal to some $\mathrm{Gr}^{W^{i_j}}(W^{i_\ell}_b)$. Then Condition 4.7 says that this bundle is preserved by the connection $\nabla_{i_j,\mathrm{Gr}}$. This shows the next to last phrase.

Finally, choose some associated-graded piece U^x_a/U^x_{a-1}, and two other indices i_j and i_ℓ. There is an index b such that

$$W^{i_j}_{b-1} \subset U^x_{a-1} \subset U^x_a \subset W^{i_j}_b.$$

Similarly there is an index c such that

$$W^{i_\ell}_{c-1} \subset U^x_{a-1} \subset U^x_a \subset W^{i_\ell}_c.$$

Now U^x_a/U^x_{a-1} is a subquotient of one of the terms G in the associated-graded for the common refinement of W^{i_j} and W^{i_ℓ}. The connections $\nabla_{i_j,\mathrm{Gr}}$ and $\nabla_{i_\ell,\mathrm{Gr}}$ define the same connection on G, and both of them preserve the subbundles of G corresponding to U^x_{a-1} and U^x_a. Hence they induce the same connection on U^x_a/U^x_{a-1}. This proves the last phrase. □

Theorem 4.9. *Suppose $(X, E, \{(V_i, W^i, \nabla_{i,\mathrm{Gr}})\})$ is a pre-patching collection which satisfies the above patching compatibility condition 4.7. Then it has a refinement which is a patching collection, that is to say there exists a compatible patched connection for a refined pre-patching collection.*

Proof. To begin, we can choose over each V_i a connection ∇_i on $E|_{V_i}$ such that ∇_i preserves the filtration W^i and induces the connection $\nabla_{i,\mathrm{Gr}}$ on the associated-graded. One way to do this for example is to choose a $\mathcal{C}^\infty$ hermitian metric on E which induces a splitting

$$\mathrm{Gr}^{W_i}(E|_{V_i}) \cong E|_{V_i},$$

and use this isomorphism to transport the connection $\nabla_{i,\mathrm{Gr}}$.

Choose a partition of unity $1 = \sum_i \zeta_i$ with $\mathrm{Supp}(\zeta_i)$ relatively compact in U_i. Consider the *patched connection*

$$\nabla^\# := \sum_i \zeta_i \nabla_i.$$

It is well-defined as a $\mathcal{C}^\infty$ operator $E \to A^1(E)$ (where $A^\cdot$ denotes the differential forms on X), because the ζ_i are compactly supported in the open set U_i of definition of ∇_i. Furthermore, it is a connection operator, that is it satisfies Leibniz' rule:

$$\nabla^\#(ae) = \sum_i \zeta_i \nabla_i(ae) = \sum_i a\zeta_i\nabla_i(e) + (\sum_i \zeta_i d(a))e = a\nabla^\#(e) + d(a)e$$

using $\sum_i \zeta_i = 1$.

We would now like to consider compatibility of $\nabla^\#$ with the filtrations. Choose $x \in X$. Let $i_1, \ldots, i_N$ be the indices for which x is contained in $Supp(\zeta_{i_j})$. Choose a neighborhood

V''_x as in the situation of Lemma 4.6, contained in $V_{i_1} \cap \cdots \cap V_{i_N}$ but not meeting the support of any ζ_j for j not in $\{i_1, \ldots, i_N\}$. Let U^x be the common refinement of the filtrations W^{i_j} given by Lemma 4.6.

Each of the connections ∇_{i_j} preserves every U^x_a. Indeed, U^x_a is sandwiched between $W^{i_j}_{b-1}$ and $W^{i_j}_b$, and ∇_{i_j} induces the connection $\nabla_{i_j,\mathrm{Gr}}$ on $W^{i_j}_b/W^{i_j}_{b-1}$. By hypothesis, and Lemma 4.6, the connection $\nabla_{i_j,\mathrm{Gr}}$ preserves the image of U^x_a in $W^{i_j}_b/W^{i_j}_{b-1}$, therefore ∇_{i_j} preserves U^x_a.

Furthermore, the connections ∇_{i_j} all induce the same connection on U^x_a/U^x_{a-1}, as follows from the same statement for the connections $\nabla_{i_j,\mathrm{Gr}}$ in Lemma 4.6.

The neighborhoods V''_x cover X. Together with the filtrations U^x and the connections induced by any of the ∇_{i_j} on $\mathrm{Gr}^{U^x}(E|_{V''_x})$ this gives a pre-patching collection refining the original one.

The connection $\nabla^{\#}$ is compatible with this new pre-patching collection. Indeed, it is a sum of terms ∇_i and on any open set V''_x the only terms which come into play are the ∇_{i_j} which preserve the filtration U^x and induce the given connections on $\mathrm{Gr}^{U^x}(E|_{V''_x})$. This $\nabla^{\#}$ preserves the filtration U^x. By the partition of unity condition $\sum \zeta_{i_j} = 1$ on V''_x, the patched connection $\nabla^{\#}$ induces the given connection on each $\mathrm{Gr}^{U^x}(E|_{V''_x})$. $\square$

4.4. **The patched connection for a representation unipotent along a smooth divisor.** If we have tried to be somewhat general in the previous presentation, we only use the construction of the patched connection in the simplest case. Suppose X is a smooth variety and $D \subset X$ is a closed smooth irreducible divisor. Choose the basepoint $x \in X - D$ and suppose we have a representation $\rho : \pi_1(X - D, x) \to GL_r(\mathbb{C})$.

Let γ denote the path going from x out to a point near D, once around counterclockwise, then back to x. We assume that ρ is *unipotent at infinity*, that is to say that the $\rho(\gamma)$ is a unipotent matrix.

As usual, fix the following two neighborhoods covering X. First, $U := X - D$ is the complement of D. Then B is a tubular neighborhood of D. Let $B^* := U \cap B$, it is the complement of D in B otherwise known as the *punctured tubular neighborhood*. We have a projection $B \to D$, making B into a disc bundle and B^* into a punctured-disk bundle over D. In terms of the previous notations, the index set is $I = \{0, 1\}$ and $U_0 = U$, $U_1 = B$ with $U_0 \cap U_1 = B^*$.

Let $\overline{E}$ denote the holomorphic vector bundle on X which is the Deligne canonical extension of the flat bundle associated to ρ. Let ∇ denote the flat connection on E. In particular, (E, ∇) is the flat bundle over U associated to ρ.

Fix the trivial filtration $W^0_0 := E$ and $W^0_{-1} = 0$ over the open set $U = U_0$. The associated-graded is the whole bundle E and we take $\nabla_{0,\mathrm{Gr}} := \nabla$.

Recall that a *graded-extendable filtration* on $E|_{B^*}$ is a filtration $\{W_k\}$ by strict ∇-flat subbundles, such that the induced connection ∇_{Gr} on $Gr^W(E|_{B^*})$ extends to a connection over B. Note that the W_k extend to strict subbundles of $\overline{E}|_B$, indeed we take the canonical extension of W_k with respect to the connection induced by ∇. Hence we are given natural bundles $Gr^W(\overline{E}|_B)$ and the graded-extendability condition says that the connection induced by ∇ on these graded bundles, should be nonsingular along D.

Examples of such filtrations include the kernel filtration (see Lemma 2.3) or the monodromy weight filtration along D, using the hypothesis that ρ is unipotent at infinity.

On $U_1 = B$ let $W^1 = \{W_k\}$ denote some choice of graded-extendable filtration.

Let $\nabla_{1,\mathrm{Gr}}$ be the connection induced by ∇ over B^*, projected to the associated-graded $\mathrm{Gr}^{W^i}(E|_{B^*})$ and then extended from B^* to a connection on $\overline{E}|_B$, well-defined over all of B.

The resulting collection of neighborhoods, filtrations and connections on the associated-graded's, is a pre-patching collection on X.

Lemma 4.10. *Suppose ρ is a representation of $\pi_1(U)$ which is unipotent at infinity, and choose a graded-extendable filtration W on the corresponding flat bundle restricted to B^*. The pre-patching collection associated to (ρ, W) by the above discussion satisfies the compatibility condition 4.7, hence by Theorem 4.9 it admits a compatible patched connection denoted $\nabla^{\#}$.*

Proof. This is obvious, since the filtration on U is the trivial filtration so over the intersection B^* it clearly admits a common refinement with the filtration $\{W_k\}$ on B. □

Since there are only two open sets and a single intersection, it is easy to write down explicitly the patched connection $\nabla^{\#}$ here. Furthermore, there is no need to refine the pre-patching collection in this case.

The partition of unity consists of a single function ζ supported on B with $1-\zeta$ supported on U. We choose a $\mathcal{C}^\infty$ trivialization of the filtration over B, $\overline{E}|_B \cong Gr^W(\overline{E}|_B)$. Thus $\nabla_{1,\mathrm{Gr}} = Gr^W(\nabla)$ gets transported to a connection ∇_B on $\overline{E}|_B$. Then

$$\nabla^{\#} = (1-\zeta)\nabla + \zeta\nabla_B$$

is a $\mathcal{C}^\infty$ connection on $\overline{E}$ over X. Over B it preserves the filtration W and on $Gr^W(\overline{E}|_B)$ it induces the given connection $\nabla_{1,\mathrm{Gr}}$ which is flat. Thus, $\nabla^{\#}$ is locally nil-flat in the easy sense that, over the open set $U' \subset U$ which is the complement of the support of ζ, it is flat (equal to the original ∇), while over B it is upper triangular with strictly upper triangular curvature, with respect to the filtration W. We have $X = U' \cup B$.

Corollary 4.11. *We obtain secondary classes*

$$\widehat{c}_p(\rho, W) := \widehat{c}_p(\nabla^{\#}) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$$

from the patched connection. These are independent of the choices of neighborhoods and partitions of unity used to define $\nabla^{\#}$.

Proof. It follows directly from Proposition 4.1 and Lemma 4.2: the Chern forms of $\nabla^{\#}$ vanish identically everywhere, because the curvature is everywhere strictly upper triangular in some frame. Thus, the Cheeger-Simons class in differential characters, lies in the subgroup $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$. This argument was mentioned in Corollary 2.4 of Cheeger-Simons [Ch-Sm]. Independence of choices follows from Lemma 4.2. □

Using the extension of Deligne-Sullivan [De-Su] given by Proposition 3.2, eventually after going to a finite cover of X ramified only over D, we can assume that the canonical extension $\overline{E}$ is trivial. Thus, we can apply the variational way of understanding the Chern-Simons class of $\nabla^{\#}$ in differential characters.

We will see in §6 below (Corollary 6.2) that the class $\widehat{c}_p(\rho, W)$ is also independent of the choice of graded-extendable filtration W, so it can also be denoted $\widehat{c}_p(\rho/X)$.

In the more general normal-crossings case, one would like to apply the general considerations of the previous subsections to obtain a construction. However, we found that it is not immediately obvious how to produce a covering and appropriate filtrations such that the filtrations admit a common refinement on the intersections (Condition 4.7). The structure of the commuting nilpotent logarithms of monodromy transformations is complicated. Some structure results are known, for example the monodromy weight filtrations of $\sum a_i N_i$ are the same whenever $a_i > 0$, a result which is now generalized from the case of variations of Hodge structure to any harmonic bundle by Mochizuki [Mo]. However, this doesn't provide an immediate answer for patching the connection. This is one of the main reasons why, in the present paper, we are treating the case of a smooth divisor only.

See also Remark 7.6 below for a somewhat different difficulty in the normal crossings case.

5. Compatibility with the Deligne Chern class

Suppose X is a smooth complex projective variety. Consider the following situation: E is a holomorphic vector bundle on X with holomorphic structure operator $\overline{\partial}$. Suppose ∇_1 is a connection obtained by the patching construction. We assume that in a standard neighborhood V_x of any point x, the local filtrations W^x are by holomorphic subbundles of $(E, \overline{\partial})$, and that the holomorphic structure on the graded pieces $Gr_k^{W_x}(E)$ coincides with the $(0,1)$ part of the flat connections induced by ∇_1 on these pieces. In this situation, we claim that the Chern-Simons class in $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$ defined by the patched connection ∇_1, projects to the Deligne Chern class of $(E, \overline{\partial})$ in $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$.

For this we use the formalism of F^1-connections introduced by Dupont, Hain and Zucker. Their method fully works only when X is compact. Recall that this is a variant of the differential character construction. Let $DHZ^{k,k+1}$ denote the group of analogues of differential characters used by Dupont, Hain and Zucker. We have an exact sequence, by quotienting the exact sequence in (3) by the Hodge piece F^p;

$$0 \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to DHZ^{2p-1,2p} \to \frac{A^{2p}_{\mathrm{cl}}(X, \mathbb{Z})}{(A^{p,p} + \cdots + A^{2p,0}) \cap A^{2p}_{\mathrm{cl}}(X, \mathbb{Z})} \to 0.$$

Here $DHZ^{2p-1,2p} := \widehat{H^{2p}}(X, \mathbb{C}/\mathbb{Z}(p))/F^p$.

Suppose we have a connection ∇_0 compatible with $\overline{\partial}$; this means that $\nabla_0^{0,1} = \overline{\partial}$. In [DHZ], it is shown that the differential character defined by the connection ∇_0 projects from $\widehat{H^{2p}}(X, \mathbb{C}/\mathbb{Z})$ to $DHZ^{2p-1,2p}$, to a class which goes to zero in the term "closed forms modulo the Hodge filtration" on the right, and which thus comes from a class in the Deligne cohomology on the left; and that this is the same as the Deligne class of $(E, \overline{\partial})$.

In our case, we construct ∇_0 as follows: take

$$\nabla_0^{0,1} := \overline{\partial}, \quad \nabla_0^{1,0} := \nabla_1^{1,0}.$$

This is by definition compatible with $\overline{\partial}$, so its class in $\widehat{H^{2p}}(X, \mathbb{C}/\mathbb{Z})$ projects to the Deligne Chern class by Dupont-Hain-Zucker [DHZ].

On the other hand, notice that ∇_0 defines a connection which preserves the filtration W^x on the neighborhood of any $x \in X$, and which induces the original flat connection on the associated graded pieces. Preserving the filtration is because $\overline{\partial}$ and ∇_1 both preserve the filtration. On the graded pieces, recall that ∇_1 induces the flat connection, and also the flat connection has the same operator $\overline{\partial}$ as comes from E. In particular $\nabla_0^{0,1} = \nabla_1^{0,1}$ on the graded pieces, so ∇_0 induces the same connection as ∇_1 here.

From this we get that ∇_0 also has strictly upper triangular curvature form Ω_0, so its class in $\widehat{H^{2p}}(X, \mathbb{C}/\mathbb{Z})$ projects to zero in $A^{2p}_{\mathrm{cl}}(X, \mathbb{Z})$. Thus, ∇_0 defines a class in $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$. This class projects to the Deligne Chern class, by the result of [DHZ].

To finish the proof of compatibility, we will show that ∇_0 and ∇_1 define the same class in $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$.

Lemma 5.1. *The Chern-Simon classes $\widehat{c_p}(E, \nabla_0)$ and $\widehat{c_p}(E, \nabla_1)$ are equal.*

Proof. For this, connect the connection ∇_1 to ∇_0 by an affine path of connections

$$\nabla_t = t\nabla_1 + (1-t)\nabla_0.$$

For $t = 0, 1$ this coincides with the previous ones. Let Ω_t denote the curvature form of ∇_t and let ∇_t' denote the derivative with respect to t. The rest of the proof is the same as in Lemma 4.2. □

Denote this class by $\widehat{c_p}(E, \nabla)$, for $p \geq 1$. We have thus shown, together with Lemma 5.1 and Corollary 4.11:

Proposition 5.2. *Suppose X is a smooth complex projective variety, with a logarithmic connection (E, ∇) on X with nilpotent residues along a smooth and irreducible divisor*

D. It restricts to a flat connection (E_U, ∇_U) on the complement $U := X - D$. Let B be a tubular neighborhood of the divisor D as obtained in Lemma 2.3 and (E_B, ∇_B) be the restriction of (E, ∇) on B. Then the secondary classes $\widehat{c_p}(E_B, \nabla_B)$ and $\widehat{c_p}(E_U, \nabla_U)$ glue together to give a canonically determined class $\widehat{c_p}(E, \nabla) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$, for $p \geq 1$. The classes $\widehat{c_p}(E, \nabla)$ lift the Deligne Chern classes $c_p^{\mathcal{D}}(E)$ under the projection

$$H^{2p-1}(X, \mathbb{C}/\mathbb{Z}) \longrightarrow H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

6. Rigidity of the secondary classes

In this section we would like to show that the secondary classes are invariant under deformations of the representation. In the flat case this is a consequence of a well-known formula for the variation of the secondary class. In our case we need to be somewhat careful about the local filtrations.

Before getting to the rigidity result we look at the construction from a somewhat more general point of view. We are given an open covering of X by neighborhoods U and B. In our situation B and even $B^* := B \cap U$ are connected, and $\pi_1(B^*) \to \pi_1(B)$ is surjective.

We have a representation ρ of $\pi_1(U)$, corresponding to a flat bundle (E, ∇) and to a local system $L = E^{\nabla}$ on U. Denote by L_{B^*} the restriction of L to a local system on B^*. Suppose we are given a filtration W of L_{B^*} such that the graded pieces Gr_k^W extend to local systems over B.

The patching construction with trivial filtration over U gives a patched connection and a secondary class which we denote here by

$$\widehat{c_p}(E, \nabla, W) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$$

to emphasize dependence on the filtration.

Recall from Corollary 4.6: if $\tilde{W}$ is a different filtration such that W and $\tilde{W}$ admit a common refinement, then

$$\widehat{c_p}(E, \nabla, W) = \widehat{c_p}(E, \nabla, \tilde{W}).$$

Lemma 6.1. *Suppose W and $\tilde{W}$ are two filtrations of L_{B^*} by sub-local systems, such that the associated graded pieces extend as local systems on B. Then these are connected by a string of filtrations*

$$W(0) = W, W(1), \ldots, W(a_1) = \tilde{W}$$

such that $W(a)$ satisfies the same conditions for any $0 \leq a \leq a_1$: it is a filtration of L_{B^*} by sub-local systems, such that the associated graded pieces extend as local systems on B. Furthermore, any adjacent ones $W(a-1)$ and $W(a)$ admit a common refinement for $0 < a \leq a_1$.

Proof. The proof is by induction on the rank r of E. It is easy for $r = 1$, so we assume $r > 1$ and that it is known for representations of rank $r' < r$.

Recall that we have the *canonical Jordan-Hölder filtration* W^{JH} of L_i. The first step W^{JH}_0 is *socle* or largest semisimple subobject of L_{B^*}, and the remainder of the filtration is determined inductively by the condition that it should induce the canonical Jordan-Hölder filtration of $L_{B^*}/W^{\mathrm{JH}}_0$. We note that this filtration has the property that the associated graded pieces extend as local systems on B. Indeed, the associated-graded of $W(JH)$ is the semisimplification of L_{B^*}, but since L_{B^*} has at least one filtration W whose graded pieces extend to B, it follows that the pieces of the semisimplification all extend to B. In this argument we are using surjectivity of $\pi_1(B^*) \to \pi_1(B)$ so that an extension to B is unique if it exists.

Denote by W_b the first nontrivial piece in the filtration W. Then the socle of W_b is a nontrivial sub-local system $V \subset L_{B^*}$, so by the universal property of the socle of L_{B^*} this is contained in W^{JH}_0. Now V is a subsystem of the first elements of both filtrations W and W^{JH}. Hence, W and W^{JH} induce filtrations on L_{B^*}/V. By the inductive hypothesis, these two filtrations are connected by a sequence as in the conclusion of the lemma. Lifting and including V as the first element, we obtain a sequence of filtrations connecting $W \cup \{V\}$ to $W^{\mathrm{JH}} \cup \{V\}$. We can then add on W and W^{JH} to the ends of this sequence, so we obtain a sequence connecting W to W^{JH}. Similarly there is a sequence connecting $\tilde{W}$ to W^{JH}. Putting them together we obtain a sequence connecting W to $\tilde{W}$. This completes the proof. ∎

Corollary 6.2. *Suppose W and $\tilde{W}$ are two filtrations of L_{B^*} by sub-local systems, such that the associated graded pieces extend as local systems on B. Then*

$$\widehat{c}_p(E, \nabla, W) = \widehat{c}_p(E, \nabla, \tilde{W}).$$

Proof. Use the sequence of filtrations $W(a)$ constructed in the previous lemma. The secondary classes of adjacent elements are the same:

$$\widehat{c}_p(E, \nabla, W(a-1)) = \widehat{c}_p(E, \nabla, W(a)),$$

because the adjacent elements admit a common refinement. Therefore

$$\widehat{c}_p(E, \nabla, W) = \widehat{c}_p(E, \nabla, W(0)) = \widehat{c}_p(E, \nabla, W(a_1)) = \widehat{c}_p(E, \nabla, \tilde{W}).$$

∎

This corollary says that, while we used the monodromy weight filtrations as a canonical way of defining the secondary classes, we could have used any filtrations compatible with the flat connection and having associated-graded which extend as flat bundles on B. In view of this corollary, we now denote the secondary classes by

$$\widehat{c}_p(\rho/X) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}).$$

Corollary 6.3. *These classes are additive in ρ and contravariantly functorial in (X, D).*

Proof. Suppose ρ_1 and ρ_2 are representations on $U = X - D$ unipotent around D. Choose graded-extendable filtrations W^1 for $\rho_1|_{B^*}$ and W^2 for $\rho_2|_{B^*}$. Then $(W^1 \oplus W^2)_i :=$

$W_i^1 \oplus W_i^2$ is a graded-extendable filtration for $\rho_1 \oplus \rho_2$. From our construction of patched connections $\nabla_1^{\#}$ for (ρ_1, W^1) and $\nabla_2^{\#}$ for (ρ_2, W^2), we get a patched connection $\nabla_1^{\#} \oplus \nabla_2^{\#}$ for $(\rho_1 \oplus \rho_2, W^1 \oplus W^2)$. The associated differential Chern character is the sum, because the differential Chern characters are additive on direct sums of connections with vanishing Chern forms—the terms of the form $Tr(\ldots)Tr(\ldots)\ldots$ don't contribute in the variational formula (4). Thus

$$\widehat{c}_p((\rho_1 \oplus \rho_2)/X) = \sum_{r+s=p} \widehat{c}_r(\rho_1/X).\widehat{c}_s(\rho_2/X).$$

Suppose $f : (X', D') \to (X, D)$ is a morphism of smooth quasiprojective varieties with smooth divisors inducing a map $f : X' - D' \to X - D$, and suppose ρ is a representation of $\pi_1(X - D)$ unipotent along D. Then f^* is unipotent along D'. We can choose a tubular neighborhood B' of D' mapping into the tubular neighborhood B of D. If W is a graded-compatible filtration for ρ over B^* then f^* is a graded-compatible filtration for $f^*(\rho)$, and again by the formula for the patched connection ∇^* for (ρ, W), we get that $f^*\nabla^{\#}$ is a patched connection for $(f^*(\rho), f^*W)$. Hence

$$\widehat{c}_p(f^*(\rho)/X') = f^*\widehat{c}_p(\rho/X).$$

□

Turn now to the question of rigidity: if we deform the representation then the secondary classes stay the same.

Lemma 6.4. *Suppose we are given a $\mathcal{C}^\infty$ family of representations $\rho(t)$ of $\pi_1(U)$, for $t \in [0,1]$. Suppose we are given a $\mathcal{C}^\infty$ family of filtrations $W(t)$ by sub-local systems of $L_{B^*}(t)$, having the property that the associated-graded pieces extend across B. Then the secondary classes are constant:*

$$\widehat{c}_p(\rho(t)/X) = \widehat{c}_p(\rho(t')/X), \quad t, t' \in [0,1],$$

for $p \geq 2$.

Proof. We may localize in t to smaller intervals if necessary. Let E be the $\mathcal{C}^\infty$-bundle underlying the canonical extension of $\rho(0)$. Then we may identify the bundles corresponding to $\rho(t)$ with E, in such a way that the filtrations all correspond to the same filtration by strict subbundles. This is because the elements of the filtrations have the same ranks for all t, and as t varies we can redress the subbundles back onto the same original one by a Gramm-Schmidt process which is locally unique. Here we might cut the interval up into smaller pieces, but still a finite number by a compactness argument.

Now, $\rho(t)$ corresponds to a connection $\nabla(t)$ on $E|_U$, and $E|_B$ has a filtration by strict subbundles W which corresponds to the filtration of local systems $W(t)$ for each t. We obtain a $\mathcal{C}^\infty$ family of patched connections $\nabla^{\#}(t)$. These are all locally nil-flat, with respect to a constant filtration on each open set.

Now we apply the usual proof of rigidity of Chern-Simons classes, see Cheeger-Simons [Ch-Sm, Proposition 2.9]. They show that the difference between the Chern-Simons classes is given by

$$\widehat{c}_p(\nabla^{\#}(1)) - \widehat{c}_p(\nabla^{\#}(0)) = i. \int_0^1 P(\frac{d}{dt}\nabla^{\#}(t) \wedge \Omega_t^{p-1})dt|_{Z_{2i-1}}$$

Here P is the trace form defining the p-th Chern form and the integral is taken with endpoints 0 and 1. In our case, $\frac{d}{dt}\nabla^{\#}(t)$ are upper-triangular and Ω_t^{p-1} are strictly upper-triangular. Hence, as long as $p-1>0$ the trace vanishes. □

Finally, we obtain the rigidity in general.

Proposition 6.5. *Suppose X is covered by a Zariski open dense subset U and a tubular neighborhood B of the irreducible smooth divisor D, with $B^* := B \cap U$. Suppose we are given a continuous family of representations $\rho(t)$ of $\pi_1(U)$ for $t \in [0,1]$, whose monodromy transformations around the divisor $\rho(t)(\gamma)$ are unipotent. Then the secondary classes are constant:*

$$\widehat{c}_p(\rho(0)/X) = \widehat{c}_p(\rho(1)/X),$$

for $p \geq 2$.

Proof. There is an affine modular variety for representations ρ of $\pi_1(U)$ such that $\rho(\gamma)$ has trivial characteristic polynomial. In view of this, we may replace our continuous family of representations by a piecewise algebraic family. Then, the interval of definition can be divided up into sub-intervals of the form $[a_i, a_{i+1}]$ such that for $t \in (a_i, a_{i+1})$ the monodromy weight filtrations (or more easily, the kernel filtrations) of the $N(t) := \log \rho(t)(\gamma)$ vary in an algebraic manner with the same ranks. Then at the endpoints of these intervals, the limits of these filtrations are again filtrations by sub-local systems, such that the associated graded pieces extend across B. Note however that the limiting filtrations will not in general be the monodromy weight filtrations or kernel filtrations of the $N(t)$. Lemma 6.2 shows that the secondary classes defined with these limiting filtrations, are the same as those defined by the monodromy weight filtrations. Then Lemma 6.4 applies to give

$$\widehat{c}_p(\rho(a_i)/X) = \widehat{c}_p(\rho(a_{i+1})/X).$$

Putting these all together, from the first and last intervals we get the statement of the proposition. □

7. A deformational variant of the patching construction in K-theory

Reznikov's proof involved an aspect of arithmetical K-theory. Starting with a representation ρ defined over a number field F, he considered all embeddings $F \hookrightarrow \mathbb{C}$; for each embedding the volume piece of the regulator vanished, by differential geometric arguments. Using Borel's calculation of the rational K-theory of F, this then implied that the classifying map $X \to BSL(F)$ was trivial on the rational homology. In order to replicate this part of the proof here, we need a version of the regulator construction which is related to K-theory.

Recall that, with our usual notations, the diagram

$$\begin{array}{ccc} B^* & \to & B \\ \downarrow & & \downarrow \\ U & \to & X \end{array}$$

is a homotopy pushout. Indeed, if we were to replace U by the complement U_0 of the interior of B, consider B_0 a smaller closed tubular neighborhood, and let B_0^* be the closure of $X - U_0 - B_0$ then B_0^* is a cylinder on S which is the boundary of U_0 or B_0 (they can be identified by an isomorphism). Then X is exactly the standard cylindrical construction of the homotopy pushout for the maps $S \to B_0$ and $S \to U_0$, and the diagram

$$\begin{array}{ccc} B_0^* = S \times [0,1] & \to & B_0 \\ \downarrow & & \downarrow \\ U_0 & \to & X \end{array}$$

is homotopy equivalent to the previous one.

Suppose we have a representation ρ defined over a field F, that is $\rho : \pi_1(U) \to GL_r(F)$. Because of the homotopy pushout square, the problem of constructing a map $X \to BGL(F)^+$ is reduced to giving maps on B and U, plus a comparison over B^*. The representation ρ gives a map $U \to BGL(F)$. With the assumption that ρ is unipotent at infinity, choice of a compatible filtration W for $\rho|_{B^*} := \rho|_{\pi_1(B^*)}$ induces a representation which extends over B to

$$Gr^W(\rho|_{B^*}) : \pi_1(B) \to GL_r(F)$$

again giving a map $B \to BGL(F)$.

Deformation from a representation to its associated-graded, is a polynomial deformation. Thus we can get a map $B^* \to BGL(F[t])$ linking the maps on U and B. The deformation theorem in K-theory allows us to interpret this as a gluing datum giving rise to a map $X \to BGL(F)^+$.

7.1. **The deformation theorem.** Consider the following situation. Let F be a field. We get two morphisms $e_0, e_1 : F[t] \to F$ consisting of evaluation of polynomials at 0 and 1 respectively. Inclusion of constants is $c : F \to F[t]$, whose composition with e_i is the identity. The *deformation theorem* in K-theory (Quillen [Qu], see [Sr] or [Ro]) says that all of these maps induce homotopy equivalences of K-theory spaces

$$BGL(F)^+ \to BGL(F[t])^+ \stackrel{e_0 \text{ or } e_1}{\longrightarrow} BGL(F)^+.$$

Define a space $BGL(F)^+_{\rm def}$ to be the homotopy pushout in the diagram

$$\begin{array}{ccc} BGL(F[t])^+ & \to & BGL(F)^+ \\ \downarrow & & \downarrow \\ BGL(F)^+ & \to & BGL(F)^+_{\rm def} \end{array}.$$

Explicitly, $BGL(F)^+_{\rm def}$ is obtained by gluing the cylinder $BGL(F[t])^+ \times [0,1]$ to two copies of $BGL(F)^+$ along the evaluation maps

$$e_0 : BGL(F[t])^+ \times \{0\} \to BGL(F)^+$$

and

$$e_1 : BGL(F[t])^+ \times \{1\} \to BGL(F)^+.$$

We can express $BGL(F)^+_{\rm def}$ as a union of two open sets: the first is the gluing of

$$BGL(F[t])^+ \times [0,1) \text{ to a copy of } BGL(F)^+ \text{ by } e_0,$$

the second is the gluing of

$$BGL(F[t])^+ \times (0,1] \text{ to the other copy of } BGL(F)^+ \text{ by } e_1.$$

And their intersection is $BGL(F[t])^+ \times (0,1)$. Each of the open sets retracts to $BGL(F)^+$. The deformation theorem implies that the top and left maps in the above square are homotopy equivalences. The Van Kampen theorem plus Mayer-Vietoris implies that the maps $BGL(F)^+ \to BGL(F)^+_{\rm def}$ are homotopy equivalences.

Corollary 7.1. *There is an isomorphism on cohomology with any coefficients*

$$H^*(BGL(F)^+_{\rm def}, k) \cong H^*(BGL(F)^+, k) = H^*(BGL(F), k).$$

7.2. **Deformation patching.** Now we say that a *deformation patching datum* for our diagram of pointed connected spaces

$$(U,u) \xleftarrow{a} (B^*, b^*) \xrightarrow{b} (B, b)$$

is a triple of representations

$$\eta_U : \pi_1(U,u) \to GL(F),$$

$$\eta_B : \pi_1(B,b) \to GL(F),$$

and

$$\eta_{B^*} : \pi_1(B^*, b^*) \to GL(F[t])$$

such that

$$e_1 \circ \eta_{B^*} = a^*(\eta_U) \text{ and } e_0 \circ \eta_{B^*} = b^*(\eta_B).$$

In more geometric terms, we require representations on U and B, plus a deformation between their restrictions to B^*. Typically, the representations will go into a finite-dimensional subgroup of the form $GL_r(F)$ (resp. $GL_r(F[t])$).

Suppose X is the homotopy pushout of $U \leftarrow B^* \to B$. We can assume for example that, as in the geometric situation, this diagram is homotopic to $U_0 \leftarrow S \to B_0$ and B^* is a cylinder $S \times (0,1)$, furthermore $U = U_0 \cup B^*$ and $B = B_0 \cup B^*$ retract to U_0 and B_0 respectively.

Given a deformation patching datum $(\eta_U, \eta_B, \eta_{B^*})$, the representations give maps

$$U, B \to BGL(F) \text{ and } B^* \to BGL(F[t]).$$

By functoriality of homotopy pushout, this gives a homotopy class of maps

$$X \to BGL(F)^+_{\rm def},$$

in particular using Corollary 7.1 we get a map

$$H^*(BGL(F), k) \to H^*(X, k).$$

If $F \subset \mathbb{C}$ then we can apply this to the universal regulator class in $H^{2p-1}(BGL(F), \mathbb{C}/\mathbb{Z})$ to get a *deformation regulator class* denoted

$$\widehat{c}_p^{\rm def}(\eta_U, \eta_B, \eta_{B^*}) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}).$$

Its imaginary part will be called the *deformation volume regulator* denoted

$$Vol_{2p-1}^{\rm def}(\eta_U, \eta_B, \eta_{B^*}) \in H^{2p-1}(X, \mathbb{R}).$$

7.3. **The deformation associated to a filtration.** We now point out that we get a deformation patching in our standard canonical-extension situation. Assume here that the map $\pi_1(B^*) \to \pi_1(B)$ is surjective, as is the case if B is the tubular neighborhood of a divisor D and $B^* = B - D$.

Say that a *filtration patching datum* consists of a representation

$$\rho : \pi_1(U) \to GL(V)$$

for a finite dimensional vector space V, plus a filtration W of V such that W is invariant under the action of $\pi_1(B^*)$, and the induced action of $\pi_1(B^*)$ on $Gr^W(V)$ factors through a representation

$$Gr^W(\rho/B) : \pi_1(B) \to GL(Gr^W(V)).$$

Note that this factorization is unique because of the assumption that $\pi_1(B^*) \to \pi_1(B)$ is surjective. In the divisor situation, such a filtration will exist if and only if $\rho(\gamma)$ is unipotent.

Given a filtration patching datum (ρ, W) we can define a deformation patching datum as follows. Choose a splitting for the filtration $V = \bigoplus_i V_i$ which yields $V \cong Gr^W(V)$, and furthermore choose a compatible basis for V which gives

$$GL(V) \cong GL_r(F) \hookrightarrow GL(F).$$

Composing with ρ gives $\eta_U : \pi_1(U) \to GL(F)$. On the other hand, composing with the representation $Gr^W(\rho/B)$ gives

$$\eta_B : \pi_1(B) \xrightarrow{Gr^W(\rho/B)} GL(Gr^W(V)) \cong GL(V) \cong GL_r(F) \hookrightarrow GL(F).$$

For η_{B^*}, notice that the matrices preserving W are block upper triangular. Then define a deformation between $\rho|_{\pi_1(B^*)}$ and $Gr^W(\rho/B)|_{\pi_1(B^*)}$ as follows (this is the same deformation as was referred to in the proof of Lemma 3.3 communicated by Deligne [De3]). Using the decomposition of V we get

$$End(V) \cong \bigoplus_{i,j} Hom(V_i, V_j).$$

Let $End_W(V)$ be the subspace of endomorphisms preserving W, so

$$End_W(V) \cong \bigoplus_{i\geq j} Hom(V_i, V_j).$$

Define a map $\psi_t : End_W(V) \to End_W(V)$ by multiplying by t^{j-i} on the piece $Hom(V_i, V_j)$. At $t = 1$ this is the identity and at $t = 0$ this is the projection to the block diagonal pieces. Note that $\psi_t(MM') = \psi_t(M)\psi_t(M')$. Thus if $p : \Gamma \to End_W(V)$ is a group representation (whose image lies in the subset of invertible matrices) then the function $\psi_t \circ p$ is again a group representation. This gives a deformation $\Gamma \to GL_r(F[t])$ whose value at 0 is the original p and whose value at 0 is the associated-graded of p. Apply this to the restriction $\rho|_{\pi_1(B^*)}$ with $\Gamma = \pi_1(B^*)$. This gives a representation

$$\eta' : \pi_1(B^*) \to GL_r(F[t])$$

such that $e_1 \circ \eta'$ is the representation $\rho|_{\pi_1(B^*)}$, and whose value $e_0 \circ \eta'$ is the associated-graded, which is equal to $Gr^W(\rho/B)|_{\pi_1(B^*)}$. Letting η_{B^*} be the composition of η with the inclusion $GL_r(F) \hookrightarrow GL(F)$ we have completed our deformation patching datum $(\eta_U, \eta_B, \eta_{B^*})$ associated to the filtration patching datum (ρ, W).

Corollary 7.2. *Suppose we have a filtration patching datum (ρ, W). Then again supposing X is the homotopy pushout of $U \leftarrow B^* \to B$ we get a map $X \to BGL(F)^+$ and classes in $H^*(X, k)$ for any class in $H^*(BGL(F), k)$. In particular for $\sigma : F \to \mathbb{C}$ we get regulator classes denoted $\widehat{c}_p^{\rm def}(\rho, W) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$.*

7.4. **Comparison with the classes defined by patched connections.** We would like to compare these with the classes defined by the patched connections. As described at the beginning of this section, consider compact subsets $U_0 \subset U$ and $B_0 \subset B$, retracts of the bigger subsets, such that U_0 is the complement of an open tubular neighborhood of D and B_0 is a smaller closed tubular neighborhood. Consider $B_0^* \cong S \times [0, 1]$, the closure of $X - U_0 - B_0$. Thus, X is obtained by gluing together U_0 and B_0 with the cylinder B_0^*. In this way X can be seen as a homotopy pushout.

Recall that S is an S^1 bundle over D.

Suppose $F = \mathbb{C}$ and we have a deformation patching datum $(\eta_U, \eta_B, \eta_{B^*})$. Suppose also that the representations η_U, η_B (resp. η_{B^*}) go into a finite rank group $GL_r(F) = GL_r(\mathbb{C})$ (resp. $GL_r(F[t]) = GL_r(\mathbb{C}[t])$). Then we can define a patched up connection as follows. Let η_U and η_B correspond to flat connections on U_0 and B_0. The deformation η_{B^*} into $GL_r(\mathbb{C}[t])$ can be evaluated at $t \in [0, 1]$, via the evaluation map $e_t : \mathbb{C}[t] \to \mathbb{C}$. This gives a family of representations in $GL_r(\mathbb{C})$, which may also be viewed as a family of flat connections on S parametrized by $t \in [0, 1]$. Taking the connection form to be zero in the dt direction gives a connection over the cylinder $B_0^* = S \times [0, 1]$, a connection which on the endpoints glues together with the given flat bundles on U_0 or B_0. Putting them together we get a connection $\nabla^{\rm def}$ on X. It has the property that in U_0 and B_0 it is flat, whereas in $B_0^* = S \times [0, 1]$ it is flat along each t-level set $S \times \{t\}$.

Now note that the curvature form $\Omega = \Omega_{\nabla^{\rm def}}$ restricts to zero on the t-level sets. Since these have codimension 1, it follows that in any local coordinates of the form (t, x_i) where x_i are local coordinates on S, all terms in Ω have a dt, that is there are no terms of the form $dx_i \wedge dx_j$. It follows that $\Omega \wedge \Omega = 0$. In particular the Chern forms of Ω vanish except maybe for the first one.

The differential character $\widehat{c}_p(\nabla^{\rm def})$ associated to the connection $\nabla^{\rm def}$ therefore projects to zero in the closed $2p$-forms whenever $p > 1$, so it defines a class in $H^{2p-1}(X, \mathbb{C}/\mathbb{Z})$ for any $p > 1$.

The two things we need to know are resumed in the following lemmas.

Lemma 7.3. *The class $\widehat{c}_p(\nabla^{\rm def})$ obtained by using the deformed connection on the cylinder, is equal to the regulator class $\widehat{c}_p^{\rm def}(\eta_U, \eta_B, \eta_{B^*})$ defined using the deformation theorem in K-theory.*

Proof. It suffices to prove this for the universal case where

$$U = BGL(F) \cup^{BGL(F[t] \times \{0\}} (BGL(F[t]) \times [0, 1)) ,$$
$$B = BGL(F) \cup^{BGL(F[t] \times \{1\}} (BGL(F[t]) \times (0, 1]) ,$$
$$B^* = BGL(F[t]) \times (0, 1)$$

and $X = BGL(F)_{\rm def}$ is the homotopy pushout. In this case we know that the spaces U, B and B^* have the same homology, so the connecting map in Mayer-Vietoris is trivial. Thus we have an exact sequence

$$0 \to H^*(X, \mathbb{C}/\mathbb{Z}) \to H^*(U, \mathbb{C}/\mathbb{Z}) \oplus H^*(B, \mathbb{C}/\mathbb{Z}) \to H^*(B^*, \mathbb{C}/\mathbb{Z}) \to O.$$

Now, we the class defined by pullback under the map using the deformation triple in K-theory, restricts on U and B to the standard class. The same is true for the class defined by the previous construction. Thus, they are equal in $H^*(X, \mathbb{C}/\mathbb{Z})$. ☐

Lemma 7.4. *Starting with (ρ, W), do the procedure of §7.3 to get η_U, η_B and η_{B^*}. The class $\widehat{c}_p(\nabla^{\rm def})$ defined using this deformation triple, is equal to the class defined in §4.4 using a patched connection $\nabla^{\#}$ for (ρ, W).*

Proof. The two classes come from connections $\nabla^{\rm def}$ and $\nabla^{\#}$ respectively. Both of these connections are compatible with the pre-patching collection associated to (ρ, W) in §4.4. By Lemma 4.2 the classes are the same. ☐

With these two lemmas we get that for $p > 1$ the patched connection class is the same as the class defined using K-theory as above.

Corollary 7.5. *Suppose we are given a filtration triple. Then the regulator classes defined on the one hand using the map $X \to BGL(F)^+_{\rm def}$ obtained by using the associated deformation triple; and on the other hand using the patching construction of Corollary 4.11, coincide.*

Proof. We have

$$\widehat{c}_p^{\mathrm{def}}(\eta_U, \eta_B, \eta_{B^*}) = \widehat{c}_p(\nabla^{\mathrm{def}}) = \widehat{c}_p(\nabla^{\#}) = \widehat{c}_p(\rho/X).$$

The first equality is by Lemma 7.3, the second equality by Lemma 7.4, and the third is the definition of $\widehat{c}_p(\rho/X)$. □

Remark 7.6. *The above argument is another place where it becomes unclear how to generalize our procedure to the case of a normal crossings divisor. Near the codimension k pieces of the stratification of D, the "collar" looks like $S \times [0,1]^k$. However, if we envision a k-variable deformation, then the argument saying that the higher Chern forms vanish, no longer works. This then is another reason why we are restricting to the case of a smooth divisor in the present paper.*

8. Hermitian K-theory and variations of Hodge structure

In order to prove that the volume invariants vanish in the case of variations of Hodge structure, Reznikov used a direct calculation of the space of invariant polynomials on a group of Hodge type. In order to apply this idea to the extended regulators, we use a variant of the previous deformational construction for hermitian K-theory.

8.1. **Hermitian K-theory.** Start by recalling some of the basics of hermitian K-theory. See for example [Ka].

We work with commutative rings A with involution $a \mapsto \overline{a}$ preserving the product. The basic example is $\mathbb{C}$ with the complex conjugation involution. Given an A-module V we denote by $\overline{V}$ the same set provided with the conjugate A-module structure. On the other hand, denote by V^* the usual dual module of a projective A-module. Note that we have a natural isomorphism

$$(\overline{V})^* \cong \overline{(V^*)}$$

and these will be indiscriminately noted $\overline{V}^*$. Either one may be viewed as the module of antilinear homomorphisms $\lambda : V \to A$, that is such that $\lambda(av) = \overline{a}\lambda(v)$.

An *hermitian pairing* is a morphism

$$h : V \to \overline{V}^*$$

which may be interpreted as a form

$$u, v \mapsto \langle u, v\rangle_h = h(u)(\overline{v}) \in A$$

satisfying the properties

$$\langle au, v\rangle_h = a\langle u, v\rangle_h, \quad \langle u, av\rangle_h = \overline{a}\langle u, v\rangle_h.$$

Fix $\epsilon = \pm 1$. An *ϵ-hermitian module* over A is a pair (V, h) consisting of a projective A-module V provided with an hermitian pairing h such that

$$\langle v, u\rangle_h = \epsilon\overline{\langle u, v\rangle_h}.$$

If there exists $i \in A$ with $i^2 = 1$ and $\overline{i} = -i$ and h is an ϵ-hermitian pairing then ih is a $-\epsilon$-hermitian pairing. So, in this case the distinction between the two values of ϵ disappears. This happens for the rings we consider.

If V is any A-module then the *hyperbolic ϵ-hermitian A-module* is defined by $H^{\epsilon}(V) = V \oplus \overline{V}^*$ with h interchanging the factors with a sign determined by ϵ. Put $H^{\epsilon}_{n,n} := H_{\epsilon}(A^n)$.

Let $O(V, h)$ be the group of automorphisms of the ϵ-hermitian A-module (V, h). Let $O^{\epsilon}_{n,n}(A) := O(H^{\epsilon}_{n,n})$ and let $O^{\epsilon}_{\infty,\infty}(A)$ be the direct limit of these groups for the natural inclusion maps as $n \to \infty$. It has a perfect commutator subgroup just as is the case for $GL_{\infty}(A)$, so we can make the *plus construction*

$$BO^{\epsilon}_{\infty,\infty}(A)^+.$$

Karoubi defines the Quillen-Milnor L-groups by

$$L^{\epsilon}_n(A) := \pi_n BO^{\epsilon}_{\infty,\infty}(A)^+.$$

Recall that

$$K_n(A) := \pi_n BGL_{\infty}(A)^+.$$

The hyperbolic construction gives a map

$$H : GL_{\infty}(A) \to O^{\epsilon}_{\infty,\infty}(A), \text{ hence } H^+ : BGL_{\infty}(A)^+ \to BO^{\epsilon}_{\infty,\infty}(A)^+,$$

and on the other hand forgetting the hermitian form gives a map

$$F : O^{\epsilon}_{\infty,\infty}(A) \to GL_{\infty}(A), \text{ hence } F^+ : BO^{\epsilon}_{\infty,\infty}(A)^+ \to BGL_{\infty}(A)^+.$$

These give maps between the K-groups and the L-groups:

$$H : K_n(A) \to L^{\epsilon}_n(A),$$
$$F : L^{\epsilon}_n(A) \to K_n(A).$$

Karoubi considers the cokernel

$$W^{\epsilon}_n(A) := \mathrm{coker}\,(H : K_n(A) \to L^{\epsilon}_n(A)),$$

and on [Ka, page 392, Corollaire 5.8] he defines $\overline{W}^{\epsilon}_n(A)$ by inverting the prime 2. The polynomial ring $A[x]$ has an involution extending that of A, defined by $\overline{x} = x$. One of his main results is the following:

Theorem 8.1 (Karoubi [Ka], Corollaire 5.11)**.** *Suppose A is regular. The inclusion $A \to A[x]$ induces isomorphisms $\overline{W}^{\epsilon}_n(A) \cong \overline{W}^{\epsilon}_n(A[x])$.*

The following corollary was undoubtedly considered obvious in [Ka] but needs to be stated.

Corollary 8.2. *Suppose A is regular. Letting $\mathbb{Z}' := \mathbb{Z}[\frac{1}{2}]$ the inclusion $A \to A[x]$ induces isomorphisms on L-theory*

$$L^{\epsilon}_n(A) \otimes \mathbb{Z}' \xrightarrow{\cong} L^{\epsilon}_n(A[x]) \otimes \mathbb{Z}'.$$

Proof. Evaluation at 0 gives a splitting $A \to A[x] \xrightarrow{e_0} A$, compatible with the hermitian structure. It follows that the morphism

$$L_n^\epsilon(A) \to L_n^\epsilon(A[x])$$

is a split inclusion. Now we have a diagram with horizontal right exact sequences

$$\begin{array}{ccccccc} K_n(A)\otimes\mathbb{Z}' & \to & L_n^\epsilon(A)\otimes\mathbb{Z}' & \to & \overline{W}_n^\epsilon(A) & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ K_n(A[x])\otimes\mathbb{Z}' & \to & L_n^\epsilon(A[x])\otimes\mathbb{Z}' & \to & \overline{W}_n^\epsilon(A[x]) & \to 0 \end{array}$$

where the left vertical arrow is an isomorphism by the fundamental homotopy invariance theorem in K-theory, the middle arrow is a split inclusion, and the right vertical arrow is an isomorphism by Theorem 8.1. It follows that the middle vertical arrow is surjective, so it is an isomorphism. □

Corollary 8.3. *For any ring with involution A, the map*

$$BO_{\infty,\infty}^\epsilon(A)^+ \to BO_{\infty,\infty}^\epsilon(A[x])^+$$

induces a homotopy equivalence after localizing away from the prime 2 (in particular, for rational homotopy theory). The same is true of the evaluation maps

$$e_0, e_1 : BO_{\infty,\infty}^\epsilon(A[x])^+ \to BO_{\infty,\infty}^\epsilon(A)^+.$$

Let $BO_{\infty,\infty}^\epsilon(A)^+_{\mathrm{def}}$ denote the homotopy pushout of the evaluation maps e_0, e_1 appearing in the previous corollary. Then also the map

$$BO_{\infty,\infty}^\epsilon(A)^+ \to BO_{\infty,\infty}^\epsilon(A)^+_{\mathrm{def}}$$

is an equivalence after localizing away from 2 and in particular in rational homotopy theory.

Note that for each evaluation map e_i, $i = 0, 1$ there is a commutative diagram

$$\begin{array}{ccc} BO_{\infty,\infty}^\epsilon(A[x])^+ & \to & BGL_\infty(A[x])^+ \\ \downarrow & & \downarrow \\ BO_{\infty,\infty}^\epsilon(A)^+ & \to & BGL_\infty(A)^+ \end{array}$$

where the vertical maps are the evaluation maps. This gives a commutative diagram of homotopy pushout squares which we don't write down, in which the pushout map is

$$BO_{\infty,\infty}^\epsilon(A)^+_{\mathrm{def}} \to BGL_\infty(A)^+_{\mathrm{def}}$$

which is compatible with the rest.

8.2. **Hermitian deformation patching.** We now apply this to the case $A = \mathbb{C}$ with the involution being complex conjugation. Since $i \in \mathbb{C}$ by the above remark the choice of ϵ doesn't matter and we now take $\epsilon = 1$ and drop it from notation.

The group $O_{n,n}(\mathbb{C})$ is more commonly known as $U(n,n)$, the unitary group of the hermitian form of signature n,n. This is because the natural hermitian form on hyperbolic space $H(\mathbb{C}^n)$ has signature (n,n). Thus

$$O_{\infty,\infty}(\mathbb{C}) = \lim_{\rightarrow} U(n,n).$$

Note also that for any p,q we have $U(p,q) \subset U(n,n)$ for $n \geq \max(p,q)$ so we can also write

$$O_{\infty,\infty}(\mathbb{C}) = \lim_{\rightarrow} U(p,q).$$

So, if (X,x) is a path-connected pointed space with a representation $\rho : \pi_1(X,x) \to U(p,q)$ for some p,q, then we obtain a map

$$X \to BO_{\infty,\infty}(\mathbb{C}) \to BO_{\infty,\infty}(\mathbb{C})^+.$$

The patching construction as previously done applies in this case too.

For a given p,q let V be the $\mathbb{C}$-vector space with hermitian form h of signature p,q. Let $V[t] := V \otimes_{\mathbb{C}} \mathbb{C}[t]$ be its extension of scalars to $\mathbb{C}[t]$. Let $O_{p,q}(\mathbb{C}[t])$ denote the group of hermitian automorphisms of $V[t]$. For $p = q = n$ this coincides with the previous notation $O_{n,n}(\mathbb{C}[t])$ and For any $n \geq \max(p,q)$ we have an inclusion $O_{p,q}(\mathbb{C}[t]) \subset O_{n,n}(\mathbb{C}[t])$ obtained by direct sum with a form of signature $n-p, n-q$.

Suppose we have a diagram of pointed path-connected spaces

$$(U,u) \xleftarrow{l} (B^*,b^*) \xrightarrow{c} (B,b)$$

together with representations

$$\eta_U : \pi_1(U,u) \to U(p,q),$$

$$\eta_B : \pi_1(B,b) \to U(p,q),$$

and

$$\eta_{B^*} : \pi_1(B^*,b^*) \to O_{p,q}(\mathbb{C}[t])$$

such that

$$e_1 \circ \eta_{B^*} = l^*(\eta_U) \text{ and } e_0 \circ \eta_{B^*} = c^*(\eta_B).$$

In other words, we again have representations on U and B, plus a deformation between their restrictions to B^*. We call this an *hermitian deformation triple.*

As before, we suppose given the subsets U_0, B_0 and $B_0^* \cong S \times [0,1]$, and $X := U_0 \cup^{S\times[0,1]} B_0$ is the homotopy pushout, so we obtain a map

$$X \to BO_{\infty,\infty}(\mathbb{C})^+_{\rm def}.$$

Lemma 8.4. *Composing the above representations with the inclusions $U(p,q) \subset GL(p+q,\mathbb{C})$ or $O_{p,q}(\mathbb{C}[t]) \subset GL(p+q,\mathbb{C}[t])$ we obtain from our hermitian deformation triple a usual deformation patching datum in the previous sense. This in turn gives a map*

$$X \to BGL(\mathbb{C})^+_{\rm def},$$

which is homotopy equivalent to the composition of

$$X \to BO_{\infty,\infty}(\mathbb{C})^+_{\mathrm{def}} \xrightarrow{F} BGL(\mathbb{C})^+_{\mathrm{def}}.$$

Proof. This comes from the compatibility of the homotopy pushout squares used to define $BO_{\infty,\infty}(\mathbb{C})^+_{\mathrm{def}}$ and $BGL(\mathbb{C})^+_{\mathrm{def}}$. □

Now the key part of the present argument comes from Reznikov's fundamental observation about the cohomology degrees of generators of the cohomology theories on both sides. Recall that the *Borel volume regulators* are classes

$$r_p^{\mathrm{Bor}} \in H^{2p-1}(BGL(\mathbb{C})^+, \mathbb{R}).$$

These correspond to the imaginary parts of the $\mathbb{C}/\mathbb{Z}$ regulators we are studying.

We can repeat all the constructions in §7 and in the present §8 for the special linear group $SL(\mathbb{C})$. As in [Re2, p.377, §2.7], we will eventually reduce to the case when we look at $SL_r(\mathbb{C})$-valued representations. Thus Lemma 8.4 will give us maps

$$X \to BSL(\mathbb{C})^+_{\mathrm{def}},$$

homotopy equivalent to the composition of

$$X \to BSO_{\infty,\infty}(\mathbb{C})^+_{\mathrm{def}} \xrightarrow{F} BSL(\mathbb{C})^+_{\mathrm{def}}.$$

Lemma 8.5. *For any $p > 1$ the pullback of r_p^{Bor} via the map $BSO_{\infty,\infty}(\mathbb{C})^+ \to BSL(\mathbb{C})^+$ is zero.*

Proof. It suffices to show this for any finite stage $SU(p,q) \to SL_r(\mathbb{C})$. Then, Reznikov's argument, basically by observing that there are no $S(U(p)\times U(q))$-invariant polynomials on $SU(p,q)$, gives the statement [Re2]. □

Since $BSL(\mathbb{C})^+ \to BSL(\mathbb{C})^+_{\mathrm{def}}$ induces an isomorphism on rational homology, the volume invariant extends to an invariant denoted also r_p^{Bor} on $BSL(\mathbb{C})^+_{\mathrm{def}}$.

Corollary 8.6. *For any $p > 1$ the pullback of r_p^{Bor} via the map $BSO_{\infty,\infty}(\mathbb{C})^+_{\mathrm{def}} \xrightarrow{F} BSL(\mathbb{C})^+_{\mathrm{def}}$ is zero.*

Proof. The map is the same as in the previous lemma, on rational cohomology. □

Corollary 8.7. *Given an hermitian deformation triple, the associated volume invariant $Vol^{\mathrm{def}}_{2p-1}(\eta_U, \eta_B, \eta_{B^*})$ is zero for any $p > 1$.*

Proof. Recall that $Vol_{2p-1}(\eta_U, \eta_B, \eta_{B^*})$ is, by definition, the pullback of r_p^{Bor} via the map $X \to BGL(\mathbb{C})^+_{\mathrm{def}}$ obtained by deformation patching. This map is shown in Lemma 8.4 to factor through $BO_{\infty,\infty}(\mathbb{C})^+_{\mathrm{def}}$. Then apply Corollary 8.6, using the reduction to SL and $SO_{\infty,\infty}$ mentioned above, from [Re2, p.377, §2.7]. □

8.3. **An hermitian deformation triple associated to a VHS.** Consider a representation ρ underlying a complex variation of Hodge structure, with unipotent monodromy along an irreducible smooth divisor D. In this case there is a VMHS $(V, W, F, \tilde{F}, \langle\cdot,\cdot\rangle)$ on the divisor component. We don't need to know about the Hodge filtrations F and $\tilde{F}$. The basic information we need to know about the weight filtration and the hermitian form is what is given by the 1-variable nilpotent and SL2–orbit theorems (see [Sch]). Look at the data $(V, N, \langle\cdot,\cdot\rangle)$ where V is the vector space, $N = \log \rho(\gamma)$ is the logarithm of the monodromy around the divisor D, and $\langle\cdot,\cdot\rangle$ is the flat indefinite hermitian form preserved by ρ. We normalize to suppose that $\langle\cdot,\cdot\rangle$ is hermitian symmetric, rather than hermitian antisymmetric; by multiplying by $i = \sqrt{-1}$ we can always assume this.

The 1-variable nilpotent and SL2–orbit theorems imply that this triple $(V, N, \langle\cdot,\cdot\rangle)$ is a direct sum of standard objects. The standard objects are symmetric powers of the standard 2–dimensional case where V has basis e_1, e_2, with $Ne_1 = e_2$ and $Ne_2 = 0$; and with $\langle e_i, e_i\rangle = 0$ but $\langle e_1, e_2\rangle = 1$.

For the standard object of rank 2, the monodromy weight filtration has graded quotients Gr_1 corresponding to e_1 and Gr_{-1} corresponding to e_2, and the form

$$(u, v) \mapsto \langle u, Nv\rangle$$

is positive definite on Gr_1. The zeroth symmetric power is just the case $N = 0$. The k-th symmetric power of the standard object has basis vectors $e_0, ..., e_k$ with $Ne_i = e_{i+1}$ and $\langle e_i, e_j\rangle = 0$ unless $i + j = k$ in which case it is 1. In this case the monodromy weight filtration puts e_i in degree $k - 2i$, going from e_0 in degree k to e_k in degree $-k$.

So in general our V will be a direct sum of these kinds of things, and the full monodromy representation of the neighborhood of D will preserve the monodromy weight filtration. Each of the standard objects comes with a splitting of the monodromy weight filtration, so taking the direct sum of these splittings allows us to choose an isomorphism $V \cong Gr^W(V)$ or equivalently an expression

$$V = \bigoplus V_k$$

with V_k corresponding to the $Gr^W_k(V)$. Then $N : V_k \to V_{k-2}$, and this polarizes the hermitian form induced by $\langle\cdot,\cdot\rangle$ on $Gr^W_k(V)$ as in [Sch]. The splittings of the standard objects relate the form $\langle\cdot,\cdot\rangle$ on the original vector space with the induced form on the associated-graded pairing Gr^W_k with Gr^W_{-k}, so the same is true of our splitting of V: the form $\langle\cdot,\cdot\rangle$ on V is the same as the induced form on $Gr^W(V)$, and in terms of the decomposition of V it pairs V_k with V_{-k}. The full monodromy representation is upper triangular for this block decomposition.

Proposition 8.8. *With notations as above, given a VHS on U with unipotent monodromy around D, we can construct an hermitian deformation triple on X.*

Proof. The representation η_U is given by ρ, and η_B is given by the associated-graded $Gr^W(\rho)$ transported to a representation on V by the splitting. Note that η_B still takes values in the unitary group $U(p,q)$ of $\langle \cdot, \cdot \rangle$ on V. Indeed, η_B is a direct sum of representations which preserve the form on Gr^W_k obtained by the polarization using N. On anything of the form $Gr^W_k \oplus Gr^W_{-k}$ the form $\langle \cdot, \cdot \rangle$ is of hyperbolic type using the polarization forms on the two pieces, so η_B preserves the form $\langle \cdot, \cdot \rangle$ on each piece $Gr^W_k \oplus Gr^W_{-k}$.

Define the deformation η_{B^*} as follows (this is basically the same as in Proposition 3.2 and §7.3 above): for $t \in \mathbb{R}$, let T_t be the automorphism of V which acts by multiplication by t^k on V_k. Then conjugation with T_t gives an action on $GL(V)$ which multiplies the block diagonal pieces by 1 and the strictly upper triangular pieces by some positive powers of t. Thus, on an upper triangular monodromy representation $\rho|_{\pi_1(B^*)}$ it extends to the case $t = 0$ giving a family of representations $\eta_{B^*} := \mathrm{Ad}(T_t)(\rho|_{\pi_1(B^*)})$ defined even for $t = 0$ and at $t = 0$ the image is the associated-graded representation η_B. This gives the required deformation. Notice that T_t preserves the form $\langle \cdot, \cdot \rangle$, because if $v_i \in V_i$ and $v_j \in V_j$ then $\langle v_i, v_j \rangle = 0$ unless $i + j = 0$, and if $i + j = 0$ then

$$\langle T_t v_i, T_t v_j \rangle = \langle t^i v_i, t^{-i} v_j \rangle = \langle v_i, v_j \rangle.$$

Thus the conjugated representations $\mathrm{Ad}(T_t)(\rho|_{\pi_1(B^*)})$ are all in the unitary group of $(V, \langle \cdot, \cdot \rangle)$ (even for $t = 0$ by continuity). Hence η_{B^*} takes values in $O_{p,q}(\mathbb{C}[t])$. This completes the construction of the hermitian deformation triple. □

In this situation X is identified with the homotopy pushout $U_0 \cup^{S \times [0,1]} B_0$ (with notations as in §8.2), and we will obtain a map

$$f_\rho : X \to BO_{\infty,\infty}(\mathbb{C})^+_{\mathrm{def}}.$$

Corollary 8.9. *The pullback of the volume regulator r_p^{Bor} by the map $F \circ f_\rho$ is the same as $Vol_{2p-1}(\rho/X)$ and it vanishes.*

Proof. Consider the hermitian deformation triple obtained in Proposition 8.8. This gives the map f_ρ. The associated deformation patching datum is the same as the one used to define $\widehat{c}_p^{\mathrm{def}}(\rho, W)$, because the construction in the proof of Proposition 8.8 complexifies to the same one as in §7.3. Therefore $(F \circ f_\rho)^*(r_p^{\mathrm{Bor}}) = Vol_{2p-1}(\rho/X)$. As in Corollary 8.7, the proof now follows from Lemma 8.4 and Corollary 8.6. □

9. The generalization of Reznikov's theorem

We can now give the generalization of Reznikov's theorem for canonical extensions in the case of a smooth divisor.

Theorem 9.1. *Suppose X is a smooth quasiprojective variety, with $D \subset X$ an irreducible closed smooth divisor. Suppose $\rho : \pi_1(X - D) \to GL_r(\mathbb{C})$ is a representation such that $\rho(\gamma)$ is unipotent for γ the loop going around D. Then the extended regulator*

$$\widehat{c}_p(\rho/X) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}), \ \ p \geq 2$$

defined using the patched connection in Corollary 4.11, is torsion.

Proof. By the rigidity result 6.5, the regulator doesn't change if we deform ρ. Thus, we may assume that ρ is defined over an algebraic number field F. The regulator is a pullback of a class via the map

$$\xi_\rho : X \to BGL(F)^+_{\rm def}.$$

Suppose $\sigma : F \hookrightarrow \mathbb{C}$ is any embedding. Composing, we get a map

$$X \to BGL(\mathbb{C})^+_{\rm def}.$$

The pullback of the volume regulator by this map, is a class $Vol_{2p-1}(\rho^\sigma) \in H^{2p-1}(X, \mathbb{R})$. This class is independent of deformations of ρ^σ within representations which are unipotent along D, by Theorem 6.5. As in [Re2, p.377, §2.7], it suffices to consider the case when the representation takes values in $SL_r(\mathbb{C})$. Indeed, one can look at the representation $\rho' = \rho \oplus \det\rho^{-1}$ which is SL_{r+1}-valued. Also note that taking canonical extensions is compatible with direct sums. Let $\widehat{c}(\rho) := 1 + \widehat{c_1}(\rho) + ... + \widehat{c_r}(\rho)$. Using the identity

$$\widehat{c}(\rho_1 \oplus \rho_2) = \widehat{c}(\rho_1).\widehat{c}(\rho_2)$$

for any two representations ρ_1 and ρ_2 (see [Ch-Sm, p.64-65]), it follows that if the theorem is true for the classes of ρ' then it is also true for the classes of ρ. Also note that the constructions in §7 and §8 hold verbatim if we look at the special linear subgroups.

Mochizuki proves in [Mo] that ρ^σ may be deformed to a complex variation of Hodge structure. When X is smooth and projective with a smooth divisor D, this can also be obtained using Biquard's earlier version of the theory for this case [Bi]. This deformation preserves the condition of unipotence at infinity, since it preserves the trivial parabolic structure of the Higgs bundle, and the Higgs field is multiplied by $t \to 0$ so if the eigenvalues are zero to begin with, then they are zero in the deformation. On the other hand, by Corollary 8.9, the extended volume regulator vanishes for a complex variation of Hodge structure. Thus, $Vol_{2p-1}(\rho^\sigma) = 0$. We now apply Reznikov's argument: by Borel's theorem, the classes $\sigma^*(Vol_{2p-1})$ generate the real cohomology ring of $BSL(F)^+$ or equivalently $BSL(F)^+_{\rm def}$. The fact that their pullbacks by ξ_ρ vanish, implies that ξ_ρ induces the zero map on rational homology. This in turn implies that the pullback by ξ_ρ of the universal class in $H^{2p-1}(BSL(F)^+_{\rm def}, \mathbb{C}/\mathbb{Z})$, is torsion. By Corollary 7.5, the pullback of this class is the same as the regulator we have defined using the patched connection. □

Corollary 9.2. *Suppose X is a smooth projective variety over $\mathbb{C}$, with $D \subset X$ an irreducible closed smooth divisor. Suppose $\rho : \pi_1(X - D) \to GL_r(\mathbb{C})$ is a representation such that $\rho(\gamma)$ is unipotent for γ the loop going around D. Let (E, ∇) be the holomorphic bundle with flat connection on $X - D$ associated to ρ, and let E_X be the Deligne canonical extension to a holomorphic bundle on X with logarithmic connection having nilpotent residue along D. Then the Deligne Chern class*

$$c_p^{\mathcal{D}}(E_X) \in H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)),\ p \geq 2$$

is torsion.

Proof. We have shown in Proposition 5.2 that the regulator class $\widehat{c}_p(\rho)$ lifts the Deligne Chern class of the canonical extension. Thus, Theorem 9.1 implies that the Deligne Chern class of the canonical extension is torsion. □

Aside from the obvious problem of generalizing these results to the case of a normal-crossings divisor, another interesting question is how to generalize Reznikov's other proof of his theorem [Re2]. This passed through a direct calculation of Borel's volume invariants using the harmonic map, instead of invoking deformation to a variation of Hodge structure. It would be interesting to see how to do this calculation for the volume invariant over X, using the harmonic map associated to ρ on $X - D$. This might lead to a better way of treating the normal-crossings case.

Another circle of questions clearly raised by Reznikov's result is to determine the torsion pieces of these classes, for example is there some arithmetical construction of these? Can one bound the torsion or construct coverings on which it vanishes?

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