CHERN INVARIANTS OF SOME FLAT BUNDLES IN THE ARITHMETIC DELIGNE COHOMOLOGY

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Abstract. In this note, we investigate the cycle class map between the rational Chow groups and the arithmetic Deligne cohomology, introduced by Green-Griffiths and Asakura-Saito. We show nontriviality of the Chern classes of flat bundles in the arithmetic Deligne Cohomology in some cases and our proofs also indicate that generic flat bundles can be expected to have nontrivial classes. This provides examples of non-zero classes in the arithmetic Deligne cohomology which become zero in the usual rational Deligne cohomology.

1. Introduction

Suppose X is a nonsingular quasi-projective variety defined over the field of complex numbers. Consider an algebraic vector bundle $\mathcal{V}$ on X with a connection

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_X$$

which is flat, i.e. $\nabla^2 = 0$. We consider the primary invariants of $(\mathcal{V}, \nabla)$ in various cohomology theories. The Chern classes of $(\mathcal{V}, \nabla)$ in the Betti, de Rham and the Deligne cohomologies of X are denoted as follows:

$$c_i^B(\mathcal{V}) \in H^{2i}(X, \mathbb{Z})$$
$$c_i^{dR}(\mathcal{V}) \in H^{2i}_{dR}(X, \mathbb{C})$$
$$c_i^{\mathcal{D}}(\mathcal{V}) \in H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)).$$

Further, the Deligne Chern class is a lift of the Betti Chern class under the class map

$$H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \longrightarrow H^{2i}(X, \mathbb{Z}).$$

By the Chern-Weil theory, the de Rham Chern classes $c_i^{dR}(\mathcal{V})$ are zero. Using the de Rham isomorphism

$$H^\bullet(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^\bullet_{dR}(X, \mathbb{C})$$

we conclude that the Betti Chern classes $c_i^B(\mathcal{V})$ are torsion.

When X is projective, it was conjectured by S. Bloch and proved by A. Reznikov ([Re]) that $c_i^D(\mathcal{V})$ are torsion, for $i \geq 2$. Bloch and Beilinson [Bl1] have conjectured that there should be a descending finite filtration

$$CH^i(X) \otimes \mathbb{Q} \supset F^1 \supset ... \supset F^i \supset F^{i+1} = 0$$

of the rational Chow groups of X and compatible with products and correspondences. Furthermore, if E is a flat vector bundle then Bloch's conjecture (see [Es]) says that

$$c_i(E) \in F^i CH^i(X) \otimes \mathbb{Q}. \quad (1)$$

Green and Griffiths [Gr-Gri, p.506] have reformulated Bloch's conjecture in (1), in terms of their proposed filtration.

We do not have concrete computations of Chern classes of flat bundles having non-trivial Chow Chern classes, except the case of a direct sum of degree zero line bundles on a general abelian variety (see [Bl2]).

On the other hand, M. Green- P.Griffiths ([Gr-Gri]) and M.Asakura- S. Saito ([As1]) have introduced a category of arithmetic Hodge structures. Given a nonsingular variety X, they have defined higher Abel-Jacobi maps into some Ext-groups in this category. These groups are termed as the arithmetic Deligne cohomology and denoted by $H^a_{A\mathcal{D}}(X,\mathbb{Q}(b))$. There are cycle class maps

$$CH^p(X)_{\mathbb{Q}} \xrightarrow{\psi^p_{A\mathcal{D}}} H^{2p}_{A\mathcal{D}}(X,\mathbb{Q}(p))$$

and the usual cycle class map

$$CH^p(X)_{\mathbb{Q}} \xrightarrow{\psi^p_{\mathcal{D}}} H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))$$

factors via $\psi^p_{A\mathcal{D}}$ (see also [Sa2]).

The Bloch-Beilinson conjecture on the injectivity of $\psi^p_{\mathcal{D}}$ for varieties over number fields implies the injectivity of the map $\psi^p_{A\mathcal{D}}$ for $X/\mathbb{C}$. Very few examples are known where non-trivial cycle classes are computed in the arithmetic Deligne cohomology and which vanish in the Deligne cohomology. Asakura [As2] studied such an explicit zero-cycle on a product of two curves.

In this note, we show non-triviality of the Chern classes of some flat bundles in the arithmetic Deligne cohomology, thus supporting the conjectural injectivity of the map $\psi^p_{A\mathcal{D}}$ and providing a large class of non-trivial classes in the arithmetic Deligne cohomology.

We show

Theorem 1.1. *Suppose A is an abelian variety of dimension g defined over $\mathbb{C}$. Let E be a holomorphic connection on A of rank n and generic in the moduli space of holomorphic connections on A. Then*

$$c_i(E) \in H^{2i}_{A\mathcal{D}}(A,\mathbb{Q}(i))$$

is non-zero, for $i \leq min\{n,g\}$.

This is proved in §3 Theorem 3.11. The proof uses the existence of a flat connection on E and the structure of indecomposable flat bundles on A, due to Moromito [Mo]. Furthermore, the key point used in the proof is the non-triviality of the Betti Chern classes and knowing the explicit Chern forms of the Poincaré bundles.

A similar proof also holds whenever the above relevant facts of the Poincaré bundle associated to a variety X and a moduli space of flat connections of rank r is known.

This was generalized to the following situation by U. Jannsen (see §5).

Theorem 1.2. *Suppose X_i are nonsingular complex projective varieties, for $i = 1,...,r$, and assume that $H^1(X_i,\mathbb{Q}) \neq 0$ for all i. Consider the product variety $X = X_1 \times X_2 \times ... \times X_r$ and a flat bundle*

$$E = p_1^*E_1 \otimes p_2^*E_2 \otimes ... \otimes p_r^*E_r$$

on X. Here p_i denotes the projection to X_i and E_i is a flat bundle on X_i of rank n_i. Assume that $(\Lambda^{n_1}E_1,\ldots,\Lambda^{n_r}E_r)$ is generic among the tuples $(L_1,\ldots,L_r)$ of flat line bundles L_i on X_i. Then for all $i\leq min\{n,r\}$, with $n=\Pi_{i=1}^r n_i$, the Chern class of E in the arithmetic Deligne cohomology

$$c_i^{\mathcal{AD}}(E)\in H^{2i}_{\mathcal{AD}}(X,\mathbb{Q}(i))$$

is non-zero.

For example, the above theorem applies to the case of a product of curves which we treat first in §4, because it illuminates the general setting in §5.

In particular, we obtain the non-triviality of the Chern classes of above flat bundles in the rational Chow groups. Furthermore, these classes lie in the expected level of the Green-Griffiths filtration. Indeed, in our examples the higher Chern classes are essentially the products of the first Chern class, hence lie in the expected level of the proposed filtration on the Chow groups. The non-triviality result is in contrast with the triviality of the Chern classes of some Gauss-Manin systems ([Mu], [vdG], [Iy], [Es-Vi],[Iy-Si]) in the rational Chow groups, which may be seen as special points in the moduli space of flat connections, namely those arising from complex variations of Hodge structures (see [Si]).

2. Arithmetic Hodge structures

We briefly recall the notion of arithmetic Hodge structures and the cycle class map into the arithmetic Deligne cohomology. For further details, we refer to [As1], [Sa2].

2.1. **Spreading out a variety X.** For any quasi-projective variety X over the field of complex numbers, a model of X is a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ \mathbb{C} & \longrightarrow & S\,, \end{array}$$

where S is an affine (integral) variety over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, $\mathcal{X}$ is of finite type over S such that every component of $\mathcal{X}$ is dominant over S, and Spec $\mathbb{C}\to S$ is a $\overline{\mathbb{Q}}$-morphism which factors through the generic point of S. Here we fix an embedding $\overline{\mathbb{Q}}\subset\mathbb{C}$. Such diagrams exist by [Gr-EGA IV,3] section 8. M. Saito ([Sa1]) has shown that there is a mixed Hodge module $R^i_{dR}(\mathcal{X}/S)$ on S such that the pullback to Spec $\mathbb{C}$ is the mixed Hodge structure on the cohomology of X. The category $MHM(S)$ of mixed Hodge modules over S is an abelian category and having non-trivial higher Ext-groups when S is positive dimensional.

There is a natural spectral sequence

$$E_1^{a,b}=Ext^b_{MHM(S)}(\mathbb{Q}(c),R^a_{dR}(\mathcal{X}/S))\Rightarrow Ext^{a+b}_{MHM(\mathcal{X})}(\mathbb{Q}(c),\mathbb{Q}).$$

We note that the Ext-group $Ext^{2p}_{MHM(\mathcal{X}\times_{\mathbb{Q}}\mathbb{C})}(\mathbb{Q}(-p),\mathbb{Q})$ is the Deligne-Beilinson cohomology of $\mathcal{X}\times_{\mathbb{Q}}\mathbb{C}$.

The arithmetic Deligne-Beilinson cohomology is defined as the direct limit

$$H^r_{\mathcal{AD}}(X,\mathbb{Q}(s)) := lim_{\mathcal{X}\to S}Ext^r_{MHM(\mathcal{X})}(\mathbb{Q}(-s),\mathbb{Q})$$

where the limit is taken over all the models $(\mathcal{X} \to S)$ of X. We can also define the groups

$$H^r_{\mathcal{D},ar}(X, \mathbb{Q}(s)) \;:=\; lim_{\mathcal{X}\to S} H^r_{\mathcal{D}}(\mathcal{X} \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Q}(s)), \tag{2}$$

as well as the following groups relevant to our discussion:

$$H^r_{ar}(X, \mathbb{Q}(s)) \;:=\; lim_{\mathcal{X}\to S} H^r(\mathcal{X} \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Q}(s)). \tag{3}$$

See also the discussion in [Ja, section 11.8]. There are cycle class maps

$$H^r_{A\mathcal{D}}(X, \mathbb{Q}(s)) \to H^r_{\mathcal{D},ar}(X, \mathbb{Q}(s)) \to H^r_{ar}(X, \mathbb{Q}(s)). \tag{4}$$

Also, notice that

$$CH^p(X) = lim_{\mathcal{X}\to S}\, CH^p(\mathcal{X})$$

since any algebraic cycle on X also spreads out over some model $\mathcal{X}$ of X.

By the results of M. Asakura and M. Saito, there is a cycle class map

$$\psi^p_{\mathcal{D},ar} : CH^p(X)_{\mathbb{Q}} \xrightarrow{\psi^p_{A\mathcal{D}}} H^r_{A\mathcal{D}}(X, \mathbb{Q}(s)) \to H^{2p}_{\mathcal{D},ar}(X, \mathbb{Q}(p)) \tag{5}$$

which factorises the cycle class map

$$CH^p(X)_{\mathbb{Q}} \xrightarrow{\psi^p_{\mathcal{D}}} H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)).$$

2.2. **Spreading out vector bundles.** Suppose E is an algebraic vector bundle of rank r on X defined over $\mathbb{C}$. By the limit theorems in [Gr-EGA IV,3] (8.5.2), (8.5.5), there is a variety S over $\overline{\mathbb{Q}}$ such that there exists a model $\mathcal{X} \longrightarrow S$ of X as well as a vector bundle $\mathcal{E} \longrightarrow \mathcal{X}$ such that E is the base change of $\mathcal{E}$ via Spec $\mathbb{C} \longrightarrow S$. Hence the Deligne Chern class

$$c_i^{\mathcal{D}}(\mathcal{E}) \in H^{2i}_{\mathcal{D}}(\mathcal{X} \otimes \mathbb{C}, \mathbb{Q}(i))$$

extends the Deligne Chern class $c_i^{\mathcal{D}}(E)$ of E. We call $(\mathcal{X} \longrightarrow S, \mathcal{E})$ as a *model* for the vector bundle E.

2.3. **Arithmetic Deligne Chern class of flat bundles.** Suppose E is an algebraic flat bundle on a nonsingular projective variety X defined over $\mathbb{C}$. We denote the image of the Chow Chern classes $c_i^{CH}(E)$ of E under $\psi^i_{A\mathcal{D}}$ by

$$c_i^{A\mathcal{D}}(E) \;\in\; H^{2i}_{A\mathcal{D}}(X, \mathbb{Q}(i)).$$

By functoriality, given any model $\mathcal{E}$ of E, via the natural map $X \longrightarrow \mathcal{X}$, the Deligne class $c_i^{\mathcal{D}}(\mathcal{E})$ pullsback to the arithmetic Deligne class $c_i^{A\mathcal{D}}(E)$. Then, by [Re], the Deligne Chern classes of E are zero. Since the cycle class map $\psi^i_{A\mathcal{D}}$ is expected to be injective ([As1]), our aim here is to show non-triviality of the Chern classes $c_i^{A\mathcal{D}}(E)$ in $H^{2i}_{A\mathcal{D}}(X, \mathbb{Q}(i))$, for some (X, E).

Remark 2.1. *Suppose $S' \longrightarrow S$ is a finite surjective cover and $f : \mathcal{X}' = S' \times_S \mathcal{X} \longrightarrow \mathcal{X}$ is the pullback morphism of finite degree d. If $\mathcal{E}$ is a vector bundle on $\mathcal{X}$ and $\mathcal{E}' = f^*\mathcal{E}$ then, by projection formula* [Fu]*,*

$$f_* c_i(\mathcal{E}') \;=\; d.c_i(\mathcal{E})$$

in $CH^*(S)_{\mathbb{Q}}$.

3. Holomorphic connections on abelian varieties

Suppose E is a holomorphic connection of rank n on an abelian variety. Then in [Mo], A. Moromito has shown the existence of an integrable connection on E (more generally for holomorphic connections on complex torus). This answers positively a question due to Atiyah. Hence E is a flat bundle on A.

A vector bundle on A is said to be *indecomposable* if it cannot be written as a direct sum of two proper subbundles.

Let $E = E_1 \oplus E_2 \oplus ... \oplus E_r$ be a decomposition of E such that each E_i is an indecomposable vector bundle admitting a holomorphic connection. Since the Chern character is additive over direct sums, it suffices to compute the Chern character of indecomposable bundles.

Let E be an indecomposable flat bundle on an abelian variety A. Then, by Matsushima [Ma], we can take the monodromy representation ϕ of E to be

$$\phi = \sigma \otimes \rho$$

where σ is a one dimensional representation and ρ is a unipotent representation. Hence we can write

$$E = E_\phi = L_\sigma \otimes E_\rho \tag{6}$$

where $L_\sigma \in \mathrm{Pic}^0(A)$ and E_ρ is the flat bundle corresponding to ρ.

Lemma 3.1. *There is a sequence of vector bundles*

$$E_\rho = E_1, E_2, ..., E_{l-1}, E_l = 0$$

such that each E_i has a holomorphic connection and $E_i = E_{i-1}/E^0_{i-1}$. Here $E^0_{i-1} \subset E_{i-1}$ is a trivial subbundle, for each $i = 1, ..., l$.

Proof. This is Proposition 5.2 and Lemma 5.4 in [Mo]. □

This immediately implies that

Corollary 3.2. *In the Grothendieck ring $K(A)_Q$ of A,*

$$E_\rho = \mathcal{O}^r$$

for some $r > 0$.

Corollary 3.3. *The Chern character of E_ϕ is given as*

$$Ch(E_\phi) = r.Ch(L_\sigma) \in CH^*(A)_{\mathbb{Q}}.$$

Hence to compute the Chern character of a holomorphic bundle E on A, it suffices to compute the Chern character of a sum of line bundles of degree zero on A, i.e. when $E = \oplus L_i$, for $L_i \in \mathrm{Pic}^0 A$.

We first consider the case when all the L_i are equal to a fixed $L \in \mathrm{Pic}^0(A)$.

Next, we show the nontriviality of the powers of the Chern form of the Poincaré bundle and of its restriction on $A \times U$ for any open subset $U \subset \mathrm{Pic}^0 A$. This will be needed in our proof of Theorem 1.1.

3.1. **Nondegeneracy of the Chern form of the Poincare bundle.** Let A be an abelian variety of dimension g and $\mathrm{Pic}^0(A)$ be the dual abelian variety. Let $\mathcal{P}$ denote the Poincaré line bundle on $A \times \mathrm{Pic}^0(A)$ and let

$$c_1(\mathcal{P}) \in H^2(A \times \mathrm{Pic}^0(A), \mathbb{C})$$

denote the first Chern class of $\mathcal{P}$.

Write $A = V/\Lambda_V$ and $\mathrm{Pic}^0(A) = W/\Lambda_W$, where V is a complex vector space of dimension g and $W = \mathrm{Hom}_{anti}(V, \mathbb{C})$ is the group of $\mathbb{C}$ anti-linear forms. Furthermore Λ_V, Λ_W are lattices in V and W respectively. We assume that Λ_W is the dual lattice of Λ_V.

Let $z_1, z_2, ..., z_g$ be the complex coordinate functions on V associated to a basis $e_1, ..., e_g$ of V. Then the differentials $dz_1, dz_2, ..., dz_g, d\bar{z}_1, ...d\bar{z}_g$ are linearly independent over $\mathbb{C}$. Similarly, let $w_1, ..., w_g$ be the complex coordinate functions on W associated to the dual basis $f_1, ..., f_g$ of W. Then the differentials $dw_1, ..., dw_g, d\bar{w}_1, ..., d\bar{w}_g$ are linearly independent over $\mathbb{C}$.

Since the tangent bundle of an abelian variety and its complex conjugate are trivial, we recall some consequences on the Hodge decomposition of A and $\mathrm{Pic}^0(A)$.

Lemma 3.4. *We have the Hodge decomposition*

$$H^n(A, \mathbb{C}) = \oplus_{p+q=n} H^q(A, \Omega^p_A)$$

where Ω^p_A is the sheaf of holomorphic forms on A. Furthermore,

$$H^q(A, \Omega^p_A) \simeq \bigwedge^p T^*_e(A) \otimes \bigwedge^q \bar{T}^*_e A).$$

Here $T^*_e(A) := Hom_{\mathbb{C}}(V, \mathbb{C})$ denotes the cotangent space of A at the identity and $\bar{T}^*_e(A)$ denotes the group of $\mathbb{C}$ anti-linear forms on V (this is same as the vector space W above). Moreover, the forms $dz_{i_1} \wedge dz_{i_2} ... \wedge dz_{i_p}$ and $d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} ... \wedge d\bar{z}_{j_q}$, for different indices $i_1 < i_2 < ... < i_p$ and $j_1 < j_2 < ... < j_q$, form a basis of $\bigwedge^p T^*_e(A)$ and $\bigwedge^q \bar{T}^*_e(A)$ respectively.

A similar decomposition holds on $Pic^0(A)$.

Proof. See [Bi-La, Theorem 1.4.1, p.16]. □

Since $\mathcal{P}$ is a nondegenerate line bundle of index g on the abelian variety $A \times \mathrm{Pic}^0(A)$, we can write its explicit Chern form.

Lemma 3.5. *The first Chern form of $\mathcal{P}$ is*

$$c_1(\mathcal{P}) = \frac{i}{2}(dz_1 \wedge d\bar{w}_1 + ... + dz_g \wedge d\bar{w}_g + dw_1 \wedge d\bar{z}_1 + ... + dw_g \wedge d\bar{z}_g). \tag{7}$$

in $H^2(A \times Pic^0(A), \mathbb{C})$.

Proof. We notice that the vector space V is the tangent space $T_e(A)$ and W is actually $\bar{T}^*_e(A)$. Define a hermitian form $H : (V \times \bar{T}^*_e(A)) \times (V \times \bar{T}^*_e(A)) \to \mathbb{C}$ by the rule;

$$H((v_1, l_1), (v_2, l_2)) = \overline{l_2(v_1)} + l_1(v_2).$$

This hermitian form is the fist Chern class of the Poincaré line bundle (see [Bi-La, p.38]).

More precisely, the first Chern class is written as (see [Bi-La, p.41, Ex 2])

$$\begin{aligned} c_1(\mathcal{P}) &= \frac{i}{2}(\sum_{i,j=1}^{g} H((e_i,0),(0,f_j))dz_i \wedge d\bar{w}_j \\ &+ \sum_{i,j=1}^{g} H((0,f_i),(e_j,0))dw_i \wedge d\bar{z}_j \\ &+ \sum_{i,j=1}^{g} H((e_i,0),(e_j,0))dz_i \wedge d\bar{z}_j \\ &+ \sum_{i,j=1}^{g} H((0,f_i),(0,f_j))dw_i \wedge d\bar{w}_j). \end{aligned}$$

We notice that the third and the fourth term in the above expression is zero because the hermitian form is zero on this part, and that

$$H((e_i,0),(0,f_j) = \delta_{i,j},\ \ H((0,f_j),(e_i,0)) = \delta_{i,j},$$

since $\{f_1, f_2, ..., f_g\}$ is the dual basis of $\{e_1, e_2, ..., e_g\}$. This implies the claim. □

3.2. **Nondegeneracy on any open subset of Pic$^0(A)$.** We now want to show that the Chern forms of $\mathcal{P}$ remains nonzero when restricted to any open subset of the type $A \times U$, where $U \subset \mathrm{Pic}^0(A)$ is any open subset.

Lemma 3.6. *The Hodge–Künneth component of $c_1(\mathcal{P})$ in*

$$H^{1,0}(A) \otimes H^{0,1}(Pic^0(A))$$

is nonzero. This shows that the Hodge–Künneth component of $c_1(\mathcal{P})^i$ in $H^{i,0}(A) \otimes H^{0,i}(Pic^0(A))$ is nonzero, for each $i \leq g$.

Proof. The Chern form in (7) together with the decomposition in Lemma 3.4 implies the lemma. □

Suppose $Z \subset \mathrm{Pic}^0(A)$ is any closed subset and $U := \mathrm{Pic}^0(A) - Z$ is the open complementary subset. Consider the long exact sequence of cohomologies associated to the triple $(\mathrm{Pic}^0 A, Z, U)$;

$$\rightarrow H^i_Z(\mathrm{Pic}^0(A), \mathbb{C}) \xrightarrow{i_*} H^i(\mathrm{Pic}^0(A), \mathbb{C}) \xrightarrow{j^*} H^i(U, \mathbb{C}) \rightarrow .$$

Then the natural map j^* induces the map

$$id \otimes j^* : H^{i,0}(A) \otimes H^{0,i}(\mathrm{Pic}^0(A)) \longrightarrow H^{i,0}(A) \otimes H^i(U).$$

We want to show that the map $id \otimes j^*$ is injective. By looking at smaller open subsets U, it suffices to assume that Z is a divisor.

Lemma 3.7. *With notations as above, the image of i_* lies in the Hodge piece*

$$\bigoplus_{p+q=i, 0<p<i} H^{p,q}(Pic^0(A)).$$

Proof. With $\mathbb{Q}$-coefficients, all groups in the above long exact cohomology sequence carry natural mixed Hodge structures. As such, one can identify the group $H^i_Z(\mathrm{Pic}^0 A, \mathbb{Q})$ with the Borel–Moore homology $H_{2g-i}(Z, \mathbb{Q}(g))$ (notations as in section 7 in [Ja]), which is actually $H_{2g-i}(Z, \mathbb{Q})(-g)$ (Tate twist in Hodge theory). This group has weights i and higher, and the image of $H_{2g-i}(Z, \mathbb{Q})$ in $H^i(\mathrm{Pic}^0(A), \mathbb{Q})$ coincides with the image of $W_i H_{2g-i}(Z, \mathbb{Q}(g))$.

If $\pi : Z' \to Z$ is a desingularization, then the image of

$$\pi_* : H^{i-2}(Z', \mathbb{Q})(-1) = H_{2g-i}(Z', \mathbb{Q}(g)) \to H_{2g-i}(Z, \mathbb{Q}(g))$$

is $W_i H_{2g-i}(Z, \mathbb{Q}(g))$, see [Ja, Lemma 7.6]. Now we get our claim, because the occurring Hodge types (p, q) in $H^{i-2}(Z, \mathbb{Q})(-1)$ have $p > 0$ and $q > 0$. One may also translate the above argument, via duality, to usual cohomology and use Deligne's result [De, 8.2.5], cf. [Ja, Remark 7.7]. □

Lemma 3.8. *The image of the Hodge–Künneth component of the Chern class $c_1(\mathcal{P})^i$ is nonzero in $H^{i,0}(A) \otimes H^i(U)$, for $i \leq g$.*

Proof. By Lemma 3.7, we deduce that $\mathrm{Im}\, i_* = \mathrm{Ker} j^*$ intersects trivially with $H^{0,i}(\mathrm{Pic}^0(A))$, so that j^* is injective when restricted to this group.

This gives the injectivity of the map

$$id \otimes j^* : H^{i,0}(A) \otimes H^{0,i}(\mathrm{Pic}^0(A)) \longrightarrow H^{i,0}(A) \otimes H^i(U).$$

The claim now follows from Lemma 3.6. □

3.3. **Statement of the Main theorem.**

Theorem 3.9. *Suppose A is an abelian variety of dimension g defined over $\mathbb{C}$. Let $L \in Pic^0(A)$ be a generic line bundle on A. Consider the algebraic flat bundle $E = L^{\oplus n}$ on A. Then the Chern classes*

$$c_i^{A\mathcal{D}}(E) \in H^{2i}_{A\mathcal{D}}(A, \mathbb{Q}(p))$$

are non-trivial, for $i \leq min\{n, g\}$.

Proof. Suppose $(\mathcal{X} \to S, \mathcal{L})$ is a model for L on A. Then $\mathcal{X} \to S$ is an abelian scheme and $\mathcal{L} \to \mathcal{X}$ is a line bundle which restricts on any fiber to a degree 0 line bundle and extending $L \to A$. Consider a fine moduli scheme $\mathcal{A}_g$ of g-dimensional polarized abelian varieties (with suitable level structures) together with the universal family

$$\mathbb{A} \longrightarrow \mathcal{A}_g$$

which are defined over $\overline{\mathbb{Q}}$. Consider the family of dual abelian varieties

$$\mathrm{Pic}^0\mathbb{A} \longrightarrow \mathcal{A}_g$$

and a Poincaré line bundle

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ \mathbb{A} \times_{\mathcal{A}_g} \mathrm{Pic}^0\mathbb{A}. \end{array}$$

Furthermore, $\mathrm{Pic}^0\mathbb{A}$ is a fine moduli space defined over $\overline{\mathbb{Q}}$ of pairs (A', L'), where L' is a degree 0 line bundle on a g-dimensional abelian variety A' (see [Mu3] or [Fa-Ch]). Hence, by universality, there is a morphism (actually over some finite surjective cover of S which we assume to be S, in view of Remark 2.1)

$$\eta : S \longrightarrow \mathrm{Pic}^0\mathbb{A}$$

defining the cartesian diagram of $\overline{\mathbb{Q}}$-schemes of finite type

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\eta'} & \mathbb{A} \times_{\mathcal{A}_g} \mathrm{Pic}^0\mathbb{A} & \longrightarrow & \mathbb{A} \\ \downarrow \beta & & \downarrow p & & \downarrow \\ S & \xrightarrow{\eta} & \mathrm{Pic}^0\mathbb{A} & \longrightarrow & \mathcal{A}_g, \end{array}$$

and there is some line bundle M on S such that $\mathcal{L} = \eta'^*(\mathcal{P}) \otimes \beta^* M$. By possibly passing to an open subscheme of S we may assume that M is trivial so that $\mathcal{L} = \eta'^*(\mathcal{P})$. Let Z be the scheme-theoretic image of S in $\mathcal{A}_g$. By taking base change with Z we get an induced cartesian diagram as follows.

$$\begin{array}{ccccccc} \mathcal{X} & \xrightarrow{\eta'} & \mathbb{A}_Z \times_Z U & \hookrightarrow & \mathbb{A}_Z \times_Z \mathrm{Pic}^0\mathbb{A}_Z & \longrightarrow & \mathbb{A}_Z \\ \downarrow \beta & & \downarrow p_U & & \downarrow p & & \downarrow \\ S & \xrightarrow{\eta} & U & \hookrightarrow & \mathrm{Pic}^0\mathbb{A}_Z & \longrightarrow & Z \end{array}.$$

We assume that L is generic, which means that $\eta : S \to \mathrm{Pic}^0\mathbb{A}_Z$ is dominant. Since η is of finite type, a usual limit argument shows that, by possibly passing to open subschemes of S and $\mathrm{Pic}^0\mathbb{A}_Z$, we may assume that η is flat with dense open image $U \subseteq \mathrm{Pic}^0\mathbb{A}$ as indicated. By base change, η' is faithfully flat as well.

Consider the powers of the first Betti Chern class of the pull-back of $\mathcal{P}$ to $\mathcal{A}_Z \times_Z U$

$$(c_1^B)^i \in H^{2i}((\mathbb{A}_Z \times_Z \mathrm{Pic}^0\mathbb{A}_Z) \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Q}).$$

Take the base change with the $\mathbb{C}$-rational point $t : \mathrm{Spec}\,\mathbb{C} \to \mathcal{A}_g$ given by the given complex abelian variety A. By definition it has image in the generic point of Z. Then we get induced morphisms

$$\begin{array}{ccccc} \mathcal{X}_t & \xrightarrow{\eta'_t} & A \times U_t & \hookrightarrow & A \times \mathrm{Pic}^0 A \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_{\mathbb{C}} & \xrightarrow{\eta'_{\mathbb{C}}} & (\mathbb{A}_Z)_{\mathbb{C}} \times_{Z_{\mathbb{C}}} U_{\mathbb{C}} & \hookrightarrow & (\mathbb{A}_Z)_{\mathbb{C}} \times_{Z_{\mathbb{C}}} \mathrm{Pic}^0(\mathbb{A}_Z)_{\mathbb{C}} \end{array}$$

in which the bottom row comes via base change (over $\overline{\mathbb{Q}}$) with $\mathbb{C}$ from the top row in the previous diagram, and where the vertical arrows are closed immersions. Passing to the Betti cohomology we see that $(c_1^B(\mathcal{P}))^i$ maps to a non-zero class in $H^{2i}(\mathbb{A}_{\mathbb{C}} \times_{\mathcal{A}_{g,\mathbb{C}}} U_{\mathbb{C}}, \mathbb{Q})$ for $i \leq g$, because, by Lemma 3.8, it maps to a non-zero class in $H^{2i}(A \times U_t, \mathbb{Q})$. Finally, the faithful flatness of $\eta'_{\mathbb{C}}$ and the argument of [Ja, p. 111] tells us that

$$\eta'^*_{\mathbb{C}} : H^{2i}((\mathbb{A}_Z)_{\mathbb{C}} \times_{Z_{\mathbb{C}}} U_{\mathbb{C}}, \mathbb{Q}) \to H^{2i}(\mathcal{X}_{\mathbb{C}}, \mathbb{Q})$$

is injective, hence that the image of $(c_1^B(\mathcal{P}))^i$ is non-zero in the latter group, for $i \leq g$. In particular, we get that the Deligne Chern class

$$c_i^{\mathcal{D}}(\mathcal{L}^{\oplus n}) = \binom{n}{i} . (c_1^{\mathcal{D}}(\mathcal{L}))^i \in H^{2i}_{\mathcal{D}}(\mathcal{X} \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Q}(i))$$

is non-zero, for $i \leq \min\{n,g\}$. Since S was arbitrary, this implies that the arithmetic Deligne Chern class

$$c_i^{\mathcal{AD}}(E) \in H^{2i}_{\mathcal{AD}}(A, \mathbb{Q}(i))$$

is non-zero, for $i \leq \min\{n,g\}$.

□

Remark 3.10. *1) We actually prove a stronger statement (pointed out by Jannsen): namely that the Chern classes of E are non-zero in the group $H^{2i}_{ar}(A,\mathbb{Q})$ defined in* (3).
2) The same proof also holds if E is a direct sum of distinct bundles $L_i \in Pic^0(A)$, by considering a larger moduli space parameterizing tuples $(L_1, ..., L_n)$, where $L_i \in Pic^0(A)$. We need to replace the family of Picard groups $Pic^0(A)$ by the family of self–products (n copies) of Picard groups and repeat the argument.

This gives us the main theorem:

Theorem 3.11. *Suppose E is a generic holomorphic connection of rank n on a complex abelian variety of dimension g then*

$$c_i^{\mathcal{AD}}(E) \in H^{2i}_{\mathcal{AD}}(A, \mathbb{Q}(i))$$

is non-zero, for $i \leq min\{n,g\}$.

Proof. Using (6), Corollary 3.2 and Corollary 3.3, we deduce the conclusion. □

4. Product of curves

Let $\mathcal{M}_g$ denote a fine moduli space of nonsingular projective connected curves of genus $g>0$, with suitable level structures.

Theorem 4.1. *Suppose C_i are nonsingular projective curves of genus $g_i > 0$, for $i = 1, ..., r$. Consider the product variety $X = C_1 \times C_2 \times ... \times C_r$ and a flat bundle*

$$E = p_1^*E_1 \otimes p_2^*E_2 \otimes ... \otimes p_n^*E_r$$

on X. Here p_i denotes the projection to C_i and E_i is a flat bundle on C_i of rank n_i. Suppose $(E_1, \ldots, E_r)$ is a generic tuple of unitary flat bundles E_i of rank n_i. Then the Chern classes of E in the arithmetic Deligne cohomology are non-torsion, i.e.,

$$c_i^{\mathcal{AD}}(E) \in H^{2i}_{\mathcal{AD}}(X, \mathbb{Q}(i))$$

is non-zero, for $i \leq min\{n,r\}$, where $n = \Pi_{i=1}^r n_i$.

Proof. Consider a fine moduli space (over $\overline{\mathbb{Q}}$) of products of curves C_{g_i} of genus g_i (for $1 \leq i \leq r$)

$$\mathcal{M}_{\bar{g}} = \mathcal{M}_{g_1} \times_{\overline{\mathbb{Q}}} \mathcal{M}_{g_2} \times_{\overline{\mathbb{Q}}} ... \times_{\overline{\mathbb{Q}}} \mathcal{M}_{g_r}$$

together with the universal product variety

$$\mathbb{X} = \mathcal{C}_{g_1} \times_{\overline{\mathbb{Q}}} \ldots \times_{\overline{\mathbb{Q}}} \mathcal{C}_{g_r} \longrightarrow \mathcal{M}_{\bar{g}}.$$

Let $\mathbb{G}_i \longrightarrow \mathcal{M}_{g_i}$ be the relative fine moduli space of unitary flat connections of rank n_i [1] and let

$$\mathcal{P}_i \longrightarrow \mathcal{C}_{g_i} \times_{\mathcal{M}_{g_i}} \mathbb{G}_i$$

be a universal Poincaré bundle of rank n_i.

Consider the relative moduli space of tuples $(E_1, \dots, E_r)$ where E_i is a unitary flat bundle of rank n_i on X_i

$$\begin{array}{c} \mathbb{G} = \mathbb{G}_1 \times_{\overline{\mathbb{Q}}} \dots \times_{\overline{\mathbb{Q}}} \mathbb{G}_r \\ \downarrow \\ \mathcal{M}_{\bar{g}} \end{array}$$

together with the universal Poincaré bundle

$$\begin{array}{c} \mathcal{P} = p_1^*\mathcal{P}_1 \times_{\overline{\mathbb{Q}}} \dots \times_{\overline{\mathbb{Q}}} p_r^*\mathcal{P}_r \\ \downarrow \\ \mathbb{X} \times_{\mathcal{M}_{\bar{g}}} \mathbb{G}. \end{array}$$

Suppose (X, E) is our given $\overline{\mathbb{Q}}$-point of $\mathbb{G}$. Let $(\mathcal{X} \to S, \mathcal{E})$ be a model for E on X. Then there is a morphism (actually over a finite surjective cover of S which we assume to be S, in view of Remark 2.1)

$$S \xrightarrow{\eta} \mathbb{G}$$

and there is a cartesian diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\eta'} & \mathbb{X} \times_{\mathcal{M}_{\bar{g}}} \mathbb{G} & \longrightarrow & \mathbb{X} \\ \downarrow \beta & & \downarrow p & & \downarrow \\ S & \xrightarrow{\eta} & \mathbb{G} & \longrightarrow & \mathcal{M}_{\bar{g}}, \end{array}$$

such that $\mathcal{E} = \beta^* M \otimes (\eta')^* \mathcal{P}$ for some line bundle M on S. As in the proof of Theorem 3.9 we may assume, by shrinking S, that M is trivial. Let Z be the scheme-theoretic image of S in $\mathcal{M}_{\bar{g}}$. By base change of the above diagram with Z we obtain a cartesian diagram as follows

$$\begin{array}{ccccccc} \mathcal{X} & \xrightarrow{\eta'} & \mathbb{X}_Z \times_Z \mathcal{U} & \hookrightarrow & \mathbb{X}_Z \times_Z \mathbb{G}_Z & \longrightarrow & \mathbb{X}_Z \\ \downarrow \beta & & \downarrow p_U & & \downarrow p & & \downarrow \\ S & \xrightarrow{\eta} & \mathcal{U} & \hookrightarrow & \mathbb{G}_Z & \longrightarrow & Z. \end{array}$$

Since E was supposed to be generic, the morphism $S \to \mathbb{G}_Z$ is dominant, and hence, by possibly shrinking S, there is a dense open $\mathcal{U} \subset \mathbb{G}_Z$ as indicated such that η and hence η' are faithfully flat. Moreover, we may assume that $\mathcal{U}$ is smooth over Z.

Consider the Betti Chern class of $\mathcal{P}$

$$c_i^B \in H^{2i}((\mathbb{X}_Z \times_Z \mathbb{G}_Z) \times_{\bar{\mathbb{Q}}} \mathbb{C}, \mathbb{Q}) \quad \text{for } i \leq \min\{n, r\}\,.$$

For the point $t : \mathrm{Spec}(\mathbb{C}) \to Z \subset \mathcal{M}_{\bar{g}}$ corresponding to $C_1 \times \dots \times C_r$, let

$$\mathbb{X}_t \times \mathbb{G}_t = (C_1 \times \mathbb{G}_1(C_1)) \times \dots \times (C_{g_r} \times \mathbb{G}_r(C_r))$$

[1]The moduli spaces were constructed in [Na-Se] by Narasimhan and Seshadri, for construction of relative moduli spaces see [Si2], [Si3] over a scheme of finite type over $\mathbb{C}$, more recently see [La] over schemes over a special class of rings which includes $\overline{\mathbb{Q}}$, though the case over $\overline{\mathbb{Q}}$ is well-known and appears in the work by several authors. The construction in [Na-Se] is for $g \geq 2$, but when $g = 1$ we are essentially dealing with an abelian variety of dimension one and we can use the moduli constructions and arguments from Theorem 3.9 and Lemma 3.1.

be the base change with t, and for each j choose a desingularization $G'_j \to \mathbb{G}_j(C_j)$, i.e., a proper surjective morphism with G'_j proper and smooth, and let $G' = G'_1 \times \ldots \times G'_r$. Then the restriction of c_i^B to $\mathbb{X}_t \times G'$ has a non-zero component in

$$H^{i,0}(\mathbb{X}_t) \otimes H^{0,i}(G'),$$

since its $(j_1, \ldots, j_i)$-Künneth component (for $j_1 < \ldots < j_i$) is

$$a_{(j_1,\ldots,j_i)}\, c_1^B(P'_{j_1}) \otimes c_1^B(P'_{j_2}) \otimes \ldots \otimes c_1^B(P'_{j_i}), \tag{8}$$

where $a_{(j_1,\ldots,j_i)}$ is a positive integer and each class $c_1^B(P'_j)$ of the restriction P'_j of $\mathcal{P}_j$ to $C_j \times G'_j$ has a non-trivial Hodge-Künneth component in $H^{1,0}(C_i) \otimes H^{0,1}(G'_i)$. In fact, we have a morphism

$$\begin{array}{rcl} \mathcal{C}_i \times_{\mathcal{M}_{g_i}} \mathbb{G}_i(\mathcal{C}_{g_i}) & \to & \mathcal{C}_i \times_{\mathcal{M}_{g_i}} \mathrm{Pic}^0 \mathcal{C}_i \\ (x, E) & \mapsto & (x, \Lambda^{n_i} E). \end{array}$$

This implies that $c_1(P'_j) = c_1(\Lambda^{n_j} P'_j)$ is the pull-back of the class of the Poincaré line bundle L_j under

$$H^2(C_j \times \mathrm{Pic^0(C_j), \mathbb{Q}) \to H^2(C_j \times G'_j, \mathbb{Q})}.$$

But this map is injective by a projection formula argument, because

$$G'_j \to \mathbb{G}_j(C_j) \to \mathrm{Pic^0(C_j)}$$

is proper and surjective. Moreover, L_j is the pull-back of the Poincaré line bundle on $\mathrm{J(C_j) \times Pic^0(C_j)}$ where $\mathrm{J(C_j)}$ is the Jacobian (Albanese) of C_j, and the morphism $C_j \to \mathrm{J(C_j)}$ (depending on the choice of a point $P \in C_j$) induces an isomorphism $H^1(\mathrm{J(C_j)}, \mathbb{Q}) \to \mathrm{H^1(C_j, \mathbb{Q})}$. Now it remains to note that $\mathrm{Pic^0(C_j)}$ is the dual abelian variety of $\mathrm{J(C_j)}$, and to apply Lemma 3.6.

By shrinking $\mathcal{U}$ if necessary, we may assume that we have a factorization

$$\mathcal{U}_t \hookrightarrow G' \to \mathbb{G}_t.$$

As in the proof of Theorem 3.9 we now conclude that the restriction to $H^{2i}(\mathbb{X}_t \times \mathcal{U}_t, \mathbb{Q})$ is non-zero, and that the class

$$c_i^B(\mathcal{E}) = \eta'^* c_i^B(\mathcal{P}) \in H^{2i}(\mathcal{X}_{\mathbb{C}}, \mathbb{Q})$$

for $i \leq \min\{n, r\}$, is non-zero. Since $\mathcal{X} \longrightarrow S$ was arbitrary (up to shrinking S), we get that the arithmetic Betti Chern class

$$c_i^{A\mathcal{D}}(E) \in H^{2i}_{A\mathcal{D}}(X, \mathbb{Q}(i))$$

is non-zero, for $i \leq \min\{n, r\}$, because its image in $H^{2i}_{ar}(X, \mathbb{Q}(i))$ is non-zero.

□

5. Small Chern classes on products of varieties

This section is due to U. Jannsen and he generalized the earlier examples to the following situation.

While the expression 'generic bundle' is most familiar when one has a corresponding moduli scheme, one may consider any model $\mathcal{X} \to S$ of a complex variety X as a kind of 'moduli scheme over $\mathbb{Q}$' for X, similarly for vector bundles or line bundles. This gives the following generalization of Theorem 4.1.

Theorem 5.1. *Suppose X_i are nonsingular complex projective varieties, for $i = 1, ..., r$, and assume that $H^1(X_i, \mathbb{Q}) \neq 0$ for all i. Consider the product variety $X = X_1 \times X_2 \times ... \times X_r$ and a flat bundle*

$$E = p_1^* E_1 \otimes p_2^* E_2 \otimes ... \otimes p_r^* E_r$$

on X. Here p_i denotes the projection to X_i and E_i is a flat bundle on X_i of rank n_i. Assume that $(\Lambda^{n_1} E_1, \ldots, \Lambda^{n_r} E_r)$ is generic among the tuples $(L_1, \ldots, L_r)$ of flat line bundles L_i on X_i. Then for all $i \leq min\{n, r\}$, with $n = \Pi_{i=1}^r n_i$, the Chern class of E in the arithmetic Deligne cohomology

$$c_i^{\mathcal{AD}}(E) \in H^{2i}_{\mathcal{AD}}(X, \mathbb{Q}(i))$$

is non-zero.

Proof. For each j, let $\mathcal{X}_j \to T_j = \text{Spec } A_j$, with $A_j \subset \mathbb{C}$, be a smooth projective model for X_j. Assume that T_j has minimal dimension. There are finitely generated $\overline{\mathbb{Q}}$-algebras B_j, $A_j \subseteq B_j \subset \mathbb{C}$, such that E_j has a model

$$(\mathcal{X}'_j = \mathcal{X}_j \times_{T_j} S_j \to S_j, \, \mathcal{E}_j)$$

where $S_j = \text{Spec } B_j$. We may assume that S_j is smooth over T_j. The line bundle $\Lambda^{n_j} \mathcal{E}_j$ defines a morphism $\eta_j : S_j \to \text{Pic}^0_{\mathcal{X}_j/T_j}$ and a cartesian diagram

$$\tag{9} \begin{array}{ccccc} \mathcal{E}_j & & & & \\ \downarrow & & & & \\ \mathcal{X}'_j & \xrightarrow{\eta'_j} & \mathcal{X}_j \times_{T_j} \text{Pic}^0_{\mathcal{X}_j/T_j} & \to & \mathcal{X}_j \\ \downarrow & & \downarrow & & \downarrow \\ S_j & \xrightarrow{\eta_j} & \text{Pic}^0_{\mathcal{X}_j/T_j} & \to & T_j. \end{array}$$

Now let $t = (t(1), \ldots, t(r)) : \text{Spec}(\mathbb{C}) \to T = T_1 \times \ldots \times T_r$ be the morphism "defining $X = X_1 \times \ldots \times X_r$", i.e., induced by the inclusions $A_j \subset \mathbb{C}$. Then we get a cartesian diagram via base change with $t(j)$

$$\begin{array}{ccccc} \mathcal{E}_{j,t} & & & & \\ \downarrow & & & & \\ X_j \times_{\overline{\mathbb{Q}}} S_{j,t} & \xrightarrow{\eta'} & X_j \times_{\overline{\mathbb{Q}}} \text{Pic}^0(X_j) & \to & X_j \\ \downarrow & & \downarrow & & \downarrow \\ S_{j,t} & \xrightarrow{\eta} & \text{Pic}^0(X_j) & \to & \text{Spec}(\mathbb{C}). \end{array}$$

Let $Z = \text{Spec } A \subseteq T$, where $A \subset \mathbb{C}$ is the subalgebra generated by the A_j (this is the scheme-theoretic image of t). Furthermore let $B \subset \mathbb{C}$ be any finitely generated $\overline{\mathbb{Q}}$-subalgebra containing all B_j and let $S = \text{Spec } B$. Define $\mathcal{X}$ by the cartesian diagram

$$\begin{array}{ccccccc} \mathcal{X} & \xrightarrow{g'} & \mathcal{X}' & = & \mathcal{X}'_1 \times_{\overline{\mathbb{Q}}} \ldots \times_{\overline{\mathbb{Q}}} \mathcal{X}'_r & = \mathbb{X} \times_T S' \\ \downarrow & & \downarrow & & & \\ S & \xrightarrow{g} & S' & = & S_1 \times_{\overline{\mathbb{Q}}} \ldots \times_{\overline{\mathbb{Q}}} S_r \, , & \end{array}$$

where g is the canonical morphism, and $\mathbb{X} = \mathcal{X}_1 \times_{\overline{\mathbb{Q}}} \ldots \times_{\overline{\mathbb{Q}}} \mathcal{X}_r$. Let $\mathcal{E}$ be the pullback to $\mathcal{X}$ of the vector bundle $\mathcal{E}' = p_1^* \mathcal{E}_1 \otimes \ldots \otimes p_r^* \mathcal{E}_r$ on $\mathcal{X}'$ where $p_j : \mathcal{X}' \to \mathcal{X}'_j$ is the projection.

Then $(\mathcal{X} \to S, \mathcal{E})$ is a model for E on X. On the other hand let $\mathrm{Pic}^0_{\mathbb{X}/T} = \prod_{i=1}^r \mathrm{Pic}^0_{\mathcal{X}_j/T_j}$ (product over $\overline{\mathbb{Q}}$), and consider the cartesian diagram

$$\begin{array}{ccccccc} \mathcal{X} & \xrightarrow{g'} & \mathbb{X}_Z \times_T S' & \xrightarrow{\eta'} & \mathbb{X}_Z \times_Z (\mathrm{Pic}^0_{\mathbb{X}/T})_Z & \longrightarrow & \mathbb{X}_Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{g} & S' & \xrightarrow{\eta} & (\mathrm{Pic}^0_{\mathbb{X}/T})_Z & \longrightarrow & Z \end{array} \tag{10}$$

obtained from the obvious diagram via base change with Z. By our assumption on the genericity of the tuple $(\Lambda^{n_1} E_1, \ldots, \Lambda^{n_r} E_r)$, the morphism ηg, and hence also $\eta' g'$, is dominant. Field theory shows that this does not depend on the choices of the models. By possibly shrinking S we may assume that there is a smooth dense open $\mathcal{U} \subseteq (\mathrm{Pic}^0_{\mathbb{X}/T})_Z$ such that $\eta g : S \longrightarrow \mathcal{U}$ is faithfully flat. Moreover, we may assume that $\mathcal{U}$ is smooth over Z.

From this we get a cartesian diagram

$$\begin{array}{ccccccc} X & \xrightarrow{g'} & X \times S'_t & \xrightarrow{\eta'} & X \times \mathrm{Pic}^0 X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_{\mathbb{C}} & \xrightarrow{g} & (\mathbb{X}_Z \times_Z S')_{\mathbb{C}} & \xrightarrow{\eta} & (\mathbb{X}_Z \times_Z (\mathrm{Pic}^0_{\mathbb{X}/T})_Z)_{\mathbb{C}} & \longrightarrow & (\mathbb{X}_Z)_{\mathbb{C}}\,, \end{array}$$

where the top row comes from base change of (10) with $t : \mathrm{Spec}\mathbb{C} \to Z$, and the bottom row comes from taking the base change of the first row of (10) with $\mathbb{C}$ over $\overline{\mathbb{Q}}$. The restriction of $\mathcal{E}'$ to the fiber via t, i.e., to $X \times S'_t = (X_1 \times S_{1,t}) \times \ldots \times (X_r \times S_{r,t})$ is the exterior product $\mathcal{E}_t = p_1^* \mathcal{E}_{1,t} \otimes \ldots \otimes p_r^* \mathcal{E}_{r,t}$. Fix an $i \leq \min\{n, r\}$. Then the Chern class $c_i(\mathcal{E}_t)$ has $(j_1, \ldots, j_i)$-Künneth component (for $j_1 < \ldots < j_i$)

$$a_{j_1,\ldots,j_i}\, c_1^B(\mathcal{E}_{j_1,t}) \otimes \ldots \otimes c_1^B(\mathcal{E}_{j_i,t})$$

with a positive integer $a_{j_1,\ldots,j_i}$. For each $j \in \{1, \ldots, r\}$,

$$c_1^B(\mathcal{E}_{j,t}) = c_1^B(\Lambda^{n_j} \mathcal{E}_{j,t})$$

is the pull-back of the Poincaré line bundle L_j on $X_j \times \mathrm{Pic}^0(X_j)$. Let $\mathrm{Alb}(X_j)$ be the Albanese variety of X_j. Then L_j is in turn the pull-back of the Poincaré line bundle P_j on $\mathrm{Alb}(X_j) \times \mathrm{Pic}^0(X_j)$ via the morphism $X_j \to \mathrm{Alb}(X_j)$ (depending on the choice of a point $P_j \in X_j$), noting that we have $\mathrm{Pic}^0(X_j) = \mathrm{Pic}^0(\mathrm{Alb}(X_j))$. Now let W_j be any smooth compactification of $S_{j,t}$ which has the property that the morphism

$$\eta_j : S_{j,t} \to \mathrm{Pic}^0(X_j)$$

extends to a morphism

$$\tilde{\eta} : W_j \to \mathrm{Pic}^0(X_j).$$

Then this morphism is proper and surjective, and the induced map

$$H^1(\mathrm{Pic}^0(X_j), \mathbb{Q}) \to H^1(W_j, \mathbb{Q})$$

is injective. Thus, by Lemma 3.6 and the fact that

$$H^1(\mathrm{Alb}(X_j), \mathbb{Q}) \xrightarrow{\cong} H^1(X_j, \mathbb{Q}) \neq 0,$$

the pull-back of $c_1(P_j)$ to $X_j \times W_j$ has non-trivial Hodge-Künneth component in

$$H^{1,0}(X_j, \mathbb{Q}) \otimes H^{0,1}(W_j, \mathbb{Q}).$$

Let $W = W_1 \times \ldots \times W_r$. We conclude that the pull-back of $c_i(\mathcal{E}_t)$ to $X \times W$ has non-trivial component in

$$H^{i,0}(X,\mathbb{Q}) \otimes H^{0,i}(W,\mathbb{Q}).$$

Now let $\mathcal{U}' \subset S'$ be the preimage of the open $\mathcal{U} \subseteq (\mathrm{Pic}^0_{\mathbb{X}/T})_Z$, and let $\mathcal{U}'_t \subset S_t \subset W$ be the fiber of $\mathcal{U}$ over t. Then, by the same argument as in the proof of Theorem 3.9 we conclude that the restriction of the mentioned pull-back to $H^{2i}(X \times \mathcal{U}_t, \mathbb{Q})$ is non-zero. But this is also the pull-back of $c_i(\mathcal{E}_t)$. This shows the non-vanishing of

$$c_i^B(\mathcal{E}') \in H^{2i}((\mathcal{X}'_{\mathbb{C}}, \mathbb{Q}).$$

Since the chosen schemes S are cofinal in the limit to be considered, we conclude that $c_i^{A\mathcal{D}}(E)$ is non-zero as claimed. □

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