Basic Kernels
Problem Sheet

1. A graph G is a *cluster graph* if every connected component of G is a clique. In the Cluster Editing problem, we are given as input a graph G and an integer k, and the objective is to check whether one can edit (add or delete) at most k edges in G to obtain a cluster graph. That is, we look for a set $F \subseteq E(G)$ of size at most k, such that the graph $(V(G), (E(G) \setminus F) \cup (F \setminus E(G)))$ is a cluster graph.

   (a) Show that a graph G is a cluster graph if and only if it does not have an induced path on three vertices (a sequence of three vertices u, v, w such that $uv, vw \in E(G)$ and $uw \notin E(G)$).

   (b) Show a kernel for Cluster Editing with $\mathcal{O}(k^2)$ vertices.

2. In the Min-Ones-2-SAT problem, we are given a 2-CNF formula ϕ and an integer k, and the objective is to decide whether there exists a satisfying assignment for ϕ with at most k variables set to true. Show that Min-Ones-2-SAT admits a polynomial kernel.

3. In the d-Bounded-Degree Deletion problem, we are given an undirected graph G and a positive integer k, and the task is to find at most k vertices whose removal decreases the maximum vertex degree of the graph to at most d. Obtain a kernel of size polynomial in k and d for the problem. (Observe that Vertex Cover is the case of $d = 0$.)

4. In the d-Hitting Set problem, the input consists of a set U, called the universe, a family $\mathcal{A}$ of subsets of U such that $|A| \leq d$ for every $A \in \mathcal{A}$ and a positive integer k, and the objective is to decide whether $(U, \mathcal{A})$ has a *hitting set* of size at most k. (A set $X \subseteq U$ is said to be a hitting set of $(U, \mathcal{A})$ if $X \cap A \neq \emptyset$ for every $A \in \mathcal{A}$.)

   (a) Consider a restriction of d-Hitting Set, called Ed-Hitting Set, where we require every set in the input family $\mathcal{A}$ to be of size exactly d. Show that this problem is not easier than the original d-Hitting Set problem, by transforming a d-Hitting Set instance into an equivalent Ed-Hitting Set instance without changing the number of sets.

   (b) Show a kernel with at most $f(d)k^d$ sets for the Ed-Hitting Set problem.

5. In the Connected Vertex Cover problem, we are given an undirected graph G and a positive integer k, and the objective is to decide whether there exists a vertex cover C of G such that $|C| \leq k$ and $G[C]$ is connected.

   (a) Explain where the Buss kernel for Vertex Cover breaks down for the Connected Vertex Cover problem.

   (b) Show that Connected Vertex Cover admits a kernel with at most $2^k + \mathcal{O}(k^2)$ vertices.

   (c) Show that if the input graph G does not contain a cycle of length 4 as a subgraph, then the problem admits a kernel with at most $\mathcal{O}(k^2)$ vertices.

6. A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. In the Vertex Disjoint Paths problem, we are given an undirected graph G and k pairs of vertices (s_i, t_i), $i \in \{1, \ldots, k\}$, and the objective is to decide whether there exists paths P_i joining s_i to t_i such that these paths are pairwise vertex disjoint. Show that Vertex Disjoint Paths admits a polynomial kernel on split graphs (when parameterized by k).

7. Byteland, a country of area exactly n square miles, has been divided by the government into n regions, each of area exactly one square mile. Meanwhile, the army of Byteland divided its area into n military zones, each of area again exactly one square mile. Show that one can build n airports in Byteland, such that each region and each military zone contains one airport. (Hint use Hall's theorem for bipartite graphs)

8. In the Point Line Cover problem, we are given a set of n points in the plane and an integer k , and the goal is to check if there exists a set of k lines on the plane that contain all the input points. Show a kernel for this problem with $O(k^2)$ points