Entanglement entropy and higher spin holography

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Based on :

- Shouvik Datta, Justin R. David, Michael Ferlaino, S. Prem Kumar
  Higher spin entanglement entropy from CFT
  [arXiv:1402.0007] JHEP 1406 (2014) 096.

- Shouvik Datta
  Relative entropy in higher spin holography
  [arXiv:1406.0520] To appear in Phys. Rev. D.

Introduction and Motivation

- A major theme in the context of the holography is the study of dualities between theories of higher spin gravity and CFTs with extended symmetries.
  [Klebanov-Polyakov `02, Sezgin-Sundell `02, Gaberdiel-Gopakumar `11]
- Higher spin gravity has considerably lesser number of fields compared to full fledged string theories.
- At the same time one hopes to go beyond the classical supergravity regime and capture some features of tensionless strings.
- We have better analytical control over the dual CFTs.
- We can hope to learn a lot from both sides of the duality!

Higher spin AdS_3/CFT_2

- Vasiliev higher spin gravity in AdS_3 is conjectured to be dual to $\mathcal{W}_N$ minimal models. [Gaberdiel-Gopakumar `11]

higher spin gravity	$N, k \to \infty$ ⟷	coset CFT
$\mathrm{hs}[\lambda]$	$0 \leq \lambda = \frac{N}{N+k} \leq 1$	$\frac{SU(N)_k \otimes SU(N)_1}{SU(N)_{k+1}}$

- Evidence in favour of the duality : matching of symmetries, one-loop determinants, correlation functions, black hole partition functions, ...

Higher spin black holes and CFT thermodynamics

- There exists explicit constructions of classical solutions in higher spin gravity in 3d — black holes, conical defects etc.
  [Gutperle-Kraus `11; Kraus-Perlmutter `11; Castro-Gopakumar-Gutperle-Raeymaekers `11; ...]
- The higher spin black holes are generalizations of the BTZ black holes with higher spin charges (and with conjugate chemical potentials).
- The dual description of this is that of CFT at finite temperature and at finite chemical potentials for conserved higher spin currents.

Higher spin black holes and CFT thermodynamics

- Striking agreement of black hole thermodynamics with that of the CFT

$$\log Z = \frac{i\pi c}{12\tau}\left[1 - \frac{4}{3}\frac{\alpha^2}{\tau^4} + \frac{400}{27}\frac{\lambda^2-7}{\lambda^2-4}\frac{\alpha^4}{\tau^8} - \frac{1600}{27}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\frac{\alpha^6}{\tau^{12}} + \cdots\right]$$

[Gaberdiel-Hartman-Jin `12] also [SD-David-Ferlaino-Kumar `14; Long `14]

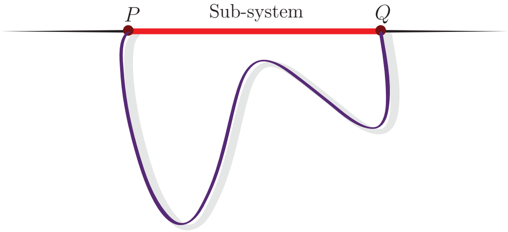
- What about more refined observables like entanglement entropy?

Entanglement entropy and higher spin holography

- The Ryu-Takayanagi minimal area proposal offers a simple and elegant route to calculate the entanglement entropy of the dual CFT.
  [Ryu-Takayanagi `06]
- For AdS_3, the HEE is proportional to the length of the geodesic anchored at the endpoints of the subsystem.
- Higher spin gravity goes beyond diffeomorphism invariance.
- We need to rethink usual notions of spacetime geometry — horizons, singularities, minimal surfaces etc.
- What is the bulk observable which captures the entanglement entropy of a $\mathcal{W}$-algebra CFT?

Wilson lines as holographic entanglement entropy functionals

- Wilson lines form the basic objects in the Chern-Simons description of higher spin gravity.
- They also generalize the notion of trajectories of massive particles i.e. geodesics.
- It has been conjectured that in higher spin gravity functionals involving Wilson lines in capture the entanglement entropy.



Wilson lines as holographic entanglement entropy functionals

- **Proposal I** : Wilson lines in the infinite dimensional representation [Ammon-Castro-Iqbal `13]

$$S_{\text{EE}} = -\log(W_{\mathcal{R}}(C)) \, , \qquad W_{\mathcal{R}}(C) = \text{tr}_{\mathcal{R}} \left(\mathcal{P} \exp \int_C (A + \bar{A}) \right)$$

The path-integral representation of the Wilson line is used to evaluate the EE.

- **Proposal II** : Composite Wilson line [de Boer-Jottar `13]

$$S_{\text{EE}} = \frac{k}{\sigma_{1/2}} \log \left[\lim_{\rho_0 \to \infty} W^{\text{comp}}_{\mathcal{R}_N}(C)|_{\rho_0 = \rho_A = \rho_B} \right] , \; W_{\mathcal{R}}(C) = \text{tr}_{\mathcal{R}_N} \Big(\mathcal{P} e^{(\int_C A)} \mathcal{P} e^{(-\int_C \bar{A})} \Big)$$

The dimension of the representation is $2^{N(N-1)/2}$ for the principal embedding. It also has a holomorphic version.

- These proposals have been shown to be equivalent. [Castro-Llabrés `14]

Testing the holographic entanglement entropy proposals

In this talk, I shall focus on some tests of the holographic EE proposal

- Comparison with universal CFT results (from Justin's talk).
- Short distance behaviour and relative entropy.

Entanglement entropy from higher spin holography

- Higher spin black holes describe the CFT at finite temperature and a finite higher spin chemical potential.
- We shall consider black holes in the simplest higher spin theory $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$. [Gutperle-Kraus `11]
- These black holes have a rich phase structure.
  We focus on the 'BTZ branch'. [David-Ferlaino-Kumar `12; Chowdhury-Saha `13; ...]
- The EE is computed via the holomorphic Wilson line functional to be

$$S_E = \frac{c}{3}\log\left|\frac{\pi}{\beta}\sinh\left(\frac{\pi\Delta}{\beta}\right)\right| + c\,\frac{\mu^2}{\beta^2}\left[\frac{32\pi^2}{9}\left(\tfrac{\pi\Delta}{\beta}\right)\coth\left(\tfrac{\pi\Delta}{\beta}\right) - \frac{20\pi^2}{9} - \frac{4\pi^2}{3}\operatorname{csch}^2\left(\tfrac{\pi}{\beta}\right)\left\{\left(\tfrac{\pi}{\beta}\coth\left(\tfrac{\pi}{\beta}\right) - 1\right)^2 + \left(\tfrac{\pi}{\beta}\right)^2\right\}\right] + \mathcal{O}(\mu^4)$$

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The first correction to the entanglement entropy matches exactly with CFT calculations. [SD-David-Ferlaino-Kumar `14]

Relative entropy

Definition and properties

- Relative entropy is a measure of distinguishability of two states for a quantum system.
- For two density matrices σ and ρ, the relative entropy is defined as

$$\mathcal{S}(\sigma||\rho) = \mathrm{tr}(\sigma \log \sigma) - \mathrm{tr}(\sigma \log \rho)$$

- Properties
  1. Non-negativity : $\mathcal{S}(\sigma||\rho) \geq 0$.
  2. Invariance under unitary trans : $\mathcal{S}(\sigma||\rho) = \mathcal{S}(U^\dagger \sigma U || U^\dagger \rho U)$.
  3. Monotonicity under partial traces : $\mathcal{S}(\sigma||\rho) \geq \mathcal{S}(\mathrm{tr}_P \sigma || \mathrm{tr}_P \rho)$
  4. Additivity : $\mathcal{S}(\sigma_A \otimes \sigma_B || \rho) = \mathcal{S}(\sigma_A || \rho) + \mathcal{S}(\sigma_B || \rho)$

[Vedral `02]

Relative entropy

Relationship with the modular Hamiltonian and entanglement entropy

We wish express relative entropy in terms of thermodynamic-like quantities.

For a given (reduced) density matrix, the modular Hamiltonian is defined as

$$\rho = \frac{e^{-H}}{\mathrm{tr}(e^{-H})}$$

It can then be shown that the relative entropy is

$$\mathcal{S}(\sigma||\rho) = \Delta\langle H\rangle - \Delta S$$

The relative entropy vanishes in the limit of small sub-system sizes

$$\lim_{\frac{\dim(A)}{\dim(A')}\to 0} \left(\Delta\langle H_A\rangle - \Delta S_A\right) = 0 \qquad \Longrightarrow \qquad \Delta\langle H\rangle = \Delta S$$

The first law of entanglement

[Blanco-Casini-Hung-Myers `13]

Relative entropy in a $\mathcal{W}$-algebra CFT and its holographic dual

- We shall try to calculate the relative entropy between a high temperature state and the vacuum in a CFT with $\mathcal{W}$ symmetries in presence of a chemical potential for the spin-3 current.

  The CFT is at large central charge and on a finite system of size R and the high temperature state is at temperature T.

- As we had seen earlier such a CFT is describable in terms of higher-spin gravity.

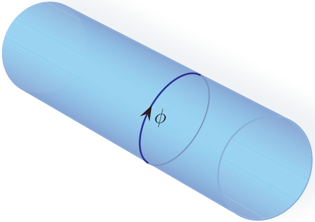
- It is possible to calculate $\langle H_A \rangle$ from the holographic stress tensor. The EE (S_A) is also calculable in terms of Wilson lines.

- We shall try to verify $\Delta\langle H_A \rangle = \Delta S$ in the short distance regime.

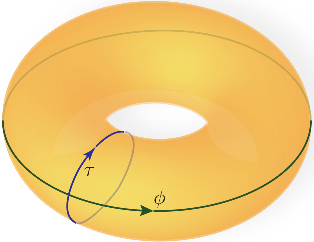
The bulk configurations

The gravity configurations dual to the vacuum and high temperature state of the CFT are the higher spin vacuum and black hole respectively.

[Gutperle-Kraus `11; Kraus-Perlmutter `11; Castro-Gopakumar-Gutperle-Raeymaekers `11; Li-Lin-Wang `13; Compere-Jottar-Song `13; Chowdhury-Saha `13]



Vacuum



High temperature state

The higher spin vacuum is a higher spin generalization of global AdS. It has trivial holonomy along the spatial ϕ cycle.

The higher spin black hole generalizes the BTZ. Its temporal cycle τ has trivial holonomy.

Modular Hamiltonian

The modular Hamiltonian is not a local quantity in general. However, there exist special cases where it is local and calculable.

[Casini-Huerta-Myers `11]

One such example is that of the vacuum state of any QFT restricted to the half-space. H_{vac} then generates boost orbits in the Rindler wedge.

[Bisognano-Wichmann `75]

There exists a conformal transformation which maps the Rindler wedge to the causal domain of a spherical entangling region.

The modular Hamiltonian associated with the vacuum in a 1+1 d CFT is

$$H_{\mathsf{vac}} = 2\pi R^2 \int_{-\frac{\phi}{2}}^{\frac{\phi}{2}} d\theta \, \frac{\cos\theta - \cos\frac{\phi}{2}}{\sin\frac{\phi}{2}} \, T_{00}(\theta)$$

Here, $T_{00} = (L_0 - \frac{c}{24}) + (\bar{L}_0 - \frac{c}{24})$. These can be obtained from the holographic stress tensor for specific states.

[Balasubramanian-Kraus `99, de Haro-Solodukhin-Skenderis `00]

Modular Hamiltonian from the holographic stress tensor

The stress tensors corresponding to the hs-vacuum and the hs-black hole can be obtained by solving holonomy conditions.

The expectation values of the modular Hamiltonian are therefore

$$\langle H \rangle_{\text{state}} = \text{tr}(\rho_{\text{state}} H_{\text{vac}}) = 8\pi R^2 \left[1 - \frac{\phi}{2} \cot\left(\frac{\phi}{2}\right)\right] \mathcal{L}_{\text{state}}$$

where,

$$\mathcal{L}_T = \frac{c\pi T^2}{12}\left[1 + \frac{80(\pi\mu T)^2}{3} + \frac{2560(\pi\mu T)^4}{3} + \frac{905216(\pi\mu T)^6}{27} + \cdots\right]$$

$$\mathcal{L}_{\text{vac}} = -\frac{c}{48\pi R^2}\left[1 - \frac{20}{3}\left(\frac{\mu}{R}\right)^2 + \frac{160}{3}\left(\frac{\mu}{R}\right)^4 - \frac{14144}{27}\left(\frac{\mu}{R}\right)^6 + \cdots\right]$$

The difference $\Delta\langle H \rangle$ can then be calculated.

Holographic entanglement entropy

The EEs — computed via Wilson lines — corresponding to higher spin black holes and the vacuum in the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ theory are

$$\begin{aligned} S_T(\phi) =& \frac{c}{3}\log\left|\frac{\sinh(\pi R T\phi)}{\Lambda^{-1}\,\pi T}\right| \\ &+ \frac{c}{18}(\pi\mu T)^2 \operatorname{csch}^4(\pi R T\phi)\,\Big[8\left(1-3\pi^2R^2T^2\phi^2\right)\cosh(2\pi R T\phi) \\ &\qquad +8\pi R T\phi\left(\sinh(2\pi R T\phi)+\sinh(4\pi R T\phi)\right) \\ &\qquad -5\cosh(4\pi R T\phi)-3\,\Big] + \mathcal{O}((\pi\mu T)^4) \end{aligned}$$

$$\begin{aligned} S_{\mathsf{vac}}(\phi) =& \frac{c}{3}\log\left|\frac{2R}{\Lambda^{-1}}\,\sin\left(\frac{\phi}{2}\right)\right| \\ &+ \frac{c}{72}\left(\frac{\mu}{R}\right)^2 \csc^4\left(\frac{\phi}{2}\right)\Big[3-2\left(3\phi^2+4\right)\cos(\phi)+4\phi(\sin(\phi)+\sin(2\phi)) \\ &\qquad +5\cos(2\phi)\Big] + \mathcal{O}((\mu/R)^4) \end{aligned}$$

One can systematically keep track of terms to higher orders.

Relative entropy in holographic CFTs with a $\mathcal{W}$-symmetry

We can now employ the thermodynamic-like relation to calculate the relative entropy between the high-temp state and the vacuum.

$$\mathcal{S}(\rho_T||\rho_{\text{vac}}) = (\langle H\rangle_T - \langle H\rangle_{\text{vac}}) - (S_T - S_{\text{vac}})$$

We shall focus on the small-subsystem size regime where we expect $\Delta\langle H\rangle = \Delta S$.

[Blanco-Casini-Hung-Myers `13]

Relative entropy in holographic CFTs with a $\mathcal{W}$-symmetry

At the leading order in short-subsystem sizes ...

$$\Delta S\Big|_{\text{to }\phi^2} \overset{\text{matches}}{=} \Delta\langle H\rangle\Big|_{\text{to }\phi^2}$$

$$= c\,\phi^2\Bigg[\frac{\left((\ell T)^2+1\right)}{72}+\frac{5\left((\ell T)^4-1\right)}{54}\frac{\mu^2}{R^2}+\frac{20\left((\ell T)^6+1\right)}{27}\frac{\mu^4}{R^4}$$
$$+\frac{1768\left((\ell T)^8-1\right)}{243}\frac{\mu^6}{R^6}+\frac{57664\left((\ell T)^{10}+1\right)}{729}\frac{\mu^8}{R^8}+\ldots\Bigg]$$

At the leading order in entangling interval sizes, $\Delta H = \Delta S$ in a large-c CFT with a $\mathcal{W}_3$ symmetry at finite higher spin chemical potential. ($\ell = 2\pi R$)

If the AdS is considered as the ultimate vacuum, $\Delta\langle H\rangle = \Delta S$ can be verified for that case as well.

Relative entropy in holographic CFTs with a $\mathcal{W}$-symmetry

- The relative entropy was calculated between two non-perturbative states in the CFT Hilbert space. These corresponds to two distinct instanton-like saddles in the dual gravity theory.

- One can also find what's the relative entropy between any member in the $SL(2,\mathbb{Z})$ family of smooth solutions and the vacuum. These smooth solutions have a non-zero contribution to the modular invariant gravity partition function.

$$Z[\tau] = \sum Z_{AdS_3}\left[\frac{a\tau+b}{c\tau+d}\right]$$

[Dijkgraaf-Maldacena-Moore-Verlinde `00, Manschot-Moore `07, Li-Lin-Wang `13]

Summary & Outlook

- The holomorphic Wilson line proposal correctly reproduces the universal correction to higher spin EE.
- We have verified the first law of entanglement holographically — in the regime of short intervals and at finite chemical potential for a higher spin current.
- This ensures the vanishing of the relative entropy which is expected to be true for any quantum mechanical system.
- The relative entropy in (1+1)d is independent of the UV cut-off. It's a refined observable in this sense.
- We have also probed the short-distance behaviour of the holographic EE and seen that it has the desired behaviour.
- All this lends strong support in favour of the Wilson line functional as the bulk observable which captures entanglement entropy.

Summary & Outlook

- Generalizing results to $\mathrm{hs}[\lambda]$.
- Multi-sheeted correlators from holography.
- Equations for higher spin fields from entanglement.
  [Hijano-Kraus `14]
- Is there a non-perturbative way to treat the higher spin chemical potential? [Kaneko-Zagier `95]

Thank you.

Backup slides

A thermodynamic relation for relative entropy

$$\begin{aligned}
S(\sigma||\rho) &= \mathrm{tr}(\sigma \ln \sigma) - \mathrm{tr}(\sigma \ln \rho) \\
&= \mathrm{tr}(\sigma \ln \sigma) - \mathrm{tr}(\rho \ln \rho) + \mathrm{tr}(\rho \ln \rho) - \mathrm{tr}(\sigma \ln \rho) \\
&= -S_\sigma + S_\rho - \mathrm{tr}(\rho H_\rho) + \mathrm{tr}(\sigma H_\rho) \\
&= \Big(\big\langle H \big\rangle_\sigma - \big\langle H \big\rangle_\rho \Big) - (S_\sigma - S_\rho) \\
&= \Delta \big\langle H \big\rangle - \Delta S
\end{aligned}$$

On conformal invariance

It is true that CFT partition functions/EE are calculated in conformal perturbation theory. But this does not mean conformal invariance is broken.

Turning on a chemical potential for a higher spin conserved current is on the same footing as having the system at a finite temperature.

For the free fermion theory in presence of a $U(1)$ chemical potential the partition function is

$$Z = \mathrm{Tr}(z^{J_0} q^{L_0}) = \left| \frac{\vartheta_3(\mu\beta|\tau)}{\eta(\tau)} \right|$$

When a higher spin chemical potential is turned on perturbatively, the $\mathcal{W}_3 \times \mathcal{W}_3$ asymptotic symmetry is unbroken.

[Compere-Song `13; Compere-Jottar-Song `13]

We need to specify boundary conditions for the higher spin black hole — fall-off conditions and initial data for higher spin charges.

The $\mathcal{W}_3 \times \mathcal{W}_3$ symmetry can then be shown to be remain intact by a proper redefinition of the generators.