# Dilaton Effective Action and Entanglement Entropy

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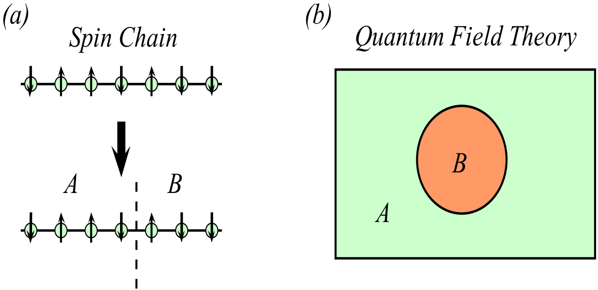
# Introduction

- Entanglement entropy is a measure of quantum correlation in a pure state.
- Take a system and divide it into two parts $A$ and $B$.




- The system is in a pure state $|\psi>$ and the Hilbert space of the system is factorized into the Hilbert space of $A$ and $B$.

$$H = H_A \otimes H_B \tag{1}$$

- One can define the reduced density matrix, $\rho_B$ for system $B$ by tracing over the $A$ Hilbert space.

$$\rho_B = Tr_{H_A} |\psi >< \psi| \tag{2}$$

- The entropy associated with density matrix $\rho_B$ is called the entanglement entropy of system $B$.

$$S_{EE} = -Tr_{H_B} \ \rho_B ln \rho_B \tag{3}$$

# Replica Trick

- Replica trick is a useful way to compute entanglement entropy in quantum field theory.
- The trick is to use the identity

$$lnX = lim_{n \to 1} \frac{X^{n-1} - 1}{n - 1} \tag{4}$$

- Using this one can write,

$$S_{EE} = -Tr\rho_A ln\rho_A = -lim_{n \to 1} \frac{Tr\rho_A^n - 1}{n - 1} = -\frac{\partial}{\partial n}|_{n=1} Tr\rho_A^n \tag{5}$$

- We can decompose $\rho_A$ as,

$$\rho_A = \int D\phi_A^{'} D\phi_A^{''} \, |\phi_A^{'} >< \phi_A^{'}|\rho_A|\phi_A^{''} >< \phi_A^{''}| \qquad (6)$$

where $|\phi_A >$ denotes a field eigenstate on the subsystem $A$.

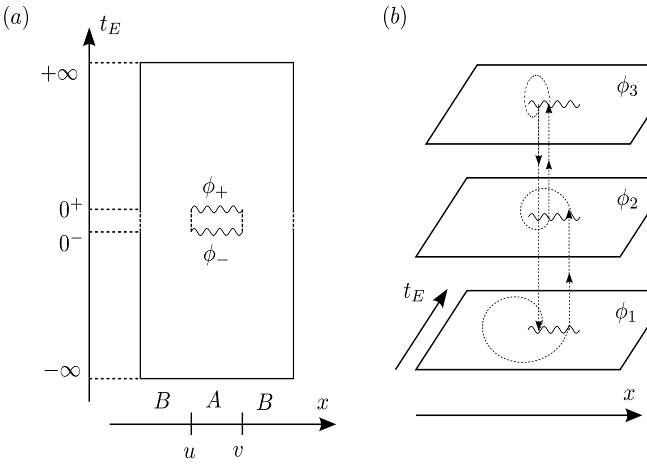
- To compute the matrix element of the reduced density matrix, let us specialize to the case of ground state of the field theory.
- The ground state wave functional can be written as,

$$\psi[\bar{\phi}(\vec{x})] = \frac{1}{\sqrt{Z}} \int_{\phi(\vec{x},0)=\bar{\phi}(\vec{x})} D\phi \; e^{-\int d^3\vec{x} \int_{-\infty}^{0} d\tau L_E(\phi)} \qquad (7)$$

where $Z$ is the Euclidean partition function of the field theory and $L_E$ is the Euclidean Lagrangian.

- Using this one can write,

$$< \phi_- | \rho_A | \phi_+ > = \frac{1}{Z} \int_{\phi(\vec{x}, 0^+)|_A = \phi_+, \phi(\vec{x}, 0^-)|_A = \phi_-} D\phi(\vec{x}, \tau) e^{-S_E(\phi)} \tag{8}$$



- So the matrix element of the reduced density operator is computed by the Euclidean path-integral with a branch cut along the subsystem, $A$, located on the $\tau = 0$ spatial slice.
- $$Tr\rho_A^n \sim \sum <\phi_1|\rho_A|\phi_n><\phi_n|\rho_A|\phi_{n-1}> ........ <\phi_2|\rho_A|\phi_1> \tag{9}$$
- With the help of the path-integral expression this expression can be written as,
$$Tr\rho_A^n = \frac{Z_n}{Z^n} \tag{10}$$

- $Z_n$ is the partition function of the Euclidean theory on the $n$-fold replicated space and $Z$ is the partition function of the theory on the originial space-time.
- The replica space is a space with conical singularities along the entangling surface.
- The opening angle at each singularity is $2\pi n$.
- The expression of the entanglement entropy can be simplified to,

$$S_{EE}(A) = n\frac{\partial}{\partial n}|_{n=1}(F_n - nF_1) \tag{11}$$

where $F = -lnZ$.
- This is the main formula that we will use.

# Massive Theories in two dimensions

- Consider a massive two dimensional theory.
- When the subsystem is an infinite half-line, Calabrese and Cardy proved that,

$$S_{EE} = \frac{c_{UV}}{6} ln \frac{\xi}{\epsilon} \qquad (12)$$

  where $\xi$ is the correlation length and $\epsilon$ is the short-distance cutoff.
- Let us first go through their proof.

# Calabrese-Cardy Proof

- The replica geometry for an infinite half-line is that of a flat cone with only one conical singularity with opening angle $2\pi n$.
- Our field theory lives on this cone.
- The field theory has a stress tensor, $T_{\mu\nu}$ which satisfies the conservation law,

$$\partial_\mu T^{\mu\nu} = 0 \tag{13}$$

- If we use complex coordinates then the conservation equation can be written as,

$$\bar{\partial} T + \frac{1}{4}\partial\Theta = 0 \tag{14}$$

where $T = T_{zz}$ and $\Theta = T^{\mu}_{\mu}$.

- On the cone translational invariance is broken and there is only rotational invariance.
- $$< T(z,\bar{z}) >_n = \frac{F_n(z\bar{z})}{z^2} \tag{15}$$
- $$< \Theta(z,\bar{z}) >_n - < \Theta >_1 = \frac{G_n(z\bar{z})}{z\bar{z}} \tag{16}$$
- Substituting them in the conservation equation we get,

$$F_n^{'} + \frac{1}{4}G_n^{'} = \frac{1}{4}(< \Theta >_n - < \Theta >_1) \tag{17}$$

where, $' = \frac{d}{d(z\bar{z})}$

- $$\frac{1}{4}\int_{cone} d^2x(< \Theta >_n - < \Theta >_1) = n\pi(F_n(r^2) + \frac{1}{4}G_n(r^2))|_0^\infty \tag{18}$$

  where we have used the fact that $d^2x = rdrd\theta$ and $0 \leq \theta \leq 2\pi n$, on the cone.
- Now the contribution form infinity is zero because it is a gapped theory.
- The contribution from zero is given the UV-CFT values.
- So,

$$G_n(0) = 0, \;\; F_n(0) = \frac{c_{UV}}{24}(1 - \frac{1}{n^2}) \tag{19}$$

- We have used the Callan-Symanzik equation and the fact that $T_{\mu\nu}$ being a conserved current has zero anomalous dimension.
- We have also used that on a cone with opening angle $2\pi n$,

$$< T(z) >_n^{CFT} = \frac{c_{UV}}{24}(1 - \frac{1}{n^2})\frac{1}{z^2} \tag{20}$$

which can be easily derived by using the map, $z \rightarrow z^{\frac{1}{n}}$ , to the complex plane and using the transformation rule of the Stress tensor.

- This leads to the result,

$$\int_{cone} d^2x(< \Theta >_n - < \Theta >_1) = -\pi \frac{c_{UV}}{6}(n - \frac{1}{n}) \qquad (21)$$

- Integrated trace of the stress tensor is the generator of rigid scale transformation.
- Now Rigid scale transformation is equivalent to changing the mass parameter of the theory.
- So,

$$\frac{1}{2\pi} \int_{cone} d^2x(< \Theta >_n - < \Theta >_1) = m\frac{d}{dm}(F_n - nF_1) \qquad (22)$$

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$$F_n - nF_1 = -\frac{c_{UV}}{12}(n - \frac{1}{n}) \ ln(ma) \qquad (23)$$

- So using the definition of entanglement entropy we get,

$$S_{EE} = \frac{\partial}{\partial n}|_{n=1}(F_n - nF_1) = -\frac{c_{UV}}{6}\ ln(ma) = \frac{c_{UV}}{6}\ ln\frac{\xi}{a} \quad (24)$$

- For example a massive scalar filed in two dimensions will have $c_{UV} = 1$, $c_{IR} = 0$ and so EE given by, $-\frac{1}{6}ln(ma)$.
- This result is valid when $\xi << L$, where $L$ is the subsystem size. In general there are finite size corrections of the form $f(\frac{\xi}{L})$.
- This correction term cannot be calculated so simply.
- Unfortunately this method of proof does not work in higher dimension.
- So let us look for an alternative derivation which can be generalized to higher dimensions.

# Review of the Komargodski-Schwimmer method

- Consider a UV-CFT which has been deformed by a relevant operator.
- Our deformed field theory is not conformal but it can be made conformally invariant by coupling to a background dilaton field $\tau(x)$.
- The dilaton, $\tau$, couples to the deformed theory as,

$$S = S_{CFT}^{UV} + \int d^2x\sqrt{h}\, g(e^{\tau(x)}\mu)\mu^{2-\Delta}O_\Delta \tag{25}$$

  where $\mu$ is the renormalization scale. We have also introduced a background metric $h_{ab}$.
- This is conformally invariant if the metric and the background fields are transformed as,

$$h_{ab} \to e^{2\sigma}h_{ab},\ \ \tau(x) \to \tau(x) + \sigma \tag{26}$$

- There are two sources of breaking of the conformal invariance.
- One is an explicit breaking by the relevant operator and the other one is the conformal anomaly.
- By coupling to the background field $\tau(x)$ we got rid of the explicit breaking.
- So the remaining source of the conformal symmetry breaking is the conformal anomaly.
- Now the partition function $Z[h_{ab}, \tau(x)]$ is invariant along the RG flow, by construction.
- So its transformation property under the combined Weyl transformations of $h_{ab}$ and $\tau(x)$ should not depend on the scale.
- In particular this means that anomalous transformation property will be the same as the anomaly of the UV-CFT.

- The free energy after integrating out the dynamical fields is then given by,

$$F(h,\tau) = F_{dil}(h,\tau) + F_{IR-CFT}(h) \tag{27}$$

- Now according to our previous argument, under an infinitesimal Weyl transformation,

$$\delta_\sigma F(h,\tau) = \int \sqrt{h}\sigma(x) A_{UV-CFT}(h) = \delta_\sigma F_{dil}(h,\tau) + \int \sqrt{h}\sigma(x) A_{IR-CFT}(h) \tag{28}$$

where $A_{CFT} = - < T^\mu_\mu >_{CFT}$ is the conformal anomaly.

- So the dilaton effective action consists of two parts. One is the Weyl non-invariant universal term which is completely determined by the conformal anomaly matching between the UV and the IR.
- The other part is the Weyl invariant part of the effective action which can be written as a functional of the Weyl invariant combination $e^{-2\tau} h_{ab}$.

- To first order dilaton couples to the trace of the energy momentum tensor, $\sim \int \tau(x) T^{\mu}_{\mu}(x)$.
- $$g(e^{\tau(x)}\mu)\mu^{2-\Delta}O_{\Delta} = (g(\mu) + \tau(x)\mu\frac{dg}{d\mu})\mu^{2-\Delta}O_{\Delta} \quad (29)$$
$$= g(\mu)\mu^{2-\Delta}O_{\Delta} + \tau(x)\beta(g)\mu^{2-\Delta}O_{\Delta} = \tau(x)T^{\mu}_{\mu} \quad (30)$$
- So to compute the integrated trace we can couple to a constant dilaton field.
- We need to compute the dilaton effective action for a constant dilaton background field.

# Massive Scalar In Two Dimwnsions

- Let us consider a massive scalar field of mass $m$ in two dimensions described by the Euclidean action,

$$S = \frac{1}{2} \int ((\partial\phi)^2 + m^2\phi^2) \tag{31}$$

- We want to compute the entanglement entropy of a subsystem which want to keep arbitrary.
- It could be an infinite half-line or it could be an interval of finite length.
- One way to do this is to use the identity (Calabrese-Cardy, Casini),

$$\frac{\partial}{\partial m^2} lnZ_n = -\frac{1}{2} \int G_n(\vec{r}, \vec{r}) d^2\vec{r} \tag{32}$$

- $G_n(\vec{r}, \vec{r}\,')$ is the Green's function of the operator $(-\nabla^2 + m^2)$, on the singular space.

- Now instead of doing this one could also use the following identity,

$$m^2 \frac{\partial}{\partial m^2} lnZ_n = -\frac{1}{2}\frac{\partial}{\partial \tau}|_{\tau=0}\ lnZ_n(\tau) \qquad (33)$$

- $-lnZ_n(\tau)$, is the free energy computed on the cone for the theory defined by the euclidean action,

$$S(\tau) = \frac{1}{2}\int((\partial\phi)^2 + m^2 e^{-2\tau}\phi^2) \qquad (34)$$

- Now this is precisely the coupling of the dilaton to the massive theory.
- So we can interpret the number $\tau$ as a constant background dilaton field.
- This shows that we can calculate the entanglement entropy once we know the dilaton effective action on the cone.

# Universal Part In Two dimensions

- Naively one would expect that the trace of the energy-momentum of a CFT of central charge $c$ on the cone is given by

$$\int_{cone} \sqrt{h} < T^{\mu}_{\mu} >= \frac{c}{24\pi} \int_{cone} \sqrt{h} R(h) \tag{35}$$

- But the correct expression is given by (Cardy-Peschel, Nucl.Phys. B300 (1988) 377),

$$\int_{cone} \sqrt{h} < T^{\mu}_{\mu} >= \frac{c}{24\pi} \frac{1}{2}(1 + \frac{1}{n}) \int_{cone} \sqrt{h} R(h) \tag{36}$$

- *In particular this shows that if one uses a smoothed out cone one will generically get wrong answers.*
- *This is also true for other anomalies, in particular Gravitational Anomaly.*

- This measures the response of the 2-D CFT on the cone to a scale transformation.
- Using this and the anomaly matching condition gives us the universal (Weyl non-invariant) part of the dilaton effective action for a constant dilaton field to be,

$$F_{dil}(n,\tau) = -\frac{c_{UV}}{24\pi}\ \frac{1}{2}(1+\frac{1}{n})\ \tau \int_{cone} \sqrt{h}R(h) \qquad (37)$$

- So we get,

$$\int_{cone} < T^{\mu}_{\mu} >_{n,universal} = -\frac{c_{UV}}{24\pi}\,\frac{1}{2}(1+\frac{1}{n})\int_{cone}\sqrt{h}R(h) \quad (38)$$

- The non-universal contribution is purely bulk contribution in this case because there is no other length scale in the problem and hence cancelled in the combination $\int_{cone}(< T^{\mu}_{\mu} >_n - < T^{\mu}_{\mu} >_1)$.
- Hence we arrive at the Calabrese-Cardy result once we note that,

$$\int_{cone}\sqrt{h}R(h) = 4\pi(1-n) \quad (39)$$

- Now let $m$ denote the mass scale associated with the relevant operator.
- A scale transformation is equivalent to a change in the parameter $m$. (Calabrese-Cardy)
- So,

$$m\frac{d}{dm}S_{EE} = n\frac{\partial}{\partial n}|_{n=1}\;(\mu\frac{d}{dm}F(n) - n\mu\frac{d}{dm}F(1)) \tag{40}$$

- And,

$$\mu\frac{d}{dm}F = -\int\sqrt{h} < T^{\mu}_{\mu} > \tag{41}$$

- This gives us,

$$m\frac{d}{dm}S_{EE} = -\frac{c_{UV}}{6} \tag{42}$$

- This is precisely the Calabrese-Cardy answer,

$$S_{EE} = -\frac{c_{UV}}{6}ln(ma) \tag{43}$$

# Four Dimensions

- In four dimensions the trace anomaly of a conformal field theory is give by,

$$< T^{\mu}_{\mu} >= -\frac{c}{16\pi^2}\ W^2 + 2a\ E_4 \tag{44}$$

where

$$W^2 = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2 \tag{45}$$

and

$$E_4 = \frac{1}{32\pi^2}(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) \tag{46}$$

- So the the universal (Weyl non-invariant) part of the dilaton effective action for a constant dilaton filed is given by,

$$F(n,\tau) = -\tau \int_{cone} d^4x\sqrt{h}\ (\frac{c_{UV}}{16\pi^2}W^2 - 2a_{UV}E_4) \tag{47}$$

- This gives rise to a term which is universal,

$$S_{EE} \supset -n\frac{\partial}{\partial n}|_{n=1} \int_{cone} d^4x\sqrt{h}\,(\frac{c_{UV}}{16\pi^2}W^2 - 2a_{UV}E_4)\; ln(ma) \tag{48}$$

- This for of the expression is valid for any arbitrary shape of the entangling surface in four dimensions. This can be written in closed form in terms of the intrinsic and extrinsic geometric quantities of the entangling surface.
- For example if we take our entangling surface to be a sphere in flat space then this reduces to,

$$S_{EE} \supset 4a_{UV}ln(ma) \tag{49}$$

- For a cylindrical entangling surface of radius $R$ and length $L(L >> m^{-1})$ in flat space we get,

$$S_{EE} \supset \frac{L}{2R}c_{UV}ln(ma) \tag{50}$$

- In fact, this term always appears if you compute holographic entanglement entropy in RG-flow geometries.
- Our method extends this to any field theory and explains this as the consequence of trace-anomaly matching.

# Universal Contributions From The Weyl Invariant Part

- By universal contribution we mean logarithmically divergent part of the entanglement entropy.
- The coefficient of the UV-divergent term will be a local function of the various geometric quantities.
- The Weyl invariant part of the dilaton effective action can be expanded in terms of tensors built out of $\hat{h}_{ab} = e^{-2\tau} h_{ab}$. Let us arrange these terms in order of increasing mass dimensions of the integrand.
- The first term is

$$\int_{cone} d^4x \sqrt{\hat{h}} = e^{-4\tau} \int_{cone} d^4x \sqrt{h} \tag{51}$$

This term does not contribute to the entanglement entropy because the volume of the cone does not get any contribution from the tip.

- The second term is,

$$\int_{cone} d^4x\sqrt{\hat{h}}R(\hat{h}) = e^{-2\tau}\int_{cone} d^4x\sqrt{h}R(h) \qquad (52)$$

- This gives rise to a universal term of the form,

$$a_2\ m^2\ A_\Sigma ln(ma) \qquad (53)$$

  where $A_\Sigma$ is the area of the entangling surface.
- The coefficient $a_2$ cannot be calculated form any symmetry argument. One needs to do the Feynman graphs.

- The dimension four terms can be written as linear combinations of $R^2(\hat{h})$, $R^2_{ab}(\hat{h})$ and $R^2_{abcd}(\hat{h})$. So a general dimension four term in the dilaton effective action has the structure,

$$\int_{cone} d^4x\sqrt{\hat{h}}(AR^2(\hat{h}) + BR^2_{ab}(\hat{h}) + CR^2_{abcd}(\hat{h})) \qquad (54)$$

where $A$, $B$ and $C$ are dimensionless constants. Since this term is marginal, it does not couple to a constant dilaton and so does not contribute to the universal term.

# Scale versus conformal invariance

- One of things which we assumed was that the the fixed points are CFTs .
- In a scale invariant theory,

$$T^{\mu}_{\mu} = \partial_{\nu} V^{\nu} \tag{55}$$

  where $V^{\mu}$ is called the virial current.
- To first order the coupling of the dilaton to a scale but non-conformally invariant theory is then given by,

$$\int \tau(x) T^{\mu}_{\mu} = \int \tau(x) \partial_{\mu} V^{\mu} \tag{56}$$

- This clearly shows that a space-time independent dilaton cannot decouple from a scale-invariant theory.
- This also shows that a constant dilaton still decouples from a scale invariant theory and so our method extends also to scale-invariant but non-conformally invariant theories.

- In a four dimensional scale invariant theory the trace anomaly is given by,

$$< T^{\mu}_{\mu} >=< T^{\mu}_{\mu} >_{CFT} + e \; R^2 \tag{57}$$

where $e$ is another number like $a$ and $c$.

- In a CFT the additional term does not appear because it does not satisfy the Wess-Zumino consistency condition.
- So if the UV theory is only scale invariant then we get an extra term given by,

$$n \frac{\partial}{\partial n}|_{n=1} \; e_{UV} \int_{cone} d^4x \sqrt{h} \; R^2(h) \; ln(ma) \tag{58}$$

where $e$ is the "central charge" corresponding to the $R^2$ term.

- For a general entangling surface this can be written as,

$$S_{EE} \supset e_{UV} \int_{\Sigma} R \; ln(ma) \tag{59}$$

where $R$ is the space-time Ricci scalar.

# Comparison With The Known Results

- In four dimensions the universal part of the entanglement entropy was computed for any free field of mass $m$ and spin $s$, in terms of three unknown coefficients , (Phys.Rev.Lett. 106 (2011) 050404; Hertzberg-Wilczek, Phys.Rev. D88 (2013) 4, 044054; Fursaev, Patrushev, Solodukhin )

$$S_{EE} = \frac{1}{2\pi}\int_{\Sigma}(aR_{\Sigma} - bK_{\Sigma} - cR + \frac{1}{12}m^2 D_s)ln(ma) \qquad (60)$$

  Where $D_s$ is the dimension of spin-$s$ representation.

- The coefficients $a$, $b$ and $c$ cannot be computed by the standard approach for arbitrary entangling surface.
- Our method gives the values,

$$a = a_{UV},\ b = c_{UV},\ c = 0. \qquad (61)$$

  So we have the complete answer.

- The answer for any four dimensional field theory which is described in the UV by a CFT is given by,

$$S_{EE} = \frac{1}{2\pi} \int_{\Sigma} (a_{UV} R_{\Sigma} - c_{UV} K_{\Sigma} + \alpha m^2 D_s) ln(ma) \qquad (62)$$

- This also reproduces the AdS-CFT answer exactly. (JHEP 1108 (2011) 039; Hung-Myers-Smolkin)