Entropy current and equilibrium partition function in fluid dynamics

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Aim of the talk

In this talk we would analyse the relation between two apparently disjoint physical conditions that we expect a fluid system to satisfy

1. The existence of an equilibrium solution for fluid equations in every static background

   We shall call it the 'equilibrium condition'.

2. Local entropy production on every non-equilibrium solutions of fluid equations

   We shall refer to it as the 'entropy condition'

Plan of the talk

- A brief description of what we mean by 'fluid dynamics'
- An operational definition of the 'equilibrium condition' in terms of the form of fluid equations.
- Similarly an operational definition of the 'entropy condition'
- Statement of the relation we would like to show
- A sketch of the proof
- Conclusion and future direction

What is fluid dynamics

- Fluid dynamics is an effective long wavelength description.
- It has much fewer degrees of freedom than that of the underlying microscopic theory.
- All variables are slowly varying comapred to some length scale determined the by the microscopic dynamics.
- Therefore fluid dynamics is treated in an expansion in derivatives.

Variables and equations

- Variables
  - Velocity: u^μ
  - Temperature: T
  - Conserved Charges: $q_i, \quad i = \{0, 1, 2, 3, \cdots\}$
- Equations
  - Conservation of stress tensor: $\nabla_\mu T^{\mu\nu} = 0$
  - Conservation of other charge currents: $\nabla_\mu J^\mu = 0$
- Basic input
  - Constitutive relations:
    Expressions of $T^{\mu\nu}$ and J^μ in terms of basic variables.

Transport coefficients

- Constitutive relations are written in derivative expansion,

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \cdots , \quad J^{\mu} = J^{\mu}_{(0)} + J^{\mu}_{(1)} + \cdots$$

- $T^{\mu\nu}_{(i)}$ (or $J^{\mu}_{(i)}$) is the most general symmetric tensor (vector) constructed out of u^{μ}, T and Q and i number of ∇_{μ}'s.
- Any term in $T^{\mu\nu}_{(i)}$ (or $J^{\mu}_{(i)}$) should not vanish after imposing lower order EOM.

So each $T^{\mu\nu}_{(i)}$ and $J^{\mu}_{(i)}$ can be expressed as a sum of finitely many onshell independent terms with arbitrary coefficients.

These coefficients are called transport coefficients.

Constraints on fluid dynamics

- It is difficult or mostly impossible to calculate the constitutive relations from the microscopic theory.
  Constitutive relations are largely phenomenological.
- Therefore it is very useful to find out some universal structure for the constitutive relations, independent of any microscopic details.
- Constraints on fluid dynamics $\Rightarrow$ Constraints on constitutive relations from universal physical consideration

The 'Equilibrium condition' and the 'Entropy condition' are two such physical requirement.

'Equilibrium Condition' (in detail)

- In every time independent background with slow (but otherwise arbitrary) variation in spcae, the fluid equations will admit at least one time independent solution.
- The equilibrium stress tensor and the current could be generated from some $W \sim \log(\text{partition function})$.
- W is a functional of the background metric and the gauge field(s).

$$T^{\mu\nu}_{eq} = \hat{T}^{\mu\nu} \sim \frac{\delta W}{\delta G_{\mu\nu}}, \quad J^{\mu}_{eq} = \hat{J}^{\mu} \sim \frac{\delta W}{\delta A_{\mu}}$$

- W could also be expanded in terms of the space derivatives of the metric functions.

$$W = \sum_i W_i$$

'Entropy condition' (in detail)

- In a fluid system, entropy should always be produced locally during time evolution.

In other words

- There should exist one current S^μ, a local function of fluid variables, whose divergence is non-negative on every solutions of fluid equations.
- S^μ should also admit a derivative expansion.

$$S^\mu = \sum_i S_i^\mu$$

- This implies that for every fluid system we should be able to find at least one S^μ such that within the validity of derivative expansion

$$\nabla_\mu S^\mu|_{on-shell} = \sum_i K_i^2$$

where K_i are again some local functions of fluid variables .

Some observations:

Studying different relativistic fluid system upto few orders (mostly second) in derivative expansion, it has been observed

- Imposing 'equilibrium condition' always results in a set of relations ('equalities') between the transport coefficients.
- 'Entropy condition' also requires the same set of 'equalities' between the transport coefficients.
- 'Entropy condition' further imposes some 'inequalities'.
- Most of the 'inequalities' involve both transport coefficients as well as some unknown coefficients in the entropy current.
- But there are also few 'inequalities' which involve only a subset of first order trasnport coefficients and nothing else.

What we wanted to see:

If any (or all) of the above observations are true for a general fluid at an arbitrary order in derivative expansion.

What we found:

Starting from an arbitrary W (log of the partition function) we can always construct an entropy current with non- negative divergence provided

1. The 'equilibrium condition' is satisfied.
2. The first order transport coefficients satisfy certain inequalities

Some comments about entropy current

Unlike stress tensor or currents, entropy current is not a microscopically defined quantity.
However, entropy in equilibrium is something well-defined.

In this analysis any current will be considered as a valid candidate for entropy current provided

1. in equilibrium its divergence vanishes
2. integration of its time component over any space-like slice in equilibrium, gives the total entropy of that region.
3. away from equilibrium its on shell divergence is always positive.

'Entropy condition' demands the existence of at least one such current but there is no need for uniqueness.

The construction of S^μ: General strategy

The construction will have three parts:

1. Constraints on S^μ from equilibrium partition function $[\partial_0 \sim 0]$.
2. Constraints on S^μ from adiabatic time dependence $[(\partial_i)^n >> \partial_0 >> \partial_0^2]$
3. Final form of the full covariant entropy current with covariant time dependence $[\partial_i \sim \partial_0]$

In each step, the space derivatives are always treated in derivative expansion.

For simplicity we shall restrict ourselves to an uncharged fluid.

Step-1: Entropy current in equilibrium: S^{μ}_{eq}

We start with an equilibrium fluid system

- in a background with a time-like Killing vector ∂_0
- and at temperature $T_0 = \frac{1}{\beta} \sim$
- with a partition function $Z = e^{W}$

The 'equilibrium condition' in fluid limit implies

- $W = \int_{space} \mathcal{L}[T_0, G_{\mu\nu}], \qquad \mathcal{L} = \sum_a \mathcal{L}_a = \mathcal{L}_0 + \mathcal{L}_{pert}$
- Stress tensor in equilibrium : $T^{\mu\nu}_{eq} \sim \frac{\delta W}{\delta G_{\mu\nu}}$
- Total entropy in equilibrium : $\int_{space} \mathcal{S} = \int_{space} \frac{\partial(T_0 \mathcal{L})}{\partial T_0}$

Constraints on S^μ in equilibrium ($\equiv S^\mu_{eq}$)

- Zero component of the equilibrium entropy current when integrated over space should reproduce the total entropy.

$$\int_{space} S^0|_{eq} = \int_{space} \mathcal{S}$$

.

- The equilibrium entropy current should be exactly conserved.

These two conditions could be trivially solved.

- Identify

$$S^0|_{eq} = \mathcal{S}, \quad S^i|_{eq} = 0$$

- $\nabla_\mu S^\mu|_{eq} = \partial_0 S^0|_{eq} = 0$
  where $S^\mu_{eq} = S^\mu$ evaluated on equilibrium solution.

Fixes those terms in S^μ that do not vanish in equilibrium.

Entropy current and adiabatic variation

- Now consider a slow time dependence in the background such that $\mathcal{O}(\partial_0^2)$ or higher time derivatives could be neglected.
- ∂_0(Total Entropy) $\neq 0$
- But at order $\mathcal{O}(\partial_0)$ the change could have either sign and hence could violate second law.
- The change in total entropy must be compensated by some by entropy inflow or outflow through the boundary of the region upto order $\mathcal{O}(\partial_0^2)$.

It implies

$$\partial_0 \int_{space} \mathcal{S} = -\int_{boundary} \mathcal{S}^i + \mathcal{O}(\partial_0^2)$$

for some current $\mathcal{S}^i$ of order $\mathcal{O}(\partial_0)$

Adiabatic variation (contd)

- By construction $\mathcal{S}$ is again a local function of metric variables.
- The change in total entropy due to adiabatic variation could be written as

$$\partial_0 \int_{space} \mathcal{S} = \int_{space} \left[\frac{\delta \mathcal{S}}{\delta G_{\mu\nu}}\right] (\partial_0 G_{\mu\nu}) + \text{Boundary terms} + \mathcal{O}(\partial_0^2)$$

- Now $\partial_0 G_{\mu\nu}$ could have either sign.
- If we want entropy to be produced always, then the first term should vanish.
- It turns out that the first term does vanish, whenever 'the equilibrium condition' is satisfied.

  that is

  Whenever the equilibrium stress tensor is generated from the same partition function that we used to determine the total entropy.

Summary

In summary

- From 'entropy condition' it follows that the entropy current must be adiabatically conserved.

$$\nabla_\mu S^\mu|_{adiabatic} = \mathcal{O}(\partial_0^2)$$

- This is always possible if
  in equilibrium there exists a partition function from which we could generate both the stress tesnor and the total entropy of the system.
- We also found the form of the entropy current in the speical case when the background slowly varies with time.
- We simply identify

$$S^0|_{adiabatic} = \mathcal{S}, \qquad S^i|_{adiabatic} = \mathcal{S}^i$$

where $\mathcal{S}^i$ is the boundary term compensating the change of $\mathcal{S}$ in the bulk.

The general fluid background

Now we want to extend the construction to any general time-dependent background.

So our next steps:

1. Start from any entropy current S^{μ}_{ad} with the property that On adiabatic background it reduces to the appropriate current determined from partition function
2. Compute its divergence explicitly.
3. Add new terms (adibatically vanishing) to it in a particular way so that its divergence remains positive definite upto a given order in derivative expansion

The general fluid background(contd)

Our claim:

We could always modify S^{μ}_{ad} to some consistent entropy current provided some first order transport coefficients satisfy certain inequalities

In particular, we do not need any more ‘equality’ type relations other than the ones imposed by the ‘equilibrium condition’

Now we shall sketch a proof for our claim.

But before getting into the details, we need to introduce some notations and classification that we are going to use.

Some Notation and Classification

- Dissipative terms: local functions of fluid variables that vanish in equilibrium
- Dissipative terms could be classified according to the number of derivatives.
- At leading order we have
  1. one tensor $\sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle}$
  2. One vector $h^\mu = \nabla^\mu T + T(u.\nabla)u^\mu$
  3. Two scalars $(u.\partial T), \quad \Theta = \nabla.u$
- Denote this set of leading order dissipative terms as D.
- Higher order dissipative terms could be classified as
  1. H^{many}: Terms with more than one factors from D.
  2. H^{one}: Terms with exactly one factor from D
  3. H^{zero}: Terms with no factors from D, but only their derivatives.

Some useful facts

- By construction $\nabla_\mu S^\mu_{ad}$ will contain only the dissipative terms.
- It turns out that the divergence always takes the following form

$$\nabla_\mu S^\mu_{ad} = T^{\mu\nu}_{diss} \times \{ \text{ Elements of } D\} + \text{Higher order terms} \quad (0.1)$$

Here $T^{\mu\nu}_{diss}$ denotes those terms in the stress tensor that vanish in equilibrium.

- $T^{\mu\nu}_{diss}$ also admits an expansion in terms of derivatives.

$$\begin{aligned} T^{\mu\nu}_{diss} =& \text{First order transport coefficients} \times \text{Elements of } D \\ &+ \text{Higher order terms} \end{aligned} \quad (0.2)$$

- Substituting (0.2) in (0.3) we get

$$\nabla_\mu S^\mu_{ad} = \text{Quadratic form in elements of D} + \text{Higher order terms} \quad (0.3)$$

The sketch of the proof (upto second order)

$$\nabla_\mu S^\mu_{ad} = \text{A quadratic form in D } + \text{Higher order terms}$$

If we are interested only upto second order in derivative expansion

- we can ignore the higher order terms.
- we have to demand that the quadratic form is positive defnite.
- since the quadratic form involves only the first order dissipative transport coefficients,
  it imposes certain inequalities among those coefficients.

The sketch of the proof (higher order)

$$\nabla_\mu S^\mu_{ad} = \text{A quadratic form in D } + H^{zero} + H^{one} + H^{many}$$

1. H^{many}: Terms with more than one factors from D.
2. H^{one}: Terms with exactly one factor from D
3. H^{zero}: Terms with only derivatives of D.

- Note
  - H^{many} is always suppressed compared to the quadratic form. It cannot affect the sign of the divergence.
  - H^{zero} could always be re written as $H^{zero} \sim H^{one} + \nabla_\mu V^\mu$ for some V^μ
- Modify S^μ_{add} by adding V^μ, $S^\mu_{ad} \to S^\mu_{ad} + V^\mu$
- Now the divergence $\sim$ A quadratic form in D $+ H^{one}$

Here we neglected the H^{many}

- Now $\nabla_\mu S^\mu \sim$ A quadratic form in D $+ H^{one}$
- $H^{one} \sim D \times H^{zero}$

  D: First order dissipative term

  H^{zero}: Higher order dissipative term that does not contain any factors of D
- H^{one} could be absorbed in quadratic form by a shift in D

$$\nabla_\mu S^\mu \sim \text{A quadratic form in shifted D} + (H^{zero})^2$$

- The second term is again a term of H^{zero} type. We can again convert it to H^{one} by modifying the entropy current.
- Note at every step the required modification is of increasing order in derivative expansion.
- We keep repeating it till we reach the desired order of accuracy.

So finaly the divergence is the same quadratic form but in the some shifted D.

The sketch of the proof (higher order)

- In the end

$$\nabla_\mu S^\mu \sim \text{Qudratic form in some shifted } D$$

- Coefficients in the quadratic form involves only the first order dissipative transport coefficients.
- So to satisfy 'the entropy condition' it is only the first order dissipative transport coefficients that have to be constrained by 'inequalities'

Future directions

- We have seen that in a fluid system the existence of equilibrium and positivity of the first order dissipative transport coefficients are sufficient to ensure local entropy production in time-evolution.

- Now dissipative transport coefficients are related to the stability of equilibrium
- In the language of QFT, they are often related to the unitarity of the underlying quantum theory
- It would be nice to rederive the inequalities from these aspects.
  Existence and stability of equilibrium might turn out to be equivalent to local entropy production.
  It could also serve as a consistency check for the statement that only the first order ones are constrained by inequalities.

Future directions

- One final goal could be to have a formulation of fluid dynamics where all consistency conditions are automatically taken care of.
- But we already know that there exist some particular relations among linear transport coefficients (Onsagar relations) that could never follow from our present analysis.
- It would be nice to be able to interpret Onsagar relations also in our language.
- In particular, it would be very intersting to see whether Onsagar relations could be generalized to non-linear order.

Future directions

- It seems that the method we have used here for the proof (particularly the first two parts) has several superficial similarities with derivation of Wald entropy for higher derivative gravity theory.
- The α' expansion of higher derivative gravity theory could be treated on a similar footing as the derivative expansion of fluid dynamics.
- It would be nice to understand this analogy in a rigorous way.
- That might be a step towards a perturbative proof for some 'wald entropy increase theorem'.

Thank You