Entanglement thermodynamics for Lifshitz systems

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First law of thermodynamics

- Thermodynamics provides a useful tool to study a system when it is in thermal equilibrium.
- Physics may be described in terms of few macroscopic quantities such as energy(E), temperaure(T), pressure(P), entropy(S) and chemical potential(μ).
- The laws of thermodynamics describe how these quantities behave under various conditions.
- The first law of thermodynamics tells us how the entropy changes as one changes the energy, volume and particle number of the system.

$$TdS = dE + PdV - \mu dN$$

Thermodynamics and Entanglement Entropy

- There are several interesting phenomena which occur when the system is far from thermal equilibrium.
- The entanglement entropy may provide a useful quantity to study excited quantum systems which are far from thermal equilibrium.
- For a generic quantum system it is difficult to compute the entanglement entropy. Nevertheless, at least, for those quantum systems which have holographic descriptions, one may use the holographic entanglement entropy to explore the behavior of the system.
- Another quantity which can be always defined is the energy of the system.

Motivation

- Is there is any relation between the entanglement entropy of an excited state and its energy as in ordinary thermodynamics?
- Motivated by this, we would like to explore the analogue of the first law of thermodynamics for entanglement entropy of excited state.
- We concentrate on excited states of a non-relativistic Lifshitz system.
- The aim is to compute the entanglement entropy of an excited state of the Lifshitz system in four dimensions.
- Non-relativistic and non-conformal symmetry.

Lifshitz holography

- Gravity on an asymptotically Lifshitz space-time provides a holographic description for a strongly coupled quantum field theory near the critical point.
- The information of quantum state in the dual field theory is encoded in the bulk geometry. In particular the Lifshitz geometry is dual to the ground state of the dual field theory.
- Exciting the dual field theory from the ground state to an excited state holographically corresponds to modifying the bulk geometry from Lifshitz solution to a general asymptotically Lifshitz solution.
- Need to solve Einstein's equation with a massive vector field.

EOM for Lifshitz space-time

- The equations of motion for this theory are

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + \frac{1}{2} F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{8} F_{\lambda\rho} F^{\lambda\rho} g_{\mu\nu} + \frac{1}{2} m^2 A_\mu A_\nu$$

$$\nabla_\mu F^{\mu\nu} = m^2 A^\nu .$$

- If we choose $\Lambda = -\frac{1}{2}(z^2 + z + 4)$ and $m^2 = 2z$, this theory has the following solution.
- The asymptotically Lifshitz metric takes the following form

$$\begin{aligned} ds^2 &= -r^{2z} dt^2 + r^2 (dx^2 + dy^2) + \frac{dr^2}{r^2} \\ A &= \alpha r^z dt, \quad \alpha^2 = \frac{2(z-1)}{z}. \end{aligned}$$

where the scaling symmetry is realised as an isometry:
$t \to \lambda^z t, x^i \to \lambda x^i, r \to \lambda^{-1} r$.

Excited State

- We construct an excited state by perturbing the metric and the gauge field as follows:

$$
\begin{aligned}
ds^2 &= -r^{2z}(1+f(r))dt^2 + r^2(1+h_{xx}(r))dx^2 + r^2(1+h_{yy}(r))dy^2 \\
&+ \frac{dr^2}{r^2} + 2h_{tx}(r)\, dtdx + 2h_{ty}(r)\, dtdy + 2h_{xy}(r)\, dxdy \\
A &= \alpha r^z(1+[j(r)+\frac{1}{2}f(r)])\, dt
\end{aligned}
$$

Ross and Saremi, JHEP **0909** (2009) 009

- The information about the excited state is encoded in the functions $f(r), h_{\mu\nu}(r), j(r)$.
- As $r \to \infty$, $f(r)$, $h_{\mu\nu}(r), j(r) \to 0$.

Linearized solution

- $$h_{xx}(r) = k(r) + t_d(r) \;\; and \;\; h_{yy}(r) = k(r) - t_d(r)$$
- The linearized equation of motion can be solved and the normalizable solutions for $j(r), k(r), f(r)$ and $t_d(r)$ take the following form : (since the solutions for $z = 2$ and $z \neq 2$ are quite different we write them separately in the following)
- For $z = 2$

$$\begin{aligned} j(r) &= -\frac{c_1 + c_2 \ln r}{r^4}, \\ f(r) &= \frac{4c_1 - 5c_2 + 4c_2 \ln r}{12r^4}, \\ k(r) &= \frac{4c_1 + 5c_2 + 4c_2 \ln r}{24r^4}, \\ t_d(r) &= \frac{t_{d2}}{r^4} \end{aligned}$$

Linearized solution

- For $z \neq 2$

$$
\begin{aligned}
j(r) &= -\frac{(z+1)c_1}{(z-1)r^{z+2}} - \frac{(z+1)c_2}{(z-1)r^{\frac{1}{2}(z+2+\beta_z)}}, \\
f(r) &= 4\frac{1}{(z+2)}\frac{c_1}{r^{z+2}} + 2\frac{(5z-2-\beta_z)}{(z+2+\beta_z)}\frac{c_2}{r^{\frac{1}{2}(z+2+\beta_z)}}, \\
k(r) &= 2\frac{1}{(z+2)}\frac{c_1}{r^{z+2}} - 2\frac{(3z-4-\beta_z)}{(z+2+\beta_z)}\frac{c_2}{r^{\frac{1}{2}(z+2+\beta_z)}}, \\
t_d(r) &= \frac{t_{d2}}{r^{z+2}}
\end{aligned}
$$

where $\beta_z^2 = 9z^2 - 20z + 20 = (z+2)^2 + 8(z-1)(z-2)$.

- c_1, c_2 and t_{d2} are integration constants.

Holographic Computation of Entanglement Entropy

- We compute the shift in the holographic entanglement entropy of the excited state due to the metric perturbation

$$\Delta S_E = S_E - S_E{}^{GroundState}$$

- The entangling region is taken to be a strip with width ℓ given by

$$-\frac{\ell}{2} \leq x \leq \frac{\ell}{2}, \qquad 0 \leq y \leq L.$$

The holographic entanglement entropy may be computed by minimizing a surface embedded in the time slice of the bulk geometry and ending at $r = \infty$ with the boundary coinciding with the entangling region.

Ryu and Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006)

Holographic Entanglement Entropy

- Assume that the bulk extension of the surface to be parameterized by $x = x(r)$.
- By the standard procedure of minimizing the embedded manifold we get

$$\ell = \frac{2}{r_t} \int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \frac{2\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)\,r_t}$$

where r_t is the turning point.

- The shift in the holographic entanglement entropy due to the metric perturbation is given by

$$\Delta S_E = \frac{L}{8G_N} \int_{r_t}^\infty dr\, \frac{1}{\sqrt{1-\left(\frac{r_t}{r}\right)^4}} \left[h_{yy} + \left(\frac{r_t}{r}\right)^4 h_{xx}\right]$$

Holographic Entanglement Entropy

- For $z = 2$, the shift in the entanglement entropy is given by

$$\Delta S_E = \frac{\ell L \pi}{24 {r_t}^2} \frac{1}{16\pi G_4} \Big[\frac{32 c_1}{5} - \frac{48 t_{d2}}{5} + c_2 (\frac{352}{25} - \frac{8\pi}{5}) + \frac{32}{5} c_2 \ln r_t \Big]$$

$$\begin{aligned} \langle T_{00} \rangle &= \frac{1}{16\pi G_4} \frac{4 c_2}{3} \\ \langle T_{xx} \rangle &= \frac{1}{16\pi G_4} \Big(\frac{4 c_2}{3} + 4 t_{d2} \Big). \end{aligned}$$

The total energy, entanglement pressure and entanglement chemical potential can be defined as,

$$\begin{aligned} \Delta E &= \int_0^L \int_{-\ell/2}^{\ell/2} dy \, dx \, \langle T_{00} \rangle = L \, \ell \, \langle T_{00} \rangle \\ \Delta P_x &= \langle T_{xx} \rangle \\ \Delta \mu &= \frac{\frac{4}{3}(c_1 + c_2 \ln r)}{16\pi G_4} \end{aligned}$$

First Law of Entanglement Thermodynamics for $z = 2$

- Defining an entanglement temperature as

$$T_{\rm ent} = \frac{24 r_t^2}{\pi} \frac{25}{(324 - 30\pi)} = \frac{96\Gamma^2\left(\frac{3}{4}\right)}{\ell^2 \Gamma^2\left(\frac{1}{4}\right)} \frac{25}{(324 - 30\pi)}$$

we obtain the following relation:

$$\Delta E = T_{\rm ent} \Delta S + \frac{60}{(324 - 30\pi)} \Delta P_x V - \frac{90}{(324 - 30\pi)} \Delta \mu Q$$

where $V = L\ell$ is the volume of the entangling region and $Q = m^2 \alpha V$.

- We have identified $m^2\alpha$ as some charge density j^0 using the equation of motion.
- Due to its similarity with the first law of thermodynamics we would like to consider this expression as the first law of entanglement thermodynamics.

Holographic computation for $z \neq 2$

- For $z \neq 2$, the change in holographic entanglement entropy is given by

$$\Delta S_E = \frac{L\ell\Gamma(\frac{1}{4})\pi}{2\Gamma(\frac{3}{4}){r_t}^z} \frac{1}{16\pi G_4} \left[\frac{\Gamma(\frac{1+z}{4})}{\Gamma(\frac{3+z}{4})} \frac{1}{(z+3)} (2c_1 - t_{d2}) \right.$$
$$\left. + \quad {r_t}^{\frac{1}{2}(z+2-\beta)} \frac{\Gamma(\frac{z+\beta}{8})}{\Gamma(\frac{z+4+\beta}{8})} \frac{2(4+\beta-3z)}{4+z+\beta} c_2 \right]$$

The holographic stress energy tensor for $z \neq 2$ is given by

$$\langle T_{00} \rangle = \frac{1}{16\pi G_4} \frac{4(z-2)}{z} c_1$$
$$\langle T_{xx} \rangle = \frac{1}{16\pi G_4} [2(z-2)c_1 + (z+2)t_{d2}].$$

First law for $z \neq 2$

- $$T_{\rm ent} = \frac{2 r_t^z \Gamma\left(\frac{3}{4}\right)}{\pi \Gamma\left(\frac{1}{4}\right) A_1}$$

  we can rewrite as,

  $$\Delta E = T_{\rm ent}\Delta S + \frac{A_2}{A_1}\Delta P_x V - \frac{A_3}{A_1}\Delta \mu Q$$

  where

  $$A_1 = \frac{z^2}{(z+3)(z^2-4)}\frac{\Gamma\left(\frac{1+z}{4}\right)}{\Gamma\left(\frac{3+z}{4}\right)} - \frac{2}{(z-2)(4+z+\beta_z)}\frac{\Gamma\left(\frac{z+\beta_z}{8}\right)}{\Gamma\left(\frac{z+4+\beta_z}{8}\right)},$$

  $$A_2 = \frac{1}{(z+3)(z+2)}\frac{\Gamma\left(\frac{1+z}{4}\right)}{\Gamma\left(\frac{3+z}{4}\right)}, \quad A_3 = \frac{1}{2z(4+z+\beta_z)}\frac{\Gamma\left(\frac{z+\beta_z}{8}\right)}{\Gamma\left(\frac{z+4+\beta_z}{8}\right)}$$

  This is the modified first law of entanglement thermodynamics.

Relativistic limit

- For $z = 1$,

$$
\begin{gathered}
Q = 0 \\
A_1 = \sqrt{\pi}/6 \\
A_2 = \sqrt{\pi}/12 \\
A_2/A_1 = 1/2 \\
T_{\rm ent} = \tfrac{24\Gamma^2(3/4)}{\pi\ell\Gamma^2(1/4)} \ .
\end{gathered}
$$

- Reduces to the first law obtained before for the AdS case in four dimensions:

$$\Delta E = T_{\rm ent}\Delta S + \frac{1}{2}\Delta P_x V$$

Allahbakhshi *et. al.* JHEP **1308** (2013) 102

Conclusion

- We have holographically computed the change in entanglement entropy for the excitation of the ground state Lifshitz system.
- Unlike the relativistic system with conformal symmetry, the change in entanglement entropy for the Lifshitz system contains an additional term.
- Non-relativistic modification of first law of entanglement thermodynamics.

THANK YOU