Aspects of Extremal Surfaces in $(A)dS$

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- Gauge/string realizations of Lifshitz & hyperscaling violation, and Entanglement Entropy
- A lightlike limit of entanglement
- dS/CFT at uniform energy density and a de Sitter bluewall
- Speculations on de Sitter extremal surfaces

Based on: arXiv:1408.7021, KN, (also 1212.4328, KN, Tadashi Takayanagi, Sandip Trivedi),

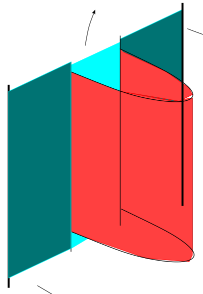
1312.1625, Sumit Das, Diptarka Das, KN; to appear.

Introduction

Over the years, we have seen many explorations and generalizations of AdS/CFT: $e.g.$ to nonrelativistic systems (holographic condensed matter), time-dependent systems, cosmology, $\ldots$

$\rightarrow$ geometric handle on physical observables.

A striking example is entanglement entropy : entropy of reduced density matrix of subsystem. Ryu-Takayanagi bulk prescription for EE: area of minimal surface in gravity dual.



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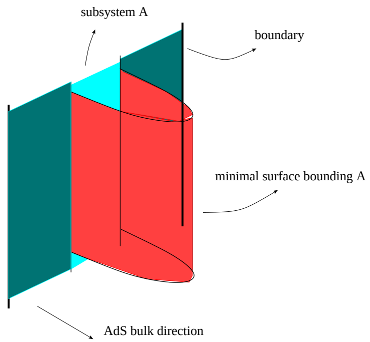

[Operationally: (i) define a spatial subsystem on the boundary, (ii) consider corresponding constant time slice in the bulk $(d+1)$-dim geometry, and a surface bounding the subsystem and dipping into the bulk, (iii) extremize area functional $\rightarrow$ minimal area (in Planck units).]

Non-static situations: extremal surfaces. (Hubeny, Rangamani, Takayanagi)

Leading EE scaling: d-dim area law $\frac{V_{d-2}}{\epsilon^{d-2}}$. (Bombelli, Koul, Lee, Sorkin; Srednicki)

Nonrelativistic Holography

Generalizations of AdS/CFT with reduced symmetries.

Lifshitz spacetime: $ds_4^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}$. (Kachru,Liu,Mulligan; Taylor)

t, x_i-translations, x_i-rotations, scaling $t \to \lambda^z t,\ x_i \to \lambda x_i$. [dynamical exponent z]

(smaller than Schrodinger symmetries: $e.g.$ Galilean boosts broken)

$[z = 1 : AdS]$ 4-dim gravity, $\Lambda < 0$, and massive gauge field $A \sim \frac{dt}{r^z}$

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More general gravity phases: $ds^2 = r^{2\theta/d_i}\Big(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}\Big)$.

θ = hyperscaling violation exponent; d_i = boundary spatial dim (x_i).

[Conformally Lifshitz. Effective Einstein-Maxwell-Dilaton theories (Trivedi et al; Kiritsis et al, $\ldots$)

$S \sim T^{(d_i-\theta)/z}$. Thermodynamics $\sim$ space dim $d_{eff} = d_i - \theta$: actual space is d_i-dim.]

$\theta = d_i - 1$: entanglement entropy $\sim$ $\log l$, logarithmic behaviour.

Gravity duals of Fermi surfaces? (Ogawa,Takayanagi,Ugajin; Huijse,Sachdev,Swingle)

Aspects of hyperscaling violating holography (Dong,Harrison,Kachru,Torroba,Wang)

$d_i - 1 \leq \theta < d_i$: entanglement entropy shows area law violations.

[Energy conditions: $(d_i - \theta)(d_i(z-1) - \theta) \geq 0, \quad (z-1)(d + z - \theta) \geq 0$.]

Lif/h.v., gauge/string realizations

Various string constructions involve x^+-dimensional reduction of

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2 g_{++}(dx^+)^2 + R^2 d\Omega_S^2$$

where $g_{++} > 0$. In lower dim'nal theory, time is $t \equiv x^-$.

(i) $z = 2$ Lifshitz (Balasubramanian,KN; Donos,Gauntlett; ...):

$$AdS \;+\; g_{++}[\sim r^0] \quad \xrightarrow{x^+-\mathrm{dim.redn.}} \quad z = 2 \text{ Lifshitz.}$$

x^+-reduction of non-normalizable null deformations of $AdS \times X$.

g_{++} sourced by lightlike matter, *e.g.* $g_{++} \sim (\partial_+ c_0)^2$ with lightlike axion $c_0 = Kx^+$:

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + K^2R^2(dx^+)^2 \;\longrightarrow\; ds^2 = -\frac{dt^2}{r^4} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}.$$

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(ii) Hyperscaling violation: AdS_{d+1} plane waves (KN)

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2}(dx^+)^2 \quad \longrightarrow$$

$$ds^2 = r^{\frac{2\theta}{d_i}}\left(-\frac{dt^2}{r^{2z}} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}\right), \quad z = \frac{d-2}{2} + 2, \quad \theta = \frac{d-2}{2}\,, \quad d_i = d-2.$$

Normalizable g_{++} mode $\;\Rightarrow\;$ dual CFT excited state, energy-momentum density $T_{++} = Q$.
Large boost, low temperature limit (Singh) of boosted black branes (Maldacena,Martelli,Tachikawa).

AdS_5 plane wave: $d = 4,\; d_i = 2,\; \theta = 1,\; z = 3$. Logarithmic behaviour of EE.

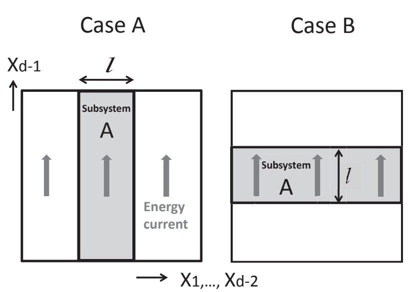
Entanglement, AdS plane waves

AdS_{d+1} plane wave: $ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2} (dx^+)^2$

EE here matches EE in hyperscaling violating lower dim'nal theory if upstairs theory entangling surface lies on $x^- = const$ surface (which is $t = const$ below). Null EE?

EE, spacelike strips (width l, $\Delta x^+ > 0 > \Delta x^-$). (KN, Takayanagi, Trivedi)

Non-static spacetime → use covariant HEE (Hubeny,Rangamani,Takayanagi).
(stationary point of area functional; if several extremal surfaces exist, choose minimal area).



Spacelike subsystem, UV cutoff ϵ:
leading divergence is area law $\sim \frac{V_{d-2}}{\epsilon^{d-2}}$

Case A: width direction x_i. Strip along energy flux.

Finite EE $\pm \frac{R^{d-1}}{G_{d+1}} V_{d-2} \sqrt{Q} l^{2-d/2}$.

$N^2 V_2 \sqrt{Q} \log(l Q^{1/4})$ [d=4]: less than $N^2 T^3 V_2 l$

(thermal entropy), larger than $-N^2 \frac{V_2}{l^2}$ (ground state).

Case B: Strip ⊥ flux.

Phase transition (no connected surface if $\Delta x^+ > 0 > \Delta x^-$).

S_A saturated for $l \gtrsim Q^{-1/4}$.

AdS_{d+1} plane waves, EE

Uniformize notation with nonconformal case: redefine $Q \to Q\frac{G_{d+1}}{R^{d-1}}$.

$$[Q \to \frac{Q}{N^2}\ \ (D3),\ \ Q \to \frac{Q}{N^{3/2}}\ \ (M2),\ \ Q \to \frac{Q}{N^3}\ \ (M5)]$$

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + \frac{G_{d+1}Q}{R^{d-3}}r^{d-2}(dx^+)^2 + R^2 d\Omega^2$$

Plane wave excited states: EE^{finite} (strip along flux direction):

$$\pm\sqrt{Q}V_{d-2}l^{2-\frac{d}{2}}\ \sqrt{\frac{R^{d-1}}{G_{d+1}}} \qquad [+:\ d<4,\quad -:\ d>4];$$

$\sqrt{Q}V_2N\ \log(lQ^{1/4})$ (D3), $\sqrt{Q}L\sqrt{l}\,\sqrt{N^{3/2}}$ (M2), $-\sqrt{Q}\frac{V_4}{l}\,\sqrt{N^3}$ (M5).

3d, 4d: finite entanglement grows with width l (large for fixed cutoff).

[spacelike strip subsystem: leading divergence is area law, $\frac{V_2}{\epsilon^2}$ $(4d)$, $\frac{V_1}{\epsilon}$ $(3d)$]

[Strip $\perp$ flux: phase transition.]

[EE^{fin} scaling estimates $\leftarrow$ approximate r_*, S^{fin} for large Q, l from EE area functional]

[Ground state EE: $S_A \sim \frac{R^{d-1}}{G_{d+1}}\big(\frac{V_{d-2}}{\epsilon^{d-2}} - c_d\frac{V_{d-2}}{l^{d-2}}\big)$]

[Temperature parameters: $r_0^4 \sim G_{10}\varepsilon_4\ (D3),\ \ r_0^6 \sim G_{11}\varepsilon_3\ (M2),\ \ r_0^3 \sim G_{11}\varepsilon_6\ (M5)$

$\lambda \to \infty,\ \varepsilon_{p+1} \to 0,$ with $\frac{\lambda^2\varepsilon_{p+1}}{2} \equiv Q = \text{fixed}.$ Boundary $T_{++} = Q$.]

$[G_5 \sim G_{10}R_{D3}^5, G_{4,7} \sim G_{11}R_{M2,M5}^{7,4},$ with $R_{D3}^4 \sim g_sNl_s^4, R_{M2}^6 \sim Nl_P^6, R_{M5}^3 \sim Nl_P^3]$

Nonconformal brane plane waves

(KN) (Recall Dp-brane phases, Itzhaki,Maldacena,Sonnenschein,Yankielowicz)

$$ds^2_{st} = \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}} dx^2_{\parallel} + \frac{G_{10}Q_p}{R_p^{(7-p)/2}} \frac{(dx^+)^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} \frac{dr^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} r^{(p-3)/2} d\Omega^2_{8-p}$$

$$e^{\Phi} = g_s \left(\frac{R_p^{7-p}}{r^{7-p}}\right)^{\frac{3-p}{4}}, \quad g^2_{YM} \sim g_s \alpha'^{(p-3)/2}, \quad R_p^{7-p} \sim g^2_{YM} N \alpha'^{5-p} \sim g_s N \alpha'^{(7-p)/2}.$$

[g_{++}-deformation obtained from double scaling limit of boosted black Dp-branes
$r_0^{7-p} = (U_0\alpha')^{7-p} \sim G_{10}\varepsilon_{p+1}; \quad \lambda \to \infty, r_0 \to 0$, with $\frac{\lambda^2 \varepsilon_{p+1}}{2} \equiv Q_p$ fixed.]

Strongly coupled Yang-Mills theories with constant energy flux T_{++}.

Dimensionally reducing on S^{8-p} and x^+, Einstein metric $ds^2_E = e^{-\Phi/2} ds^2_{st}$ gives hyperscaling violating metrics with $\theta = \frac{p^2-6p+7}{p-5}, \;\; z = \frac{2(p-6)}{p-5}$ (Singh).

D-brane plane waves, EE

$$ds_{st}^2 = \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}} dx_\parallel^2 + \frac{G_{10} Q_p}{R_p^{(7-p)/2}} \frac{(dx^+)^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} \frac{dr^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} r^{(p-3)/2} d\Omega_{8-p}^2$$

$$e^\Phi = g_s \left(\frac{R_p^{7-p}}{r^{7-p}}\right)^{\frac{3-p}{4}}, \quad g_{YM}^2 \sim g_s \alpha'^{(p-3)/2}, \quad R_p^{7-p} \sim g_{YM}^2 N \alpha'^{5-p} \sim g_s N \alpha'^{(7-p)/2}.$$

[$r_0^{7-p} = (U_0 \alpha')^{7-p} \sim G_{10} \varepsilon_{p+1}; \quad \lambda \to \infty, r_0 \to 0$, with $\frac{\lambda^2 \varepsilon_{p+1}}{2} \equiv Q_p$ fixed.

Strongly coupled Yang-Mills theories with constant energy flux T_{++}.]

Ground state: Ryu-Takayanagi, Barbon-Fuertes

$$S_A = N_{eff}(\epsilon) \frac{V_{d-2}}{\epsilon^{d-2}} - c_d N_{eff}(l) \frac{V_{d-2}}{l^{d-2}}, \qquad N_{eff}(\epsilon) = N^2 \left(\frac{g_{YM}^2 N}{\epsilon^{p-3}}\right)^{\frac{p-3}{5-p}}$$

Plane wave excited states (KN): leading divergence as above.

Scaling estimates from entanglement entropy area functional:

$$l \sim \frac{R_p^{\frac{7-p}{2}}}{r_*^{\frac{5-p}{2}}}, \qquad S_A^{finite} \sim \frac{V_{p-1}\sqrt{Q}}{(3-p)\sqrt{G_{10}}} \frac{R_p^{7-p}}{r_*^{(3-p)/2}} \qquad \text{(strip along flux)}$$

EE^{finite}: $\displaystyle \frac{1}{3-p} \frac{V_{p-1}\sqrt{Q}}{l^{(p-3)/2}} N \left(\frac{g_{YM}^2 N}{l^{p-3}}\right)^{\frac{p-3}{2(5-p)}} = \frac{\sqrt{N_{eff}(l)}}{3-p} \frac{V_{p-1}\sqrt{Q}}{l^{(p-3)/2}}$

[involves dimensionless combination $\frac{V_{p-1}\sqrt{Q}}{l^{(p-3)/2}}$ and $N_{eff}(l)$]

D-brane plane waves, EE

Plane wave excited states: leading divergence $N_{eff}(\epsilon)\frac{V_{d-2}}{\epsilon^{d-2}}$ as for ground states (area law).

EE^{finite}: $\frac{\sqrt{N_{eff}(l)}}{3-p}\ \frac{V_{p-1}\sqrt{Q}}{l^{(p-3)/2}}, \qquad N_{eff}(l) = N^2\left(\frac{g_{YM}^2 N}{l^{p-3}}\right)^{\frac{p-3}{5-p}}$

D2-M2: $V_1\sqrt{l}\sqrt{Q}\ \sqrt{\frac{N^2}{(g_{YM}^2 Nl)^{1/3}}}\ (D2); \qquad V_1\sqrt{l}\sqrt{Q}\ \sqrt{N^{3/2}}\ (M2).$

IIA regime of validity (IMSY) for turning point r_* gives $1 \ll g_{YM}^2 N l_{D2} \ll N^{6/5}$, and so $N^{3/2} \ll \frac{N^2}{(g_{YM}^2 Nl)^{1/3}} \ll N^2$. Thus $S_A^{D2,sugra}$ betw free 3d SYM (UV) and M2 (IR) (RG-consistent).

D4-M5: $-\frac{V_3\sqrt{Q}}{\sqrt{l}}\ \sqrt{N^2\frac{g_{YM}^2 N}{l}}\ (D4); \quad -\sqrt{Q}\frac{V_4}{l}\ \sqrt{N^3}\ (M5).$

The finite parts for D4-sugra and M5-phases are actually same expression: D4 is wrapped M5 ($R_{11} = g_s l_s = g_{YM}^2$) and $V_4 = V_3 R_{11},\ Q_{D4} = Q_{M5} R_{11}$. IIA: $1 \ll \frac{g_{YM}^2 N}{l} \ll N^{2/3}$.

D1: $l\sqrt{Q}\ \sqrt{\frac{N^2}{(g_{YM}^2 N l^2)^{1/2}}}$

Strip orthogonal to flux: indications of phase transitions, constrained however by IIA regime of validity.

Mutual Information

MI (disjoint subsystems A & B): $I[A,B] = S[A] + S[B] - S[A\cup B]$.

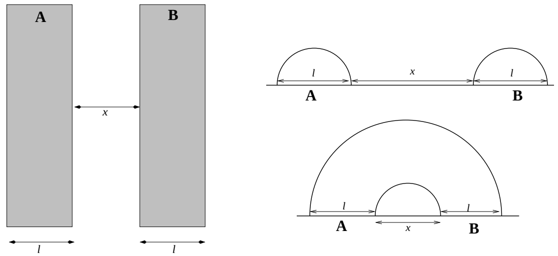
$I[A,B] \geq 0$. Cutoff-dependent divergences cancel. Gives bound for correlation fns.

Holographic mutual information: find extremal surface for $A \cup B$.

Subsystems far, two disjoint minimal surfaces: $MI = 0$.

Subsystems nearby, connected surface has lower area.

Ryu-Takayanagi $\rightarrow$ MI disentangling transition (Headrick).



Similar disentanglement for thermal states (Fischler,Kundu,Kundu): $\frac{x_c}{l} \sim 0$ (for $x, l \gg \frac{1}{T}$).

(Mukherjee, KN) MI for AdS plane wave excited states $\rightarrow$
critical separation $\frac{x_c}{l}$ between subsystems smaller than in ground state (pure AdS): $e.g.\ \frac{x_c}{l} \simeq 0.732$ (AdS_5) whereas $\frac{x_c}{l} \simeq 0.414$ (AdS_5 plane wave). [For wide strips ($Ql^d \gg 1$), critical $\frac{x_c}{l}$ independent of flux Q.]

[Narrow strips $Ql^d \ll 1$: perturbative corrections ΔS ($\sim$ EE thermodynamics) $\rightarrow$ MI decreases.]

Mutual information disentangling occurs faster.

Suggests energy density disorders system.

Lightlike limit of entanglement

(KN)

AdS_{d+1} null deformation: $ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 g_{++}(dx^+)^2$

Recall EE here matches EE in lower dim'nal theory if entangling surface in upstairs theory lies on $x^- = const$ surface (which is $t = const$ below).

Subsystem: $x^+ = \alpha\chi, \quad x^- = -\beta\chi, \quad -\frac{l}{2} < x \le \frac{l}{2}, \quad -\infty < \chi, y_i < \infty.$

$$S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{2\alpha\beta + \alpha^2 g_{++} r^2}\, \sqrt{1 + (\partial_r x)^2} \qquad [V_{d-2} = \int (\prod_{i=1}^{d-3} dy_i) d\chi]$$

$$\rightarrow \quad S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2\alpha\beta + \alpha^2 g_{++} r^2}{\sqrt{2\alpha\beta + \alpha^2 g_{++} r^2 - A^2 r^{2d-2}}}$$ (and width $l \sim r_*$).

[Spacelike strip $\alpha = \beta = 1 \rightarrow \frac{x^+ + x^-}{\sqrt{2}} \equiv t = const$ surface. $S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$, area law.]

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EE, null time x^- slices ($\beta = 0$)
$$\boxed{S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{g_{++} r^2}{\sqrt{g_{++} r^2 - A^2 r^{2d-2}}}.}$$

Lightlike limit of EE $\equiv$ highly boosted limit of EE for spacelike strips: boost $x^\pm \rightarrow \lambda^{\pm 1} x^\pm \Rightarrow \alpha = \lambda$ and $\beta = \frac{1}{\lambda} \rightarrow 0 \rightarrow$

$$S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2 + \lambda^2 g_{++} r^2}{\sqrt{2 + \lambda^2 g_{++} r^2 - A^2 r^{2d-2}}}.$$

In regime $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1 \quad \rightarrow \quad$ EE on null time x^- slices ($\beta = 0$).

Lightlike limit of entanglement

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[Spacelike strip $\alpha = \beta = 1 \to \frac{x^+ + x^-}{\sqrt{2}} \equiv t = const$ surface. $S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$, area law.]

Lightlike limit of EE ($\equiv$ highly boosted limit of spacelike EE: $x^\pm \to \lambda^{\pm 1} x^\pm$)

$$\xrightarrow{large\ \lambda} \qquad \boxed{S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{\lambda^2 g_{++} r^2}{\sqrt{\lambda^2 g_{++} r^2 - A^2 r^{2d-2}}}}$$

$i.e.$ EE on null time x^- slices in regime $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1$.

Subsystem stretched on x^+-plane: $x^- = 0, \ -\frac{l}{2} < x \le \frac{l}{2}, \ -\infty < x^+, y_i < \infty.$

Note: $\boxed{\text{leading divergence milder,} \quad S^{div} \sim \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \frac{\sqrt{\lambda^2 g_{++}(\epsilon)}}{\epsilon^{d-3}}}$.

Note also: $g_{++} = 0 \Rightarrow$ lightlike EE (on x^- slices) vanishes.

[Similar structure for boosted black branes, nonconformal brane plane waves etc.]

Null EE, AdS_{d+1} plane waves

$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2}(dx^+)^2$, dual to CFT excited states, energy-momentum density $T_{++} \sim Q$: spacelike EE gives area law, $S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$.

EE on null time x^- slices if $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1$, $i.e.$ $\lambda^2 Q\epsilon^d \gtrsim 1$, $i.e.$ elemental lightcone momentum $P_+ = T_{++}\Delta x^+ \Delta^{d-2}x|_\epsilon$ after boost is comparable to UV cutoff $\frac{1}{\epsilon}$.

In bulk: UV surface $r = \epsilon$ dips in sufficiently to feel g_{++} presence.

$$S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{\lambda^2 g_{++} r^2}{\sqrt{\lambda^2 g_{++} r^2 - A^2 r^{2d-2}}} \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2}\sqrt{\lambda^2 Q}}{d-4} \left(\frac{1}{\epsilon^{\frac{d}{2}-2}} - c_d \frac{1}{l^{\frac{d}{2}-2}}\right)$$

Milder leading divergence $S^{div} \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$ $\quad [d_{eff} = d - 1 - \theta = \frac{d}{2}]$

[Resembles spacelike EE in hyperscaling violating theory ($\theta = \frac{d-2}{2}$) from x^+-red'n.]

$g_{++} = 0$ (ground state) $\Rightarrow$ lightlike EE (on x^- slices) vanishes.

Null EE, AdS_{d+1} plane waves

$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2}(dx^+)^2$, dual to CFT excited states, energy-momentum density $T_{++} \sim Q$: spacelike EE gives area law, $S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$.

EE on null time x^- slices if $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1$, $i.e.$ $\lambda^2 Q\epsilon^d \gtrsim 1$, $i.e.$ elemental lightcone momentum $P_+ = T_{++}\Delta x^+ \Delta^{d-2}x|_\epsilon$ after boost is comparable to UV cutoff $\frac{1}{\epsilon}$.

In bulk: UV surface $r=\epsilon$ dips in sufficiently to feel g_{++} presence.

$$S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}}\int_\epsilon^{r_*}\frac{dr}{r^{d-1}}\frac{\lambda^2 g_{++}r^2}{\sqrt{\lambda^2 g_{++}r^2 - A^2 r^{2d-2}}} \sim \frac{R^{d-1}}{4G_{d+1}}\frac{V_{d-2}\sqrt{\lambda^2 Q}}{d-4}\left(\frac{1}{\epsilon^{\frac{d}{2}-2}} - c_d\frac{1}{l^{\frac{d}{2}-2}}\right)$$

Milder leading divergence $S^{div} \sim \frac{R^{d-1}}{4G_{d+1}}\frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$ $[d_{eff} = d-1-\theta = \frac{d}{2}]$

[Resembles spacelike EE in hyperscaling violating theory ($\theta = \frac{d-2}{2}$) from x^+-red'n.]

$g_{++}=0$ (ground state) $\Rightarrow$ lightlike EE (on x^- slices) vanishes.

Reminiscent of ultralocality in lightcone QFT (Wall).

Ground state: n-pt functions (fields at distinct locations) vanish. Suggests vanishing EE.

Excited states, $P_+ \neq 0$: can show free-field correlators non-vanishing. Suggests EE nonzero.

Boundary space: $ds^2 = -2dx^+dx^- + g^2(dx^+)^2 + \sum_{i=1}^{d-2} dx_i^2$, with $g^2 = T_{++}\epsilon^d \gtrsim 1$.

Usual area law $S_{div} \sim N^2\frac{V_{x^+}V_{y_i}}{\epsilon^{d-2}} = N^2 V_{d-2}\frac{\sqrt{T_{++}\epsilon^d}}{\epsilon^{d-2}} = N^2\sqrt{Q}\frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$.

Null EE, AdS_{d+1}-Lifshitz

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2K^2(dx^+)^2,$$

AdS_{d+1} deformed by non-normalizable Lifshitz deformation, at scale K.

Lower dim Lifshitz theory arises on lengthscales $\gg \frac{1}{K}$.

Subsystem: $x^+ = \alpha\chi, \quad x^- = -\beta\chi, \quad -\frac{l}{2} < x \leq \frac{l}{2}, \quad -\infty < \chi, y_i < \infty.$

$\rightarrow \quad S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2\alpha\beta+\alpha^2K^2r^2}{\sqrt{2\alpha\beta+\alpha^2K^2r^2-A^2r^{2d-2}}}$ (and width $l \sim r_*$).

$S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$, d-dim area law if $\epsilon \ll \frac{1}{K}$ — in this case UV is original CFT_d dual to AdS_{d+1}.

EE, null time x^- slices ($\beta = 0$) $\quad S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{K^2r^2}{\sqrt{K^2r^2-A^2r^{2d-2}}}.$

For larger lengthscales, $i.e.$ $\epsilon \gtrsim \frac{1}{K}$, this is approximated as a lightlike limit (EE on null x^- slices) $\rightarrow$ we see Lifshitz structure: $S^{div} \sim \frac{V_{d-2}K}{\epsilon^{d-3}}$.

Compactify $\rightarrow V_{d-2} = \int(\prod_{i=1}^{d-3} dy_i)d\chi = V_{d-3}V_+$ and $V_+ \sim \frac{1}{K}$ $\rightarrow$ lower dim area law.

Compared with AdS: $S^{fin} \sim \frac{V_{d-2}K}{l^{d-3}}$ larger for $l \gg \frac{1}{K}$ $\rightarrow$ perhaps reflection of more soft modes in Lifshitz theory (recall Lifshitz singularities (Horowitz,Way)).

Summary so far

- Some gauge/string realizations of Lifshitz & hyperscaling violation involve x^+-reduction of AdS deformations with g_{++}.
Lower dim'nal time is x^-.
Entanglement in lower dim'nal theory arises on null time x^- slices upstairs $\rightarrow$ lightlike limit of entanglement entropy.
Better understanding in lightcone QFT, ultralocality, ...

de Sitter space and dS/CFT

de Sitter space: $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$.

Fascinating for various reasons. dS/CFT: fluctuations about dS encoded thro dual Euclidean non-unitary CFT on $e.g.$ boundary at future timelike infinity $\mathcal{I}^+$ (Witten; Strominger; Maldacena, '01-'02).

Still mysterious, but perhaps interesting to explore.

(Maldacena '02) analytic continuation $r \to -i\tau,\ R_{AdS} \to -iR_{dS}$ from Eucl $AdS_4 \to$ Hartle-Hawking wavefunction of the universe $\Psi[\varphi] = Z_{CFT}$.

[Bulk EAdS regularity conditions, deep interior $\to$ Bunch-Davies initial conditions in deSitter, $\varphi_k(\tau) \sim e^{ik\tau}$ for large $|\tau|$).] $[Z_{CFT} = \Psi[\varphi] \sim e^{iI_{cl}[\varphi]}$ (semiclassical)].

[Bulk expectation values $\langle f_1 f_2' \rangle \sim \int D\varphi f_1 f_2' |\Psi|^2$.]

Wavefunction $\Psi[\varphi]$ not pure phase $\to$

complex saddle points contribute to observables.

[Operationally, dS/CFT usefully defined by analytic continuation.]

dS/CFT at uniform energy density

(Sumit Das, Diptarka Das, KN)

$$ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}\Big(- \frac{d\tau^2}{1+\alpha\tau_0^d\tau^d} + (1+\alpha\tau_0^d\tau^d)dw^2 + \sum_{i=1}^{d-1} dx_i^2\Big),$$

α is a complex phase and τ_0 is some real parameter of dimension length inverse.

Analog of regularity in the interior for asymptotically AdS solution:
Wick rotate $\tau = il$ and demand that resulting spacetime (thought of as saddle point in path integral) in the interior approaches flat Euclidean space in the (l, w)-plane with no conical singularity $\Rightarrow$ w-coordinate angular with fixed periodicity, l is radial coordinate, with

$$\alpha = -(-i)^d,\ l \geq \tau_0,\ w \simeq w + \frac{4\pi}{(d-1)\tau_0}.$$

$\Rightarrow$ *complex* metric, solving dS gravity $R_{MN} = \frac{d}{R^2_{dS}} g_{MN}$.

This is equivalent to analytic continuation $r \to -i\tau,\ R_{AdS} \to -iR_{dS}$ from $EAdS$ black brane $ds^2 = \frac{R^2_{AdS}}{r^2}\Big(\frac{dr^2}{1-r_0^d r^d} + (1-r_0^d r^d)d\theta^2 + \sum_{i=1}^{d-1} dx_i^2\Big)$.

dS/CFT at uniform energy density

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α is a complex phase and τ_0 real parameter of dimension energy, solves $R_{MN} = \frac{d}{R^2_{dS}} g_{MN}$.

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“Normalizable” metric modes $\Rightarrow$ energy-momentum tensor vev.

$$T_{ij} = \frac{2}{\sqrt{h}}\frac{\delta Z_{CFT}}{\delta h^{ij}} = \frac{2}{\sqrt{h}}\frac{\delta\Psi}{\delta h^{ij}} \propto i\frac{R^{d-1}_{dS}}{G_{d+1}} g^{(d)}_{ij} \quad \to dS \text{ black brane.}$$

[$g^{(d)}_{ij}$ = coefficient of τ^{d-2} in Fefferman-Graham expn]. [dS/CFT: $Z_{CFT} = \Psi$].

Note i arising from the wavefunction of the universe $\Psi \sim e^{iI_{cl}}$

$\Rightarrow$ energy-momentum real only if $g^{(d)}_{ij}$ pure imaginary.

dS_4/CFT_3: $\alpha = -i,\quad T_{ww} = -\frac{R^2_{dS}}{G_4}\tau_0^3$ with $T_{ww} + (d-1)T_{ii} = 0$.

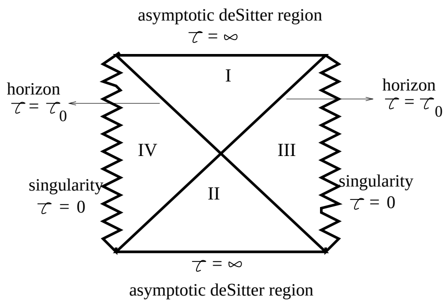
de Sitter "bluewall"

$$ds^2 = \frac{R_{dS}^2}{\tau^2}\Big(- \frac{d\tau^2}{1-\tau_0^d\tau^d} + (1-\tau_0^d\tau^d)dw^2 + dx_i^2\Big)$$

Penrose diagram resembles AdS-Schwarzschild rotated by $\frac{\pi}{2}$.

$[-\infty \le w \le \infty]$ Take $\alpha = -1$ earlier.

Equivalently, analytically continue τ_0^d parameter too.



Using Kruskal coordinates: two asymptotic dS universes $(\tau \to 0)$.

Timelike singularities $(\tau \to \infty)$. Cauchy horizons $(\tau = \tau_0)$.

$\simeq$ interior of Reissner-Nordstrom black hole (or wormhole).

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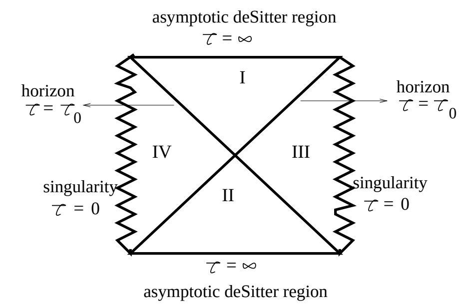


Using Kruskal coordinates: two asymptotic dS universes ($\tau \to 0$).

Timelike singularities ($\tau \to \infty$). Cauchy horizons ($\tau = \tau_0$).

$\simeq$ interior of Reissner-Nordstrom black hole (or wormhole).

Trajectories in the de Sitter bluewall and the Cauchy horizon $\to$

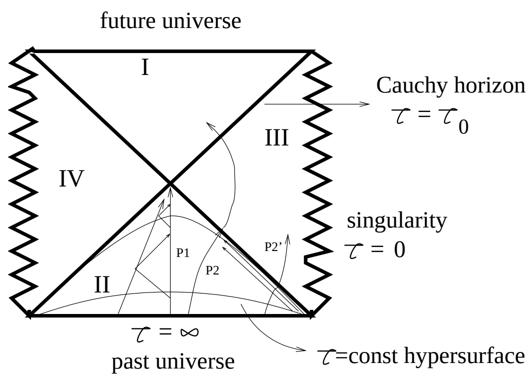
Observers P_1 are static while P_2 has w-momentum p_w,

crosses the horizon, turns around inside and appears

to re-emerge in the future universe.

Incoming lightrays from infinity "crowd near" Cauchy horizon:

Late time infalling observers P_2 see early lightrays blueshifted.



Infinite blueshift due to Cauchy horizon: instability.

de Sitter extremal surfaces

de Sitter, Poincare slicing: $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$.

A generalization of the Ryu-Takayanagi procedure in dS would start with subregion at future timelike infinity, imagine a bulk extremal surface that dips inward into the bulk (towards past). So consider Eucl time slice, $w = const$ subspace $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dx_i^2)$, bulk area functional

$$S_{dS} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{dS}dy_i}{\tau} \frac{R_{dS}}{\tau} \sqrt{d\tau^2 - dx^2} = \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - (\frac{dx}{d\tau})^2}.$$

Conserved quantity in extremization $-\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} = B\tau^{d-1}$ $i.e.$ $\dot{x}^2 = \frac{B^2\tau^{2d-2}}{1+B^2\tau^{2d-2}}$.

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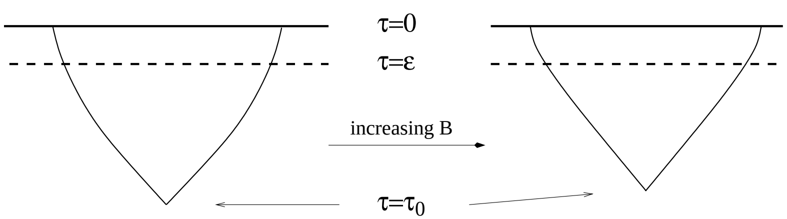
Important sign difference from AdS $\Rightarrow$ no real "turning point". $x(\tau)$ is hyperboloid.

Extremal surface $=$ two half-extremal-surfaces joined continuously at τ_0 but with sharp cusp.

$$S_{dS} = \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_{\epsilon}^{\tau_0} \frac{d\tau}{\tau^{d-1}} \frac{2}{\sqrt{1+B^2\tau^{2d-2}}}.$$

Minimize area w.r.t. parameter $B \rightarrow$ increase B.

$\Rightarrow$ surface shape saturates, approaches $\dot{x}^2 \rightarrow 1$.



$B \gg \frac{1}{\epsilon^{d-1}} \rightarrow$ past lightcone wedge of subregion: $x(\tau)$ null surface, vanishing area.

$\rightarrow$ analog of causal holographic information in AdS (Hubeny,Rangamani).

dS/CFT: complex saddle points $\rightarrow$ complex extremal surfaces?

de Sitter extremal surfaces, dS/CFT?

de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$.

$$S_{dS} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - (\frac{dx}{d\tau})^2}, \quad \dot{x}^2 = \frac{B^2 \tau^{2d-2}}{1+B^2\tau^{2d-2}} .$$

dS_4/CFT_3: $B^2 > 0 \Rightarrow$ no turning point. Consider $B^2 = -A^2 \rightarrow$

$$i\dot{x} = \frac{\pm A\tau^2}{\sqrt{1-A^2\tau^4}} , \quad \rightarrow \text{ complex extremal surface: } \tau_{UV} = i\epsilon, \ \ \tau_* \sim il.$$

$$S_{dS} = -i\tilde{S}_{dS} = -i\frac{R^2_{dS}}{4G_4} V_1 \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \frac{1}{\sqrt{1-A^2\tau^4}} \ \sim \ -\frac{R^2_{dS}}{4G_4} V_1 \left(\frac{1}{\epsilon} - c\frac{1}{l}\right) .$$

(Anninos,Hartman,Strominger) $\frac{R^2_{dS}}{4G_4} \sim -N$: so $S_{dS} > 0$, real, resembles EE.

$\rightarrow$ analytic continuation from AdS Ryu-Takayanagi extremization.

$$S_{AdS}[R, x(r), r] = \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{dr}{r^{d-1}} \sqrt{1 + \left(\frac{dx}{dr}\right)^2} , \qquad (x')^2 = \frac{A^2 r^{2d-2}}{1-A^2 r^{2d-2}} \rightarrow$$

$$\dot{x}^2 = \frac{-(-1)^{d-1} A^2 \tau^{2d-2}}{1-(-1)^{d-1} A^2 \tau^{2d-2}}, \qquad S_{dS} = -i\frac{R^{d-1}_{dS}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1-(-1)^{d-1} A^2 \tau^{2d-2}}}.$$

Although these complex extremal surfaces show up in all dS_{d+1}, resulting analytically continued area is not always real-valued: S_{dS} real in dS_4/CFT_3. Physical interpretation: EE?

[dS_4 black brane, CFT_3 at uniform energy density: S^{fin}_{dS} resembles extensive thermal entropy.]

Conclusions, questions

- Some gauge/string realizations of Lifshitz & hyperscaling violation involve x^+-reduction of AdS deformations with g_{++}.
Lower dim'nal time is x^-.
Entanglement in lower dim'nal theory arises on null time x^- slices upstairs $\rightarrow$ lightlike limit of entanglement entropy.
Better understanding in lightcone QFT, ultralocality, ...

- Deeper understanding of dS/CFT at uniform energy density.

- Deeper understanding of extremal surfaces in de Sitter, their physical interpretation (if any!).