Two Dimensional Yang-Mills Theory on Infinite Genus Surfaces

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Based on :
DK [arXiv:1407:6379] JHEP
D. Ghoshal, C. Imbimbo & DK [arXiv:1407:6380] JHEP

Introduction

- As is clear from the title, in this talk we'll look at some aspects of two dimensional Yang-Mills on a class of infinite genus surfaces.
- Action :

$$S(A) = \frac{1}{4g^2} \int_\Sigma d^2\sigma \sqrt{\eta}\, \mathrm{Tr}\left(F_{\mu\nu} F_{\alpha\beta}\right) \eta^{\mu\alpha} \eta^{\nu\beta}$$

  where g is the gauge coupling, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the field strength and A_μ is the gauge potential.
- $F = f\sqrt{\eta} d^2\sigma$. In terms of f the action reads

$$S(A) = \frac{1}{2g^2} \int_\Sigma d^2\sigma \sqrt{\eta}\, \mathrm{Tr}(f^2)$$

  Invariant under area preserving diffeomorphisms.
- Huge symmetry. Hence exactly solvable.

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Introduction contd..

- The partition function of the disc of area a and holonomy U along its boundary is given as [Migdal 1975]

$$Z(D, U, a) = \sum_{R \in \text{irreps}} \dim R \exp(-a\, g^2 C_2(R))\, \chi_R(U)$$

- For example, for the gauge group $SU(2)$, and holonomy $U = \exp(i\theta J_3)$, the partition function reads

$$Z_{SU(2)}(D, U, a) = \sum_{n>0} n \exp(-a\, g^2 n^2)\, \frac{\sin(n\theta/2)}{\sin(\theta/2)}$$

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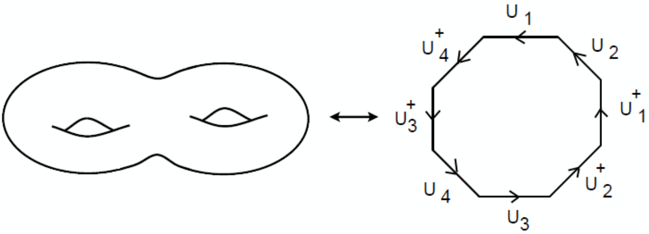
$$Z(\Sigma, a) = \int \prod_{i=1}^{4} dU_i \; Z\left(D,\; U_2^{-1} U_1^{-1} U_2\; U_1\; U_4^{-1} U_3^{-1} U_4\; U_3,\; a\right)$$



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- The partition function $Z(\Sigma^h, a)$ of a surface Σ^h with area a, genus h is given by the Migdal's formula [Migdal 1975, Witten 1991]

$$\sum_{R \in \text{irreps}} (\dim R)^{2-2h} \exp(-ag^2\, C_2(R))$$

- We have found that there exist surfaces $\Sigma_R^{h,p}$ (that we call Richards surfaces) parametrized by two nonnegative integers h and p with the partition function [DK, JHEP 2014]

$$\sum_{R \in \text{irreps}} (\dim R)^{2+\frac{2h}{p-1}} \exp(-ag^2\, C_2(R))$$

- Also, the weak coupling expansion indicates that the above partition function may have the same topological significance for the moduli spaces of flat connections on Richards surfaces as the Migdal's partition function has for finite genus surfaces. [D. Ghoshal, C. Imbimbo, DK, JHEP 2014]

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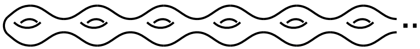
Plan of the rest of the talk..

In what follows we'll explain the following :

- What are Richards surfaces $\Sigma_R^{h,p}$?
- How is their partition function calculated ?
- Does the partition function of a Richards surface has a similar topological significance for the corresponding moduli spaces of flat connections as in case of finite genus surfaces ?

$\Sigma_R^{h,p}$: special cases



- $\Sigma_R^{1,1}$ = Loch Ness Monster : Boundary end capped
- $\Sigma_R^{0,p}$ = Sphere.
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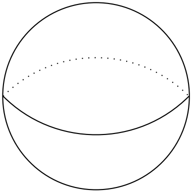

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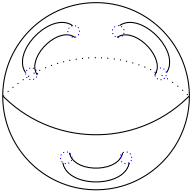
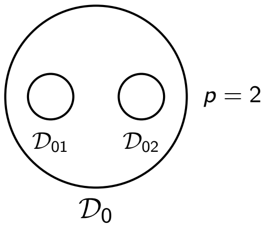


$\Sigma_R^{h,p}$: Construction

- Start with a closed disc $\mathcal{D}_0$ and mark p smaller discs $\mathcal{D}_{01}, \ldots, \mathcal{D}_{0p}$ inside it.
- Attach h handles in the complement of $\bigcup_{i=1}^{p} \mathcal{D}_{0i}$.
- Repeat the above steps (done in $\mathcal{D}_0$) in each $\mathcal{D}_{0i}$ for $i = 1, \ldots, p$, and so on recursively. The topological surface obtained after an infinite number of recursion and finally capping the boundary is $\Sigma_R^{h,p}$.



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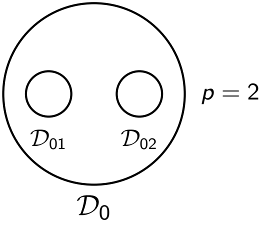

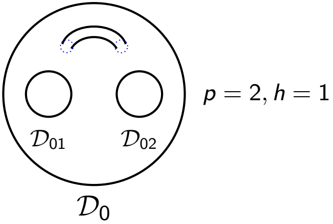

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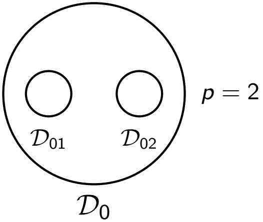
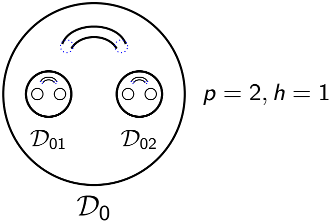
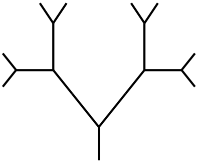




$\Sigma_R^{h,p}$: More intuitive construction

- Start with an infinite rooted tree of valance $p+1$.



- Make its edges thicker and take its outer surface.
- Attach h number of handles to each vertex and cap all the boundary ends.

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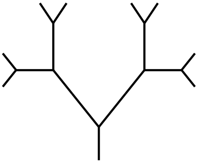

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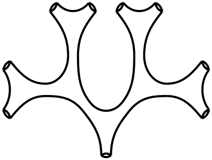

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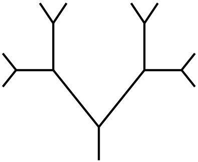

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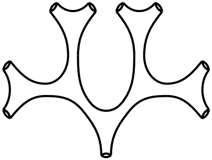

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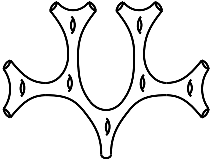

YM partition function of $\Sigma_R^{h,p}$

- Partition function of finite genus surfaces can be computed from that of a disc. But this doesn't seem possible for surfaces of infinite genus.
- However, the global self-similarity property of Richards surfaces implies a constraint on their partition function. Schematically, the equation satisfied by $Z = Z_{\Sigma_R^{h,p}}$ reads

$$Z = Z_{\text{finite genus}} * \underbrace{Z * \cdots * Z}_{p}$$

  This equation is obtained by cutting the surface at a finite level and using gluing arguments and the self similarity of $\Sigma_R^{h,p}$.
- The partition function of $\Sigma_R^{h,p}$ (reproduced below for convenience) is a unique solution of the above equation.

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Topological significance ..?

The weak coupling expansion of the YM partition function of a finite genus surface gives the intersection numbers on the moduli space of flat connections. Does the partition function of a Richards surface play a similar role for the corresponding moduli space of flat connections ?

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Moduli space of flat connections - Intro.

- A G-connection A_μ on Σ is said to be flat if its curvature form is zero.
- The space $\mathcal{M}_{(\Sigma,G)}$ of all flat G connections on a surface Σ quotiented by the action of the gauge group is called the moduli space of flat G connections corresponding to Σ.
- When Σ is a compact surface of genus $h \geq 1$, $\mathcal{M}_{(\Sigma,G)}$ is a compact symplectic manifold of dimension $\dim G\,(2h-2)$. The space $\mathcal{M}_{(\Sigma,G)}$ has singularities if the center of the group G is nontrivial.

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Topology of the moduli space

- The moduli space of flat connections has a symplectic form ω, and a 4-form Θ defined on it which are important from the topological point of view.
- The symplectic form ω is the natural 2-form

$$\omega(\delta A_1, \delta A_2) = \int_\Sigma \delta A_1 \wedge \delta A_2$$

- To know what Θ is, introduce an auxiliary Lie algebra valued 0-form ϕ and rewrite the action as

$$S = \frac{1}{2}\int_\Sigma d^2\sigma\sqrt{\eta}\left(2i\mathrm{Tr}\left(\phi f\right) + g^2\mathrm{Tr}\left(\phi^2\right)\right)$$

- The term $\int_\Sigma d^2\sigma\sqrt{\eta}\,\mathrm{Tr}\left(\phi^2\right)$ corresponds to the 4-form Θ on the moduli space.
- The numbers $\displaystyle\int_{\mathcal{M}_h} \omega^m \wedge \Theta^k$ are called the intersection numbers of forms ω and Θ and contain useful information about the topology of the moduli space.

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Relation of intersection numbers with YM partition function

- The moduli space of flat connections is the space of Global minima of the classical Yang-Mills action. Hence, in the limit of $ag^2 \to 0$ the maximal contribution to the partition function and correlation functions comes from the moduli space.
- Therefore, in the limit $ag^2 \to 0$ the Yang-Mills partition function can be expected to have some information about the topology of the moduli space.
- This is indeed true. It can be shown that the intersection numbers are related to the Yang-Mills partition function $Z_h(\epsilon = ag^2/\pi^2)$ as

$$\frac{1}{(s-k)!}\int_{\mathcal{M}_h} \omega^{s-k} \wedge \Theta^k = c\, \frac{d^k}{d\epsilon^k} Z_h(\epsilon)\Big|_{\epsilon=0}$$

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- The moduli space of flat connections is the space of Global minima of the classical Yang-Mills action. Hence, in the limit of $ag^2 \to 0$ the maximal contribution to the partition function and correlation functions comes from the moduli space.
- Therefore, in the limit $ag^2 \to 0$ the Yang-Mills partition function can be expected to have some information about the topology of the moduli space.
- This is indeed true. It can be shown that the intersection numbers are related to the Yang-Mills partition function $Z_h(\epsilon = ag^2/\pi^2)$ as

$$\frac{1}{(s-k)!}\int_{\mathcal{M}_h} \omega^{s-k} \wedge \Theta^k = c\, \frac{d^k}{d\epsilon^k} Z_h(\epsilon)\Big|_{\epsilon=0}$$

where, c denotes the number of elements in the center of G, and k and s are such that $s + k = \dim_{\mathbb{C}}\mathcal{M}_h$ [Witten 1992]

Topological significance of the partition function of Richards surfaces ?

- For the finite genus surfaces, the intersection numbers calculated from the weak coupling expansion of the Yang-Mills partition function agree with the rigorous mathematical calculations (for gauge groups SU(2) and SO(3)).

- For infinite genus surfaces mathematical results are not available. So we don't (yet) know if the weak coupling expansion of the partition function of Richards surfaces gives intersection numbers of the corresponding moduli spaces.

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Topological significance of the partition function of Richards surfaces ? contd..

- Although we cannot (yet) make any rigorous claims, if the partition functions on Richards surfaces were to have an interpretation as generators of the intersection numbers on the corresponding moduli spaces of flat connections, we would expect the following:
  - The weak coupling series should be an infinite series. (The moduli space ought to be infinite dimensional.)
  - The series should have fractional powers of ϵ for any group with a non-trivial centre. (Moduli space is singular in such cases.)
  - The series should have no fractional powers of ϵ for any group with a trivial centre. (Moduli space is smooth in such cases.)

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- We have verified that the above expectations hold true for the case of groups $SU(2)$ and $SO(3)$.
- Modulo exponentially small terms the weak coupling expansions for both these groups are given as $(\alpha = 2 + \frac{2h}{p-1})$

$$Z_{SU(2)} = \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{2\,\pi^{1+\alpha}\epsilon^{\frac{1+\alpha}{2}}} + \frac{\Gamma\left(\frac{1+\alpha}{2}\right)\sqrt{\pi}}{\pi^{1+\alpha}\Gamma\left(-\frac{\alpha}{2}\right)} \sum_{m=0}^{\infty} \frac{\left(\frac{1+\alpha}{2}\right)_m \left(\frac{2+\alpha}{2}\right)_m}{m!} \zeta(2m+1+\alpha)\,\epsilon^m$$

$$Z_{SO(3)} = \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}+\alpha}\Gamma\left(-\frac{\alpha}{2}\right)} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{1+\alpha}{2}\right)_m \left(\frac{2+\alpha}{2}\right)_m \zeta\left(2m+\alpha+1, \frac{1}{2}\right) \epsilon^m$$

- Clearly both the above expansions are infinite series for generic values of h and p. We have a fractional power of ϵ in $Z_{SU(2)}$ and no fractional powers in $Z_{SO(3)}$.

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- Also, if we analytically continue the above series expansions to negative integer values $\alpha = 2 - 2h$ of α and make use of the identities $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$ and $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$, we recover the Witten's results for the intersection numbers for a finite genus surface of genus h.

$$\int_{\mathcal{M}_{h,SU(2)}} \omega^{3h-3-2k} \wedge \Theta^k = (-1)^k \frac{(3h-3-2k)!}{2^{h-2}\pi^{2h-2-2k}} \zeta(2h-2-2k)$$

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- This serves as a consistency check for the fact that the partition function of Richards surfaces generalises Migdal's formula. It also provides a different approach to obtain the intersection numbers of finite genus surfaces (*cf* Witten's method).

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- Although the naive $h \to \infty$ limit of the Yang-Mills partition function gives a trivial result, we have found that there is a class of surfaces $\Sigma_R^{h,p}$ (generically of infinite genus) for which the partition function is nontrivial. Wilson loop expectations can also be computed.

- Analytic properties of the partition function of 2dYM on Richards surfaces suggests that the coefficients in the partition function may have an interpretation in terms of the infinite number of intersection numbers on the moduli space of flat connections, as in the case of finite genus surfaces.

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Thank you