AdS plane waves, Entanglement and Mutual Information

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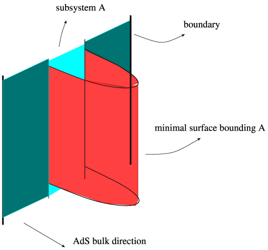
DM & K.Narayan

Entanglement Entropy and Ryu-Takayanagi prescription

AdS/CFT correspondence[Maldacena '97] in large N limit gives a map:

bulk gravity theory $\Leftrightarrow$ strongly-coupled field theory on boundary.

Geometrizes quantities in field theory otherwise difficult to compute; one such quantity is Entanglement Entropy (EE).



Ryu-Takayanagi prescription:
EE of subsystem $A \propto$ Area of bulk minimal surface bounding A (subsystem).

Entanglement Entropy

- can be used to characterize phases.
- Proportional to no. of d.o.f on bndry. b/w A and environment. Shows area law divergence. ($S_A \sim \frac{\partial A}{\epsilon^{d-2}}$ for AdS_{d+1})

Mutual Information

From a linear combination of EEs, define **Mutual Information** (MI) for disjoint subsystems A & B ($A \cap B = \varnothing$).

$$\mathcal{I}[A:B] = S[A] + S[B] - S[A \cup B]$$

where $S[A]$ is EE of subsystem A as if B was absent

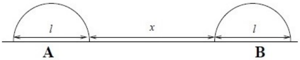
- Positive semi-definite
- Cut-off dependent divergences cancel out
- Measures correlation b/w A and B (quantum & classical)

$$\mathcal{I}[A:B] \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2}{2|\mathcal{O}_A|^2 |\mathcal{O}_B|^2}$$

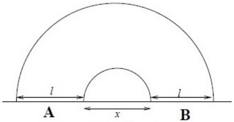
Mutual Information: Disentangling transition

$$\mathcal{I}[A : B] = S[A] + S[B] - S[A \cup B]$$

As an example, consider two **spacelike, strip subsystems** each of width ℓ $\|$ to each other separated by x. Two possible candidates of $S[A \cup B]$.



Disconnected bulk surface



Connected bulk surface

When A and B are widely separated, relevant extremal surface is simply union of disconnected surfaces.
$S[A \cup B] = S[A] + S[B] = 2S(\ell)$
$\mathcal{I}[A : B] = 0$

For nearby subsystems, connected surface has lower area.
$S[A \cup B] = S(2\ell + x) + S(x)$
$\mathcal{I}[A : B] > 0$

R-T implies *disentanglement* i.e $\mathcal{I}[A : B] = 0$ identically $\forall\ \frac{x}{\ell} > \frac{x_c}{\ell}$ [Headrick '10].

Mutual Information: Ground state and thermal state

- **Ground state** $\Leftrightarrow$ pure AdS_{d+1} $(d > 2)$:

$$\mathcal{I}[A:B] = -c\frac{V_{d-2}}{\ell^{d-2}}\left(2 - \frac{1}{(2+x/\ell)^{d-2}} - \frac{1}{(x/\ell)^{d-2}}\right)$$

$\mathcal{I}[A:B] \to \infty$ as $x \to 0$. Zero of $\mathcal{I}[A:B]$ i.e disentangling happens at

$$\mathbf{AdS_5}: \ \frac{x_c}{\ell} \simeq 0.732 \quad \& \quad \mathbf{AdS_4}: \ \frac{x_c}{\ell} \simeq 0.620$$

- **Thermal states** also show disentanglement transition [Fischler,Kundu,Kundu '12]. When $\ell T, xT \gg 1$ entanglement dominated by thermal entropy (S scales linearly as ℓ).

$$\mathcal{I}[A:B] = 0$$

$\implies$ A and B disentangle at $x > \frac{1}{T}$.

AdS plane waves

Interesting to explore non-relativistic systems with reduced symmetries. A certain class of gravity duals exhibit hyperscaling violation: $ds^2 = r^{2\theta/d_i}\left(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2}\right)$

Arise in Einstein-Maxwell-scalar theories [Trivedi et al; Kiritsis et al;...]

For $\theta = 0$ this reduces to Lifshitz metric.

$\theta = d_i - 1$ family $\rightarrow$ log **violation of area law**.

Conjectured to be gravity dual of Fermi surfaces [Ogawa,Takayanagi,Ugajin; Huijse,Sachdev,Swingle '11]. EE has been studied for hyperscaling violation geometry [Dong,Harrison,Kachru,Torroba,Wang].

Concrete string construction exists [Narayan '12]. Obtained after x^+-dimn^n red^n of AdS_{d+1} plane wave:

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2 Q r^{d-2}(dx^+)^2$$

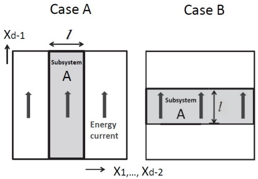
E.g: AdS_5 plane wave $\xrightarrow{x^+-\text{dimn. redn}}$ $\theta = 1 = d_i - 1$ hyperscaling violating

Dual to excited pure states with uniform energy-momentum density $T_{++} \sim Q$.

AdS plane waves and Entanglement Entropy

EE for these excited states have been studied [Narayan, Takayanagi, Trivedi '12]

Two choices for strip subsystem– depending on flux dir^n.



For large $\ell Q^{1/d} \gg 1$, we know the scaling of S_A with ℓ and Q.

- Strip $\parallel$ flux: $$S_A = \begin{cases} \pm\sqrt{Q}V_{d-2}\ell^{2-\frac{d}{2}} & \text{for} \quad [+ : d < 4, - : d > 4] \\ \sqrt{Q}V_2 \log(\ell Q^{1/4}) & \text{for} \quad d = 4 \end{cases}$$

  For $d = 4$ if we identify $Q^{1/4}$ with k_F (Fermi momentum), S_A shows log scaling, functionally similar to Fermi surface.

- Strip $\perp^r$ flux: For large ℓ, there is **NO** connected minimal surface. **Phase transition** for $\ell \gg Q^{-1/d}$.

MI for *AdS* plane waves

We have studied MI for *AdS* plane waves in three regimes: [DM, Narayan '14]

- $\ell \mathbf{Q^{1/d}} \gg \mathbf{1}$:
  - When strips are $\parallel$ to flux dirn. systems disentangle faster than ground state.
  - $\frac{x_c}{\ell}$ is independent of Q.
  - Unlike thermal states, $\exists\ x_c$ for which $\mathcal{I}[A:B] > 0$, for all ℓ.
- $\ell \mathbf{Q^{1/d}} \ll \mathbf{1}$: In this limit, we did a perturbative analysis about pure *AdS* and computed $O(Q)$ correction to MI. Faster disentanglement than ground state.
- $\ell \mathbf{Q^{1/d}} \sim \mathbf{O(1)}$: We did a numerical study when systems are $\parallel$ to flux dirn. to get a more complete parameter space.

MI for AdS plane waves: $\ell Q^{1/d} \gg 1$

1. Strip $\parallel$ to flux dirn.:

   - AdS_5 plane wave: $\mathcal{I}[A:B] \sim V_2\sqrt{Q}\log\left(\frac{\ell^2}{x(2\ell+x)}\right)$
     Disentanglement: $\mathcal{I}[A:B] = 0$ at $\frac{x_c}{\ell} \approx 0.414$. (Ground state $\frac{x_c}{\ell} = 0.732$)

   - AdS_4 plane wave: $\mathcal{I}[A:B] \sim V_1\sqrt{Q}(2\sqrt{\ell} - \sqrt{2\ell+x} - \sqrt{x})$.
     Disentanglement: $\mathcal{I}[A:B] = 0$ at $\frac{x_c}{\ell} = 0.250$. (Ground state $\frac{x_c}{\ell} = 0.620$)

   Disentangles **faster compared to ground state**.
   Critical seperation is independent of Q.

2. Strip $\perp^r$ to flux dirn.:
   At large width, there is absence of connected bulk minimal surface and $S(\ell)$, $S(2\ell+x)$ and $S(x)$ **ALL** saturate to a definite value S_{sat}. So,

$$\mathcal{I}[A:B] \sim 2S(\ell) - S(2\ell+x) - S(x) = 0$$

EE & MI for *AdS* plane waves: $\ell Q^{1/d} \ll 1$

In this regime ($\ell Q^{1/d} \ll 1$), we can compute perturbative correction to ground state entanglement.

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- For single strip, calculate deviation in turning pt. δr_* upto $O(Q)$.

  For strip $\parallel$ to flux, we find

$$\delta r_* = -\frac{\mathcal{N}_{r_*}}{4\eta} Q r_*^{d+1}$$

  where $\mathcal{N}_{r_*} = \frac{\sqrt{\pi}}{(d-1)^2}\left(\frac{\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d+1}{2d-2})} - (d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right) > 0$ & $\eta = \frac{\sqrt{\pi}\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})} > 0$.

  For strip $\perp^r$ to flux, we find

$$\delta r_* = -\beta Q r_*^{d+1}$$

  where $\beta = \frac{2^{\frac{1}{d-1}}}{8(d-1)^3\sqrt{\pi}} \frac{\Gamma(\frac{1}{2d-2})^2}{\Gamma(\frac{3}{2}+\frac{1}{d-1})} - \frac{1}{4(d-1)} > 0$

  Essentially, for either orientation, we have $\delta r_* = -\beta Q r_*^{d+1}$ where $\beta > 0$.

EE & MI for AdS plane waves: $\ell Q^{1/d} \ll 1$

In this regime ($\ell Q^{1/d} \ll 1$), we can compute perturbative correction to ground state entanglement.

- For single strip, calculate deviation in turning pt. δr_* upto $O(Q)$. [$\delta r_* = -\beta Q r_*^{d+1}$ where $\beta > 0$]
- Compute area functional (S_{EE}) upto $O(Q)$ correction to pure AdS.

For either orientation to the flux dirn,

$$\Delta S = +\frac{R^{d-1}}{G_{d+1}} \frac{\mathcal{N}_{EE}^{\|,\perp}}{4\eta^2\sqrt{2}} \frac{V_{d-2}}{\ell^{d-2}} (Q\ell^d)$$

$$\text{where } \mathcal{N}_{EE}^{\|,\perp} = \begin{cases} \frac{\sqrt{\pi}}{8(d-1)^2}\left(\frac{(d+1)\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d+1}{2d-2})} - 2(d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right) & \text{for strips} \quad \| \quad \text{flux dirn.} \\ \frac{\sqrt{\pi}}{4\sqrt{2}(d-1)^2}\left(\frac{\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d+1}{2d-2})} - (d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right) & \text{for strips} \quad \perp \quad \text{flux dirn.} \end{cases}$$

and $\eta = \frac{\sqrt{\pi}\Gamma(\frac{d}{d-2})}{\Gamma(\frac{1}{2d-2})}$.

For $d > 2$, $\mathcal{N}_{EE} > 0$

EE & MI for AdS plane waves: $\ell Q^{1/d} \ll 1$

In this regime ($\ell Q^{1/d} \ll 1$), we can compute perturbative correction to ground state entanglement.

- For single strip, calculate deviation in turning pt. δr_* upto $O(Q)$.
  [$\delta r_* = -\beta Q r_*^{d+1}$ where $\beta > 0$]
- Compute area functional (S_{EE}) upto $O(Q)$ correction to pure AdS.

  For either orientation to the flux dirn,

$$\Delta S = + \frac{R^{d-1}}{G_{d+1}} \frac{\mathcal{N}_{EE}^{\parallel,\perp}}{4\eta^2\sqrt{2}} \frac{V_{d-2}}{\ell^{d-2}} (Q\ell^d)$$

$$\text{where } \mathcal{N}_{EE}^{\parallel,\perp} = \begin{cases} \frac{\sqrt{\pi}}{8(d-1)^2}\left(\frac{(d+1)\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d+1}{2d-2})} - 2(d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right) & \text{for strips } \parallel \text{ flux dirn.} \\ \frac{\sqrt{\pi}}{4\sqrt{2}(d-1)^2}\left(\frac{\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d+1}{2d-2})} - (d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right) & \text{for strips } \perp \text{ flux dirn.} \end{cases}$$

and $\eta = \frac{\sqrt{\pi}\Gamma(\frac{d}{d-2})}{\Gamma(\frac{1}{2d-2})}$ For $d > 2$, $\mathcal{N}_{EE} > 0$.

- Similar to *Entanglement Thermodynamics*
  [Takayanagi et al;Alishahiha et al; Faulkner,Guica,Hartman,Myers,Van Raamsdonk]

MI for AdS plane waves: $\ell Q^{1/d} \ll 1$

For either orientation of strips, we get a **positive** correction to EE.
When the strips are $\parallel$ to flux dirn,

$$\text{MI:} \quad \mathcal{I}[A:B] = \mathcal{I}_{AdS}[A:B] - 2\frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{N}_{EE}^{\parallel}}{4\eta^2\sqrt{2}}V_{d-2}Q\ell^2\left(1+\frac{x}{\ell}\right)^2$$

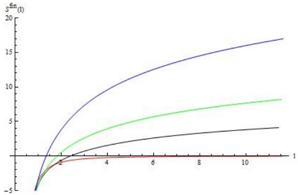
When the strips are $\perp^r$ to flux dirn,

$$\text{MI:} \quad \mathcal{I}[A:B] = \mathcal{I}_{AdS}[A:B] - 2\frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{N}_{EE}^{\perp}}{4\eta^2\sqrt{2}}V_{d-2}Q\ell^2\left(1+\frac{x}{\ell}\right)^2$$

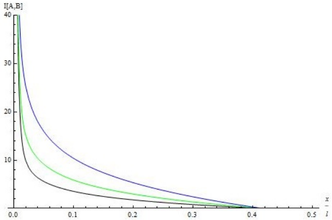
1. MI is lesser compared to ground state $\implies$ faster disentanglement. Suggesting, energy density flux Q disorders system.
2. Disentangling transition in this regime depends on flux Q.

EE and MI for AdS_5 plane waves: Numerics

We have done a numerical analysis when $\ell Q^{1/d} \sim O(1)$.



Entanglement Entropy vs subsystem width



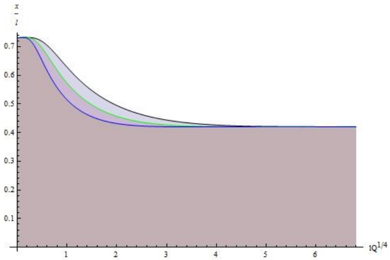
Mutual Information vs x/l

Figure: Red: $Q = 0$, Black: $Q = 1$, Green: $Q = 3$, Blue: $Q = 10$

- At large ℓ, EE is dominated by effect of energy flux Q.
- For any Q, x_c/ℓ is roughly the same.

Parameter space for AdS_5

Parameter space for AdS_5 (Strip $\parallel$ flux):

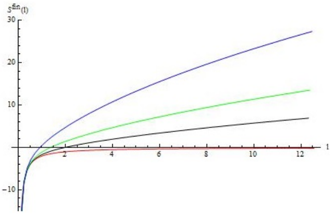


$\frac{x}{l}$ vs $lQ^{1/4}$

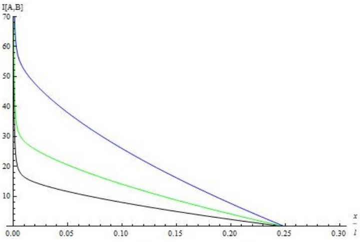
- At large ℓ, all curves flatten out $\implies$ x_c/ℓ is independent of Q.
- Near $\ell Q^{1/d} \sim O(1)$, curves are distinct $\implies$ x_c/ℓ depends on Q.
- Different from **thermal case** where we have finite parameter space. In the regime of large ℓ, subsystems remain disentangled for *any* x.

EE and MI for AdS_4

For AdS_4, we did a similar numerical analysis and obtained the following:



Entanglement Entropy vs subsystem width



Mutual Information vs x/l

Figure: Red: $Q = 0$, Black: $Q = 1$, Green: $Q = 3$, Blue: $Q = 10$

At large ℓ, there is a deviation from pure AdS for non-zero Q.
x_c/ℓ is independent of Q.

Parameter space for AdS_4

Parameter space for AdS_4:

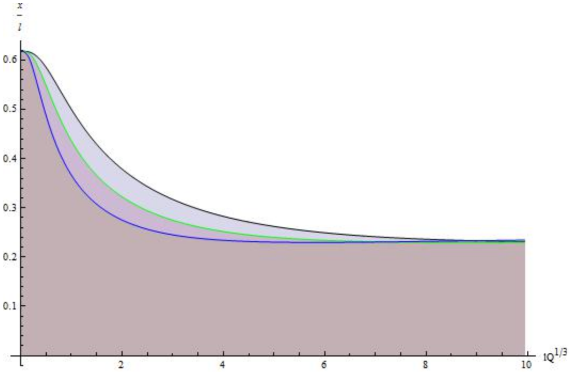


Figure: Black: $Q = 1$, Green: $Q = 3$, Blue: $Q = 10$

Conclusions

- We have studied EE and MI for AdS plane waves dual to CFT excited states with $T_{++} \sim Q$ for two strips of width ℓ, seperation x, $\|$ and $\perp^r$ to flux
- For wide strips ($Q\ell^d \gg 1$) $\|$ to flux, x_c/ℓ is independent of Q. Even for large ℓ, $\exists\, x_c$ below which A & B are entangled (unlike thermal states).
- For wide strips $\perp$ to flux, there is phase transition for $\ell \gg Q^{-1/d}$. EE saturates, MI is zero.
- In perturbative regime $Q\ell^d \ll 1$, $\Delta S \sim +V_{d-2}Q\ell^2 \implies$ faster disentangling than ground state. Probably, Energy density "disorders" system.
- Numerics show non-trivial dependence of x_c/ℓ with Q when $Q\ell^d \sim O(1)$. At large ℓ, agrees with calculations.
- In some sense, "partially ordered" states.