$3d$ gravity and dual (warped) CFTs

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References

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- "Chiral Liouville Gravity", G.C., W. Song and A. Strominger, arXiv :1303.2662
- "New Boundary Conditions for AdS_3", G.C., W. Song and A. Strominger, arXiv :1303.2660
- "Two Virasoro symmetries in stringy warped AdS_3", G.C., M. Guica and M. Rodriguez, arXiv :1407.7871

Preambule

Two dimensional critical behavior is often associated with the emergence of infinite-dimensional conformal symmetries

$$\delta t^- = \epsilon^-(t^-), \qquad \delta t^+ = \epsilon^+(t^+).$$

forming two copies of the (centerless) Virasoro algebra.

In the last years, evidence has been accumulating for another possible critical behavior associated with the so-called “warped conformal symmetries”

$$\delta t^- = \sigma(t^+), \qquad \delta t^+ = \epsilon^+(t^+),$$

which act in a chiral fashion. It forms a semi-direct product of a Virasoro algebra with a $U(1)$ Kač-Moody algebra.

The two-dimensional Virasoro $\times$ Virasoro algebra is represented both as a

- Symmetry of conformal field theories (CFTs)
- Subgroup of diffeomorphisms of gravitational theories which acts non-trivially [Brown-Henneaux, 1986]

In some cases, the two representations are related by the gauge/gravity correspondence. The symmetries are powerful enough to allow a universal microstate counting [Cardy, 1986]; [Strominger, 1998] .

The simplest semi-classical AdS_3/CFT_2 dual pair is

- Gravity with $\Lambda = -\frac{1}{\ell^2}$ with Dirichlet boundary conditions $\Leftrightarrow$ Liouville theory with central charge $c = \frac{3\ell}{2G_3}$.

  [Coussaert-Henneaux-van Driel, 1995]

(The simplest known quantum AdS_3/CFT_2 dual pair involves $D1 - D5$ branes whose near horizon limit is $AdS_3 \times S^3 \times T^4$ [Maldacena, 1998])

Warped AdS_3 as a new playground

$AdS_3 = SL(2,\mathbb{R})$ group invariant under $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. We can break the symmetry to $U(1)_L \times SL(2,\mathbb{R})_R$ by warping S^1 on AdS_2 :

- Squashed warped AdS_3 :

$$g^{AdS}_{\mu\nu} - \lambda^2 \xi^L_\mu \xi^L_\nu, \qquad \lambda > 0$$

  No black holes are known but black strings exist. Consistent supergravity truncations exist.

- Stretched warped AdS_3 :

$$g^{AdS}_{\mu\nu} + \lambda^2 \xi^L_\mu \xi^L_\nu, \qquad \lambda > 0$$

  Black holes and strings exist. No known consistent supergravity truncations.

Motivation : the Near Horizon of Extremal Kerr is squashed AdS_3 around the poles and stretched AdS_3 around the equator.

In the last years, it has been shown that the Virasoro $\times$ Kač-Moody algebra is represented as well as

- Subgroup of diffeomorphisms of gravitational theories which acts non-trivially [G.C., Detournay, 2008]
- Symmetry of field theories (called warped CFTs) [Hofman, Strominger, 2011]

The symmetries are powerful enough to allow a universal microstate counting [Detournay, Hartman, Hofman, 2012].

One simple semi-classical $AdS_3/WCFT_2$ dual pair

- Gravity with $\Lambda = -\frac{1}{\ell^2}$ with Dirichlet-Neumann boundary conditions at fixed $T_{--} = \Delta \Leftrightarrow$ Chiral Liouville theory with central charge $c = \frac{3\ell}{2G_3}$ and level $k = -\Delta$.

(A tentative quantum $WAdS_3/WCFT_2$ dual pair involves TsT deformed $D1 - D5$ branes whose near horizon limit is $WAdS_3 \times WS^3 \times T^4$ [Song, Strominger, 2011])

Outline

1. Liouville theory as dual to AdS_3 and dS_3

2. Chiral Liouville theory

3. Warped AdS_3 / “nearly a CFT” correspondence

Einstein gravity with $\Lambda = -\frac{1}{\ell^2}$ is equivalent to two copies of Chern-Simons theory with gauge group $SL(2,\mathbb{R})$, [Achucarro et al. 1986]

$$S_E = \int_{\mathcal{M}} d^3x\sqrt{-g}(R + \frac{2}{\ell^2}) = S[A] - S[\bar{A}] + (\text{boundary terms}),$$

$$S[A] = \frac{\ell}{16\pi G}\int_{\mathcal{M}} \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$$

With Dirichlet boundary conditions for the metric,

$$\frac{ds^2}{\ell^2} = \frac{dr^2}{r^2} + r^2(-dt^2 + d\phi^2) + (subleading)$$

the (correct) action is equal to the one of **Lorentzian Liouville theory** on the boundary cylinder $\mathcal{C} \equiv \partial\mathcal{M}$,

$$S_E \equiv \frac{\ell^2}{64\pi G}\int_{\mathcal{C}} d\phi\, dt\left((\partial_t\Phi)^2 - \frac{1}{\ell^2}(\partial_\phi\Phi)^2 - \frac{16}{\ell^2}e^{\Phi}\right)$$

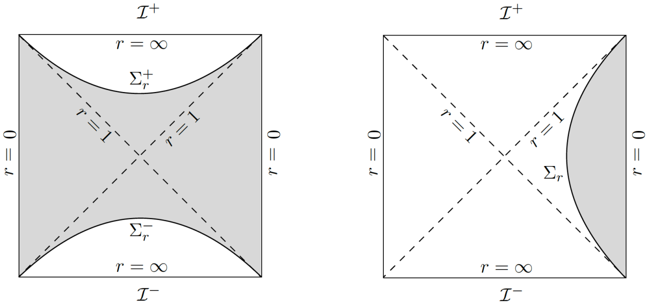
[Coussaert-Henneaux-van Driel, 1995]

Einstein gravity with $\Lambda = +\frac{1}{\ell^2}$ is equivalent to two copies of Chern-Simons theory with gauge group $SL(2,\mathbb{C})$. Similarly, one find **Euclidean Liouville theory** at $\mathcal{I}^+ \cup \mathcal{I}^-$,

$$S_E \equiv -\frac{\ell^2}{64\pi G}\int_{\mathcal{I}^+\cup\mathcal{I}^-} d\phi\, dt \left((\partial_t \Phi)^2 + \frac{1}{\ell^2}(\partial_\phi \Phi)^2 + \frac{16}{\ell^2} e^{\Phi} \right)$$

[Cacciatori, Klemm, 2001]

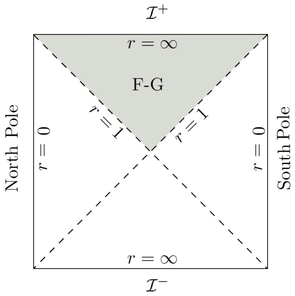
This motivates the dS/CFT conjecture [Strominger, 1998]

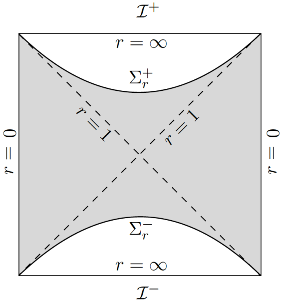


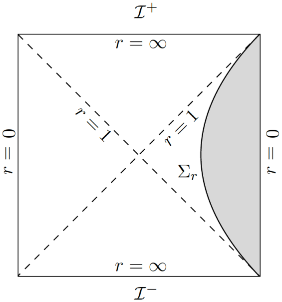
What about the static patch ?

First, one can prove that $\omega(\delta g, \delta g; g) = 0$. Therefore surface charges are conserved at any radius.
$\Rightarrow$ The two Virasoro algebras are defined everywhere in the bulk.

Second, reaching the static patch implies going to Eddington-Finkelstein coordinates which leads to some subtelties







After the reduction procedure, we find

$$S_E \equiv -\frac{\ell^2}{64\pi G}\int_{\Sigma_r} d\phi\, dt\left((\partial_t\Phi)^2 + \frac{1}{\ell^2}(\partial_\phi\Phi)^2 + \frac{16}{\ell^2}e^{\Phi}\right)$$

This confirms the dS/CFT conjecture. The boundary Euclidean time t is the Lorentzian static time in spacetime.

2. Chiral Liouville theory

Chiral Boundary Conditions

Asymptotically locally AdS_3 metrics are

$$ds^2 = \frac{dr^2}{r^2} + (r^2 g^{(0)}_{ab} + T_{ab} + \dots)dx^a dx^b, \qquad a = \{t, \phi\}$$

(We take $R_{(0)} = 0$)

The variation of the Einstein action is

$$\delta S_E = \int_{\mathcal{C}} \sqrt{-g_{(0)}}\, T^{ab} \delta g^{(0)}_{ab}$$

An alternative to Dirichlet (Brown-Henneaux) boundary conditions are the chiral boundary conditions :

$$\begin{aligned} ds^2_{(0)} &= -dt^+(dt^- + h(t^+)dt^+), \\ T_{--} &= \Delta\,. \end{aligned}$$

(Proof : $\delta\left(S_E + \frac{\Delta}{4\pi}\int_{\mathcal{C}} dt^+ dt^- \sqrt{-g_{(0)}}\, g^{--}_{(0)}\right) = 0$.)

$$ds^2_{(0)} = -dt^+dt^- - h(dt^+)^2.$$

$$\partial_- h = 0.$$

For $h \neq 0$, the conformal Killing symmetries are the warped conformal symmetries

$$\delta t^+ = \epsilon(t^+), \qquad \delta t^- = \sigma(t^+).$$

The diffeomorphisms $\delta t^- = \epsilon^-(t^-) \neq constant$ are not allowed.

The algebra of symmetries is the Virasoro-Kač-Moody algebra, not two copies of the Virasoro algebra.

Warped conformal symmetry algebra

Using canonical quantization of surface charges associated with the asymptotic symmetries, one can compute Dirac brackets.

The current-current commutators are obtained as

$$\begin{aligned}
[j^-_\sigma(t^+), j^-_\sigma(s^+)] &= \pi k_{KM}\,\partial_{t^+}\delta(t^+ - s^+), \\
[j^-_\epsilon(t^+), j^-_\sigma(s^+)] &= 2\pi j^-_\sigma(t^+)\partial_{t^+}\delta(t^+ - s^+) \\
[j^-_\epsilon(t^+), j^-_\epsilon(s^+)] &= 2\pi\partial_{t^+}\delta(t^+ - s^-)(j^-_\epsilon(t^+) + j^-_\epsilon(s^+)) \\
&\quad -\frac{\pi c_R}{6}\partial^3_{t^+}\delta(t^+ - s^+).
\end{aligned}$$

This is a Virasoro Kač-Moody algebra with central charge and level

$$c_R = \frac{3\ell}{2G}, \qquad k_{KM} = -4\Delta.$$

Global AdS_3 has $\Delta = -\frac{c_L}{24}$ (unitary representations) ;
BTZ has $\Delta > 0$ (non-unitary representations).

A warped CFT from Einstein gravity

The Hamiltonian reduction of the Einstein action using the chiral boundary conditions leads to

$$S = \int \frac{dt^+ dt^-}{32\pi G}\left(\partial_+\Phi\partial_-\Phi + 4e^{\Phi} + h((\partial_-\Phi)^2 - 2\partial_-^2\Phi - \frac{4\Delta}{k})\right)$$

$$\partial_- h = 0.$$

which is chiral Liouville theory. The Dirac brackets of the Virasoro-Kac-Moody generators are reproduced.

Einstein gravity with chiral boundary conditions is therefore equivalent to chiral Liouville theory at the semi-classical level.

Features of chiral Liouville theory

- It can be obtained from the $2d$ Polyakov action

$$S = \frac{c}{96\pi}\int d^2x\sqrt{-g}(R\,\Box^{-1}R - 2\Lambda) + \frac{\Delta}{4\pi}\int d^2x\sqrt{-g}g^{--}.$$

  in chiral gauge

$$ds^2 = -e^{2\rho}dt^+(dt^- - h(t^+)dt^+)$$

  without reference to $3d$ gravity.
- The theory is classically integrable
- The stress-tensor is not symmetric
- Chiral Liouville is a deformation of a CFT via an irrelevant $(2,1)$ operator.
- It can be coupled to matter
- It exists at the semi-classical level. No quantum theory is known.

3. Warped AdS_3 / "nearly a CFT" correspondence

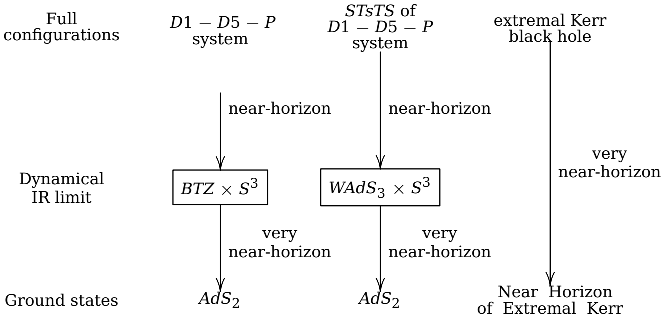
Main idea : Extend universality, with dynamics

- Black hole entropy is universal $S = \frac{A}{4G}$. Why ?
- Microscopic counting in string theory has been successful for black holes which admit AdS_3 as a near-horizon geometry. The existence of a CFT explains universality.
- One attempt for the extremal Kerr black hole is the Kerr/CFT conjecture [Guica, Hartman, Song, Strominger, 2008]
- However, the proposal faces the “no dynamics issue” [Amsel et al, 2009] [Dias et al, 2009].

We look for a middle ground : a controlled stringy holography without AdS_3 : a warped AdS_3 correspondence with a “nearly CFT” theory with universal properties.

[El-Showk, Guica, 2011]

The middle ground



Main point : It exists a consistent supergravity truncation such that

- $WAdS_3$ admits 2 Virasoros as asymptotic symmetries
- Consistent dynamics exists for matter fields

S-dual dipole truncation

Type IIB on $S^3 \times T^4$ with some fluxes lead to [Detournay, Guica, 2013]

$$L = R - 4(\partial U)^2 - \frac{4}{\ell^2} e^{-4U} A^2 + \frac{2}{\ell^2} e^{-4U}(2 - e^{-4U}) - \frac{1}{\ell} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}.$$

The theory admits the spacelike squashed warped black string solution

$$\begin{aligned} g_{\mu\nu} &= e^{4U}(\widehat{g}^{BTZ}_{\mu\nu} - A_\mu A_\nu), \\ e^{4U} &= 1 + \lambda^2 T_+^2, \qquad A = \lambda \sigma_R. \end{aligned}$$

where σ_R is a $SL(2,\mathbb{R})_R$ right-invariant 1-form.
It is a BTZ black string with deformation $\lambda > 0$.

Sector without bulk propagating modes

The black string belongs to the sector defined as

$$g_{\mu\nu} = e^{4U}(\widehat{g}_{\mu\nu} - A_\mu A_\nu), \qquad \widehat{R}_{\mu\nu} + \frac{2}{\ell^2}\widehat{g}_{\mu\nu} = 0$$
$$F_{\mu\nu} = \frac{2}{\ell}\epsilon_{\mu\nu\lambda}\widehat{g}^{\lambda\delta}A_\delta, \qquad \widehat{g}^{\mu\nu}A_\mu A_\nu = 1 - e^{-4U}.$$

One can construct this sector from a solution to vacuum Einstein gravity $\widehat{g}_{\mu\nu}$, together with one of its constant norm self-dual vector.

There are at least 2 solutions for A_μ : any linear combination of the three left-moving $SL(2,\mathbb{R})$ Killing vectors of AdS_3 with one constraint on their norm.

This sector contains the non-trivial boundary dynamics

We showed the equivalence of symplectic structures between this sector of the S-dual dipole theory and vacuum Einstein gravity,

$$\omega_{vacuum}[\delta\hat{g}, \delta\hat{g}, \hat{g}] = \omega_{\text{S-dual dipole}}[\delta g, \delta A, \delta g, \delta A, g, A].$$

Therefore, there is equivalence off-shell.

It follows that the Brown-Henneaux boundary conditions for AdS_3 lead to boundary conditions for warped AdS_3 with two Virasoro algebras.

Alternatively, chiral boundary conditions for AdS_3 lead to chiral boundary conditions for warped AdS_3 with a Virasoro-Kač-Moody algebra.

Linearized dynamics

We consider perturbations around the warped BTZ with Fourier modes

$$\delta\phi^i = e^{-i\kappa_+ t^+ - i\kappa_- t^-}\delta\phi^i(r)$$

There are 2 degrees of freedom.
The fields can be reconstructed from the solution of 2 free scalars around the BTZ black string with masses

$$m^2\ell^2 = 4(1 \pm \sqrt{1+\lambda^2\kappa_-^2}) + \lambda^2\kappa_-^2$$

The masses obeys the Breitenlohner-Freedman bound $m^2\ell^2 \geq -1$ and saturate it for $|\lambda\kappa_-| = \sqrt{3}$.
The symplectic norm in warped BTZ is equivalent to the symplectic norm of these free scalars in BTZ.

There are therefore only normal modes, no travelling waves.

The symplectic norm is finite and conserved. The phase space is well-defined. One can then argue that no dramatic backreaction will occur.

The renormalization procedure involves a non-local boundary counterterm, similar to the ones obtained in holographic renormalization of Schrödinger spacetime. [Guica, Skenderis, Taylor, van Rees, 2010]

The counterterm subtraction procedure does not introduce ghosts. Contrary to [Marolf, Andrade, 2011]

In conclusion, warped AdS_3 geometries in the S-dual dipole theory form a well-defined and self-consistent system, even with matter fields.

Semi-classical dual pairs for Einstein $3d$ gravity

(Only low bosonic spins considered. SUSY, higher spins could be added) :

	Dirichlet B.C. $\Leftrightarrow$ CFT	Dir.-Neumann B.C. $\Leftrightarrow$ warped CFT
AdS_3 (no matter)	Liouville theory	Chiral Liouville theory
(squashed) warped AdS_3 (+matter)	Deformed CFT p^μ-dependent operators no $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ inv. vacuum (relevant for Kerr/CFT)	Warped CFT p^μ-dependent operators no $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ inv. vacuum
(stretched) warped AdS_3 (+matter)	Less known (relevant for Kerr/CFT)	Less known

Take-home messages

- Euclidean Liouville theory is dual to Einstein gravity in the static patch of dS_3
- Chiral Liouville theory is an integrable warped CFT, defined semi-classically and dual to Einstein gravity in AdS_3 with chiral boundary conditions
- There is stringy truncation of Type IIB to $3d$ which admits dynamical warped AdS_3 spacetimes dual to a decoupled deformed CFT or warped CFT