Prepotential of $\mathcal{N} = 2$ conformal quiver gauge theories

Sujay K. Ashok
IMSc, Chennai

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w/E. Dell'Aquila, R. John @ IMSc
M. Billo, M. Frau, A. Lerda @ INFN, Torino

Some context

- Tremendous progress in understanding $\mathcal{N} = 2$ supersymmetric gauge theories: spectrum, correlators of non-local observables, their algebra etc. [Gaiotto-Moore-Nietzke, Cecotti-Vafa ...]
- 2d/4d connection [Alday-Gaiotto-Tachikawa]
- Relation to integrability [Nekrasov-Shatashvili-Pestun]
- Original goal of Seiberg-Witten: calculate the low energy effective action of $\mathcal{N} = 2$ gauge theories on the Coulomb branch.
- Do these advances help?
- Do a quiver case study ...

Plan

- Main goal: calculate the prepotential of Ω-deformed conformal quiver gauge theories:

$$\mathcal{L} = \frac{1}{4\pi} \Im \,\mathrm{Tr} \int d^2\theta d^2\tilde{\theta}\, F(A)\,. \tag{1}$$

- For this talk, restrict to $SU(2) \times SU(2)$ quiver.
- There are many traditional ways to obtain the prepotential.
  - Seiberg-Witten: calculate periods of the SW curve ($\epsilon_i = 0$)
  - Nekrasov: equivariant localization (Ω-deformed gauge theory)
- We use a combination of Seiberg-Witten theory and AGT to calculate the prepotential in the Nekrasov-Shatashvili limit ($\epsilon_2 = 0$).
- Goal:

$$F(q_i, a_i) = \sum_{q_i} q_1^{k_1}\, q_2^{k_2} \ldots F_{k_1,k_2,\ldots}(a_i) \tag{2}$$

- Some nice results and many interesting directions to explore...

Set-up

- We are interested in looking at "linear conformal quivers".

$$\boxed{2} \longrightarrow 2 \longrightarrow 2 \longrightarrow \cdots\cdots \longrightarrow 2 \longrightarrow \boxed{2}$$

- At each node, we have an $SU(2)$ with 4 flavours and so the gauge theory is conformal.
- When all masses are set to zero, the uv-curve for this theory takes a particularly simple form (double cover of a punctured Riemann sphere) [Gaiotto '09].
- For $SU(2) \times SU(2)$, we obtain (via M-theory):

$$x^2 = \phi_2(t) = \frac{U_1 t + U_2}{t(t-1)(t-q_2)(t-q_1 q_2)} \tag{3}$$

- $\phi_2(t)$ is a quadratic differential on C.
- $q_i = e^{2\pi i \tau_i}$, where τ_i is the uv coupling of the ith gauge group.
- $\lambda_{SW} = x\, dt$ is the SW differential.

Set-up

- Equivalent way to write the curve:

$$x^2 = \frac{U_1 q_2 t + U_2}{t(t-1)(t-q_1)(t-q_2^{-1})} \tag{4}$$

As q_i go to zero, or at weak coupling, the punctured sphere becomes [Gaiotto]:

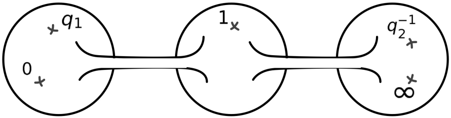

Figure: Punctured sphere in the weak-coupling limit

- Naturally defines A_1 and A_2 cycles in the weak-coupling limit.
- Find symplectic basis $A_i \cap B_j = \delta_{i,j}$.

Periods and Seiberg-Witten theory

- The period integrals are therefore defined to be

$$a_1 = \frac{1}{\pi i}\int_0^{q_1} x(t, U_i)\, dt \qquad a_2 = \frac{1}{\pi i}\int_{q_2^{-1}}^{\infty} x(t, U_i)\, dt \tag{5}$$

- Usually,
  - Calculate both these integrals.
  - Invert them to get $U_i(a_i)$.
  - Do the B_i-integrals:

$$\frac{\partial F}{\partial a_i} = \frac{1}{2\pi i}\int_{B_i} x(t, U_i(a_i))\, dt \tag{6}$$

    and integrate to get the prepotential.
- However there is a much easier way to do this if we know how U_i are related to $\frac{\partial F}{\partial q_i}$ directly.
- Analogous to finding Matone's relation.

Periods and Seiberg-Witten theory

- Claim [Krichever '94, Marshakov, Mironov, Morozov]:

$$\boxed{\mathrm{Res}_{t_i}\ \phi_2(t) = \frac{\partial F}{\partial t_i}\,.} \tag{7}$$

- This will be proved later using AGT.
- What this means is that we can write the curve as

$$x^2(t) = \frac{(q_1-1)F_1}{t(t-1)(t-q_1)} + \frac{(1-\frac{1}{q_2})F_2}{t(t-1)(t-q_2^{-1})} \tag{8}$$

where $F_i = q_i \frac{\partial F}{\partial q_i}$.

- From the period calculation $a_i = \oint_{A_i} x(t)dt$,

$$a_i(F_1, F_2) \sim \sqrt{F_i} + \ldots \tag{9}$$

- Invert (order by order in q_i) and integrate w.r.t q_i.

$$F \sim \sum_i \log(q_i) a_i^2 + \ldots \tag{10}$$

- Very easy to calculate the prepotential as a series in q_i.

The answer

$$F(a_i, q_i) = F_{\text{class.}} + \boxed{F_{\text{1-loop}}} + \sum_{m,m} q_1^m q_2^n \; F_{m,n} \,. \qquad (11)$$

$$\begin{aligned} F_{\text{inst}}(a_i, q_i) &= \frac{q_1}{2}(a_1^2 - a_2^2) + \frac{q_2}{2}(a_2^2 - a_1^2) + \frac{q_1 q_2}{4}(a_1^2 + a_2^2) \\ &+ \frac{q_1^2}{64 a_1^2}(13 a_1^4 - 14 a_1^2 a_2^2 + a_2^2) + \frac{q_2^2}{64 a_2^2}(13 a_2^4 - 14 a_1^2 a_2^2 + a_1^2) \\ &\frac{q_1^3}{192 a_1^2}(23 a_1^4 - 26 a_1^2 a_2^2 + 3 a_2^4) + \frac{q_2^3}{192 a_2^2}(23 a_2^4 - 26 a_1^2 a_2^2 + 3 a_1^4) \\ &+ \frac{q_1^2 q_2}{64 a_1^2}(3 a_1^4 - 2 a_1^2 a_2^2 - a_2^4) + \frac{q_1 q_2^2}{64 a_2^2}(3 a_2^4 - 2 a_1^2 a_2^2 - a_1^4) + \ldots \end{aligned} \qquad (12)$$

Completely symmetric if we exchange $(a_1, q_1) \leftrightarrow (a_2, q_2)$ and this also matches with Nekrasov calculation.

Summary so far:

- Derive uv curve from M-theory with non-zero masses [Witten '97].
- Find A and B cycles in the weak-coupling limit [Gaiotto].
- Use residue fomula to write the curve in terms of F_i [Marshakov et al.].
- Do *only* the A-periods.
- Invert and integrate to find $F(a_i, q_i)$. Works easily for any linear quiver.
- We can do independent checks: a) Nekrasov b) Thomae formula for roots in terms of genus-2 theta functions. This also validates our expression for the SW curve.

Directions

- One could add masses now and re-calculate periods. Harder this time, different techniques (Picard-Fuchs equations).
- One could turn on ϵ_1, keeping $\epsilon_2 = 0$ (Nekrasov-Shatshvili limit) and ask how one would calculate the corrections to the prepotential.
- Will not review the Ω-deformation. Two parameter deformation of SYM theory: can be understood as a twisted compactification of $\mathcal{N} = 1$ SYM in $d = 6$ on T^2. Breaks Lorentz invariance in $d = 4$.
- It is crucial for the AGT correspondence.
- **Next stop**: Calculate ϵ_1-deformed prepotential.

AGT

$$\left\langle \prod_{i=1}^{n} V_{m_i} \right\rangle_{a_i} = Z_{U(1)}\, Z_{\text{inst}}(a_i, m_i, \epsilon_1, \epsilon_2)\,. \tag{13}$$

- LHS is the conformal block of Liouville CFT in $d = 2$, with a specific pair of pants decomoposition.
- Z_{Nek} is the instanton partition function of a conformal $SU(2)^{n-3}$ quiver.

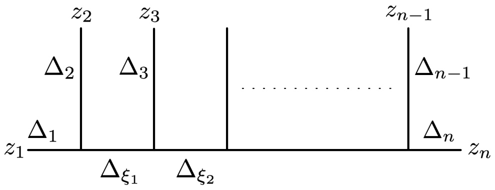

Figure: Pair-of-pants decomposition of conformal block

Map of parameters [AGGTV '09]

- All conformal dimensions are measured in units of $\hbar$. The Liouville central charge is $c = 1 + 6Q^2$, with $Q = b + 1/b$. These map to

$$\epsilon_1 = b\hbar \qquad \epsilon_2 = \frac{\hbar}{b} \tag{14}$$

- The NS limit is taking $\hbar \to 0$ and $b \to 0$ keeping

$$\epsilon_1 = \frac{\hbar}{b} \quad \text{fixed}\,. \tag{15}$$

- The Coulomb parameters map to internal momenta:

$$\hbar\xi_i = a_i + \frac{1}{2}(\epsilon_1 + \epsilon_2)\,. \tag{16}$$

- In the massless limit, all external Δ_k are equal and given by

$$\Delta_0 = \frac{Q^2}{4} = \frac{(\epsilon_1 + \epsilon_2)^2}{4\epsilon_1\epsilon_2} \to \frac{\epsilon_1}{4\epsilon_2}\,. \tag{17}$$

The SW curve

According to AGT, the quantum deformed SW curve is given by

$$x^2 = \phi_2^q(z) = \frac{\langle T(z) \prod_{i=1}^n V_{\alpha_i} \rangle}{\langle \prod_{i=1}^n V_{\alpha_i} \rangle} \ . \tag{18}$$

The classical curve is given by the limit

$$\lim_{\epsilon_{1,2} \to 0} (-\epsilon_1 \epsilon_2 \phi_2^q(z)) = \phi_2(z) \, . \tag{19}$$

To evaluate the RHS we make use of the conformal Ward identities and write it as

$$\phi_2^q(z) = \frac{1}{\langle \prod_{i=1}^n V_{\alpha_i} \rangle} \sum_{i=1}^n \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \langle \prod_{i=1}^n V_{\alpha_i} \rangle \ . \tag{20}$$

The SW curve

- $$\phi_2^q(z) = \frac{1}{\langle \prod_{i=1}^n V_{\alpha_i} \rangle} \sum_{i=1}^{n} \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle \prod_{i=1}^{n} V_{\alpha_i} \rangle \ . \tag{21}$$
  We now use invariance of the chiral conformal block under the $L_{-1,0,1}$ generators.
- We also set three points to $(0, 1, \infty)$. The remaining points are set to q_1 and q_2^{-1}.
- This allows us to solve for the three derivatives at $(0, 1, \infty)$ in terms of $\frac{\partial}{\partial q_i}$.
- Lastly, we set $\langle \prod_i V_i \rangle \sim e^{-\frac{F}{\epsilon_1 \epsilon_2}}$ to get the SW curve:

$$\boxed{\phi_2(z) = \frac{(q_1-1) q_1 \frac{\partial F}{\partial q_1}}{z(z-1)(z-q_1)} + \frac{(1-\frac{1}{q_2}) q_2 \frac{\partial F}{\partial q_2}}{z(z-1)(z-\frac{1}{q_2})} \ .} \tag{22}$$

Null-vectors vs. surface operators

- Claim [AGT-Gukov-Verlinde, Drukker-Gomis-Okuda-Teschner]

$$\Psi(z) := \Big\langle \prod_{i=1}^{n} V_{m_i}\, \Phi_{2,1}(z) \Big\rangle_{a_i} = Z_{U(1)}\, Z_{\text{inst},S}(a_i, m_i, \epsilon_1, \epsilon_2; z)\,. \tag{23}$$

- On the LHS, $\Phi_{2,1}(z)$ is a degenrate field of the Virasoro algebra, satisfying the null-vector condition

$$(\frac{1}{b^2} L_{-1}^2 + L_{-2})\Phi_{2,1} = 0\,. \tag{24}$$

- On the RHS, $Z_{\text{inst},S}$ is the instanton partition function of the quiver gauge theory in the presence of a surfce operator. Operationally, what this means is that in the NS limit:

$$Z_{\text{inst},S}(a_i, m_i, \epsilon_1, \epsilon_2; z) = {}^{-\frac{F}{\epsilon_1 \epsilon_2} - \frac{1}{\epsilon_1} W(z) + \dots} \tag{25}$$

Null-vector decoupling equation

- Plan: use null-vector decoupling to write a Schrodinger type equation for the "wavefunction" $\Psi(z)$.
- Use the AGGVT ansatz for $\Psi(z)$ and take the $U(1)$ factor to be such that

$$\Psi(z) = q_1^{\cdots} q_2^{\cdots} (1-q_1)^{\cdots} (1-q_2)^{\cdots} (1-q_1 q_2)^{\cdots} \boxed{e^{-\frac{F_{\text{inst}}}{\epsilon_1 \epsilon_2} - \frac{W(z)}{\epsilon_1} + \cdots}} \tag{26}$$

- $$\left(\frac{1}{b^2} L_{-1}^2 + L_{-2}\right) \langle \Phi_{2,1}(z) \prod_i V_i \rangle = 0 \,. \tag{27}$$
- Take the NS limit: the equation simplifies to the Schrodinger form with ϵ_1 playing the role of $\hbar$ and an $\hbar$-dependent potential:

$$\left(-\epsilon_1^2 \frac{d^2}{dz^2} + V(z, \epsilon_1) \right) \Psi(z) = 0 \,, \tag{28}$$

- The potential is of the form:

$$V(z, \epsilon_1) = \phi_2(z) + \epsilon_1^2 \, V_2(q_i, z) \tag{29}$$

Prepotential from Schrodinger equation: WKB analysis

- Use a WKB ansatz for the wavefunction:

$$e^{-\frac{1}{\epsilon_1}W(z)} =: \Phi(z,\epsilon_1) = e^{-\frac{1}{\epsilon_1}\int^z dz\, P_0(z)+\epsilon_1 P_1(z)+\ldots} \quad (30)$$

- Substituting the ansatz we find that $P_0 = \sqrt{\phi_2(z)}$: the zeroth order wavefunction is just [GMN]

$$\Phi_0(z) = e^{-\frac{1}{\epsilon_1}\int^z \lambda_{SW}}\,. \quad (31)$$

- This observation provides a basis for a natural extension of SW theory to the ϵ_1-deformed theory:

$$\Phi(z+A_i) = e^{\frac{2\pi i a(\epsilon_1)}{\epsilon_1}}\Phi(z) \qquad \Phi(z+B_i) = e^{\frac{2\pi i a_D(\epsilon_1)}{\epsilon_1}}\Phi(z)\,. \quad (32)$$

- Quantum monodromy conditions derived from fusion and braiding in Liouville [DGOT, AGGTV '09]

Ω-deformed prepotential from null-vector decoupling

- The (semi-classical) monodromy conditions imply

$$a_i(\epsilon_1) = -\frac{1}{2\pi i}\oint_{A_i}\left(P_0(z) + \epsilon_1 P_1(z) + \epsilon_1^2 P_2(z) + \ldots\right) \quad (33)$$

- The first term is just the undeformed period. The claim is that the successive WKB corrections to the wavefunction calculates the deformed SW period.
- If we expand the periods as

$$a_i(\epsilon_1) = \sum_{k=0}^{\infty} \epsilon_1^k\, a_i^{(k)}\,, \quad (34)$$

the first order term is zero. At second order, we find

$$a_1^{(2)} = -\frac{3q_1}{16\sqrt{F_1}} - \frac{1}{128 F_1^{\frac{5}{2}}}\left(q_1^2\,(17F_1^2 - 7F_1F_2 + 2F_2^2) + 40 q_1 q_2\, F_1^2\right) + \ldots \quad (35)$$

Ω-deformed prepotential from null-vector decoupling

- All odd powers vanish in this case; the next non-zero term is

$$a_1^{(4)} = \frac{q_1^2}{1024 F_1^{\frac{7}{2}}}\left(-19F_1^2 + 8F_1F_2 - 16F_2^2\right) + \dots \qquad (36)$$

- Similarly $a_2^{(k)}$ is obtained by exchanging (q_1, F_1) and (q_2, F_2). As before, invert and we obtain $F_i(a_i, q_i)$. Integrating w.r.t q_i, we obtain $F(a_i, q_i)$.

- The result matches with the Nekrasov calculation (which is much harder to do) up to 4 instantons.

- So, in the NS limit, the ϵ_1-corrected prepotential is obtained from a wavefunction that solves a quantum problem with $\phi_2(z)$ as the potential (+corrections), which satisfies monodromy conditons.

Things to do

- Doing the massive case turns out to be useful: our earlier claim needs to be modified

$$\mathrm{Res}_{t_i} x^2(t) = \frac{\partial F}{\partial t_i} + \text{corrections} \tag{37}$$

- The corrections include AGT-prefactors (now mass-dependent).
- The SW curve with masses obviously distinguishes (m_1, m_2) and (m_3, m_4). For a single gauge group, how to restore this?
- Difficult to do the general massive quiver case by factorizing polynomials. Calculate periods using Picard-Fuchs equations.

Take-home messages

- AGT helps to solve for the prepotential of undeformed conformal quiver theories by giving you Matone's relation.
- The instanton partition function is the wavefunction of a quantum mechanical system in the NS limit. This follows from null-vector decoupling in the Liouville CFT.
- The monodromy relations give you deformed periods of the gauge theory.

More interesting things to do

- How about gauge theories with no AGT-dual?
- Nekrasov-Pestun-Shatashvili have written down SW curves for any Ω-deformed (conformal) quiver gauge theory as difference equations.
- In the NS limit, for the $(SU(2))^n$ quiver, these reduce to Ward identities of Liouville theory. So this method could be applied to any gauge theory to calculate the prepotential.
- Does a WKB type solution still work?

More interesting things to do

- What about null vectors at higher level? Can they be used to solve for the prepotential along similar lines?
- What about W_N algebras for higher rank quivers?

Even more interesting things to do

- The calculation of ϵ_1 corrections reduced to a QM problem, with $\hbar = \epsilon_1$. Are there non-perturbative $O(e^{-\frac{1}{\epsilon_1}})$ corrections to the prepotential? Interpretation?
- How does one include ϵ_2 corrections systematically?

From the prepotential to the SW curve

- According to SW, the data of the low energy effective action is encoded in the SW curve. In particular,

$$\tau_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} \tag{38}$$

is the period matrix of the SW curve. Recall that the SW differential is

$$\lambda = x(t)dt = \sqrt{\frac{u_2 t + u_1}{t(t-1)(t-q_1)(t-1/q_2)}}\, dt \ . \tag{39}$$

Differentiating w.r.t the u_i we get

$$\frac{\partial \lambda}{\partial u_1} = \frac{dt}{y} \ , \quad \frac{\partial \lambda}{\partial u_2} = \frac{t\, dt}{y} \ , \tag{40}$$

where

$$y^2 = t(t-1)(t-q_1)(t-1/q_2)(t+u_1/u_2) \ . \tag{41}$$

For the quiver ...

- The SW curve is of genus two:

$$y^2 = t(t-1)(t-\zeta_1)(t-\zeta_2)(t-\zeta_3)\ . \tag{42}$$

- Suppose you know the period matrix of the curve τ_{ij} [IR].
- There are expressions for the roots in terms of theta constants of the genus-2 curve with given period matrix.
- But we know that $\zeta_1 = q_1$ and $\zeta_2 = q_2^{-1}$ etc. [uv]
- There are three dimensionless uv parameters: $(q_1, q_2, \frac{a_1}{a_2})$. There are three IR parameters τ_{ij}. The theta constants provide the map between these two sets.

Geometry of the SW curve

- Given the period matrix τ, define

$$Q_1 = e^{i\pi\tau_{11}} \ , \quad Q_2 = e^{i\pi\tau_{22}} \ , \quad \hat{Q} = e^{i\pi\tau_{12}} \ . \tag{43}$$

The genus-2 theta-constants are defined as follows:

$$\theta\left[\begin{array}{c} \epsilon \\ \epsilon' \end{array}\right] \equiv \sum_{k\in\mathbb{Z}^2} e^{\pi i\left[(k+\epsilon/2)^T\,\tau\,(k+\epsilon/2)+(k+\epsilon/2)^T\epsilon'\right)]} \ , \tag{44}$$

where ϵ, ϵ' are two 2-vectors.

- The cross ratios $\zeta_1, \zeta_2, \zeta_3$ can be written in terms of theta-constants:

$$\zeta_1 = \frac{\theta^2\left[\begin{array}{c} 10 \\ 00 \end{array}\right]\theta^2\left[\begin{array}{c} 11 \\ 00 \end{array}\right]}{\theta^2\left[\begin{array}{c} 01 \\ 00 \end{array}\right]\theta^2\left[\begin{array}{c} 00 \\ 00 \end{array}\right]} \ , \quad \zeta_2 = \frac{\theta^2\left[\begin{array}{c} 10 \\ 00 \end{array}\right]\theta^2\left[\begin{array}{c} 00 \\ 11 \end{array}\right]}{\theta^2\left[\begin{array}{c} 01 \\ 00 \end{array}\right]\theta^2\left[\begin{array}{c} 11 \\ 11 \end{array}\right]} \ . \tag{45}$$

- Is this consistent if we input the τ_{ij} obtained from the prepotential .. ? YES! $\zeta_1 = q_1$ and $\zeta_2 = q_2^{-1}$. $\zeta_3 \sim -\frac{u_1}{u_2}$.