Entanglement Rényi Entropies & their Duals

Indian Strings Meeting 2014 @ Puri

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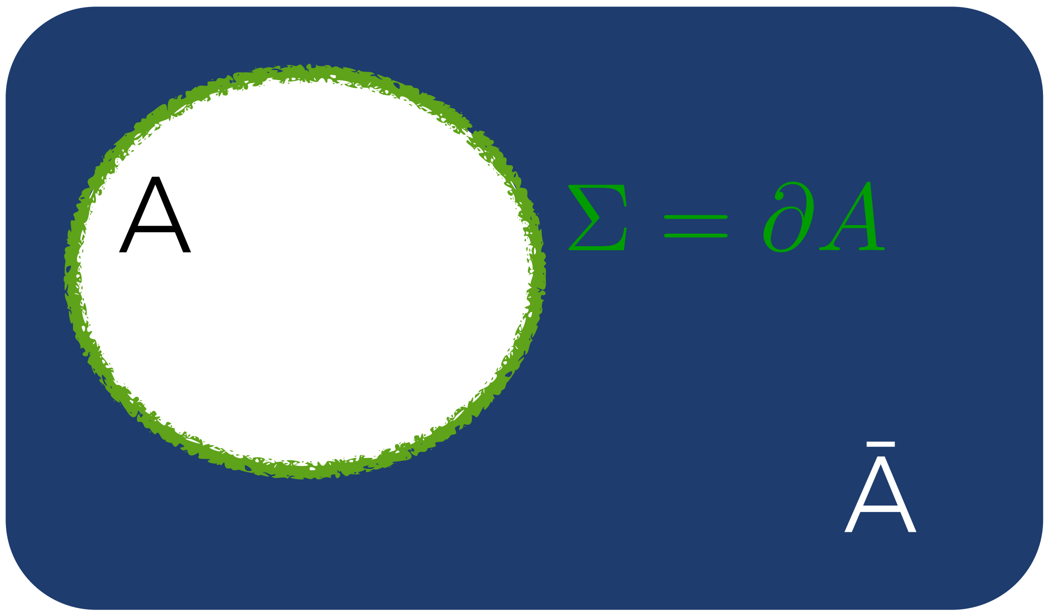
based on work with M. Crossley and E. Dyer
also: Huang & Zhou

entanglement entropy (variatio IV)



$t = t_0$

- Divide system into **spatial** region A and its complement Ā

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$$

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_{\overline{A}}} \rho$$

$$S(A) = -\mathrm{Tr}\rho_A \ln \rho_A$$

e.g.

$$\rho = |0\rangle\langle 0|$$

In QFT this is a (UV) **cutoff** dependent quantity

$$S(A) = c_2 \frac{R^{d-2}}{\delta^{d-2}} + \cdots + c_d \ln\left(R/\delta\right) + \cdots \qquad \text{(even d)}$$

area law

universal log term

why do we care?

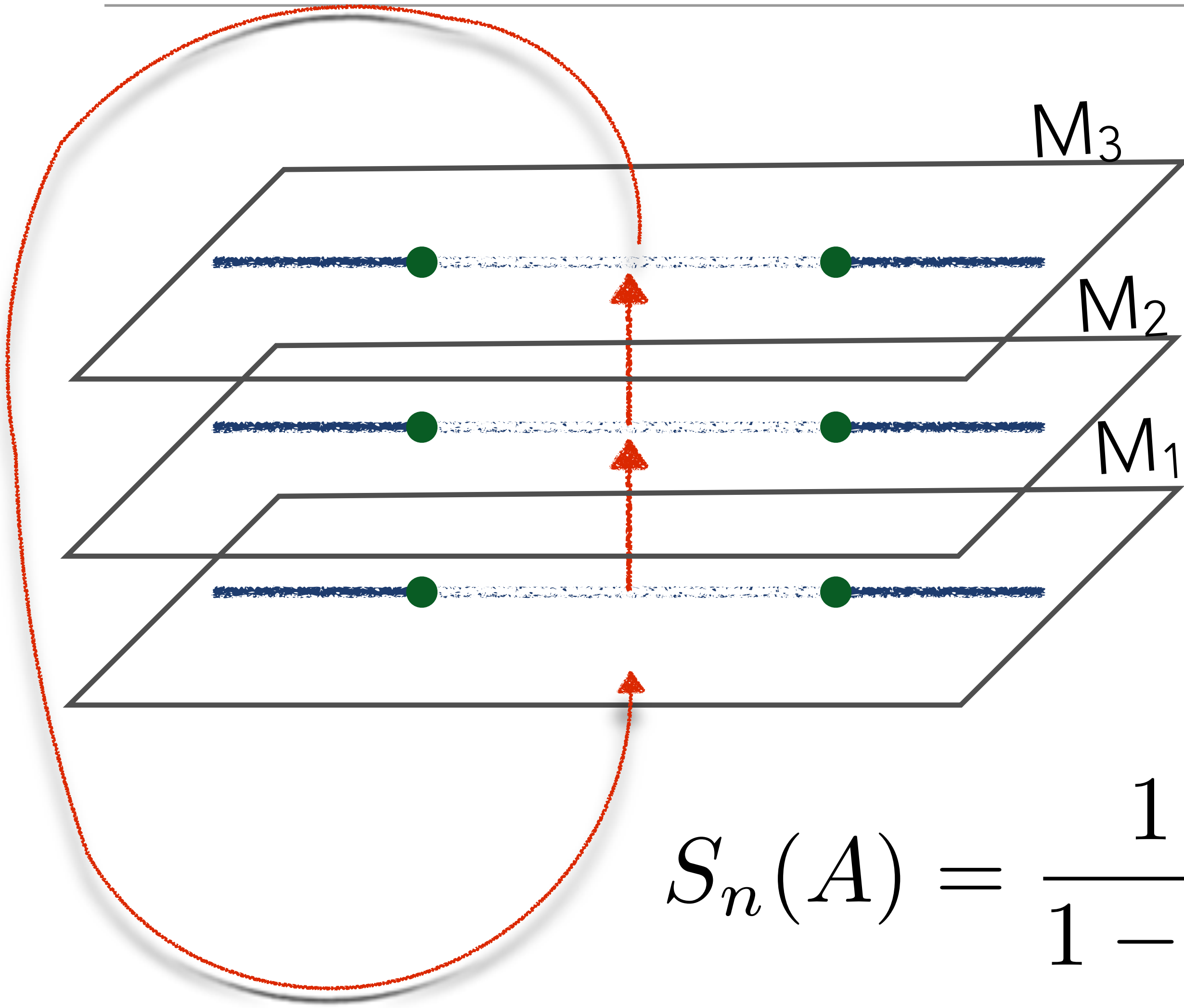
- quantum information & gravity

  - AdS/CFT: [RT] formula allows to probe **geometry**
  - Can be developed into derivation* of gravity [Raamsdonk, Myers…]
  - Qualitative considerations (e.g. tensor networks, ER = EPR,…) [Swingle; Maldacena, Susskind; …]

- important in many contexts as a **diagnostic** tool

  - C functions in odd d [Jafferis; Pufu…]
  - quantum computation
  - quantum quenches [Calabrese, Cardy;…]
  - topological order (phases beyond LGW)

entanglement basics



- Computing log(reduced density) matrix directly is very difficult

$$S(A) = -\mathrm{Tr}\rho_A \ln \rho_A$$

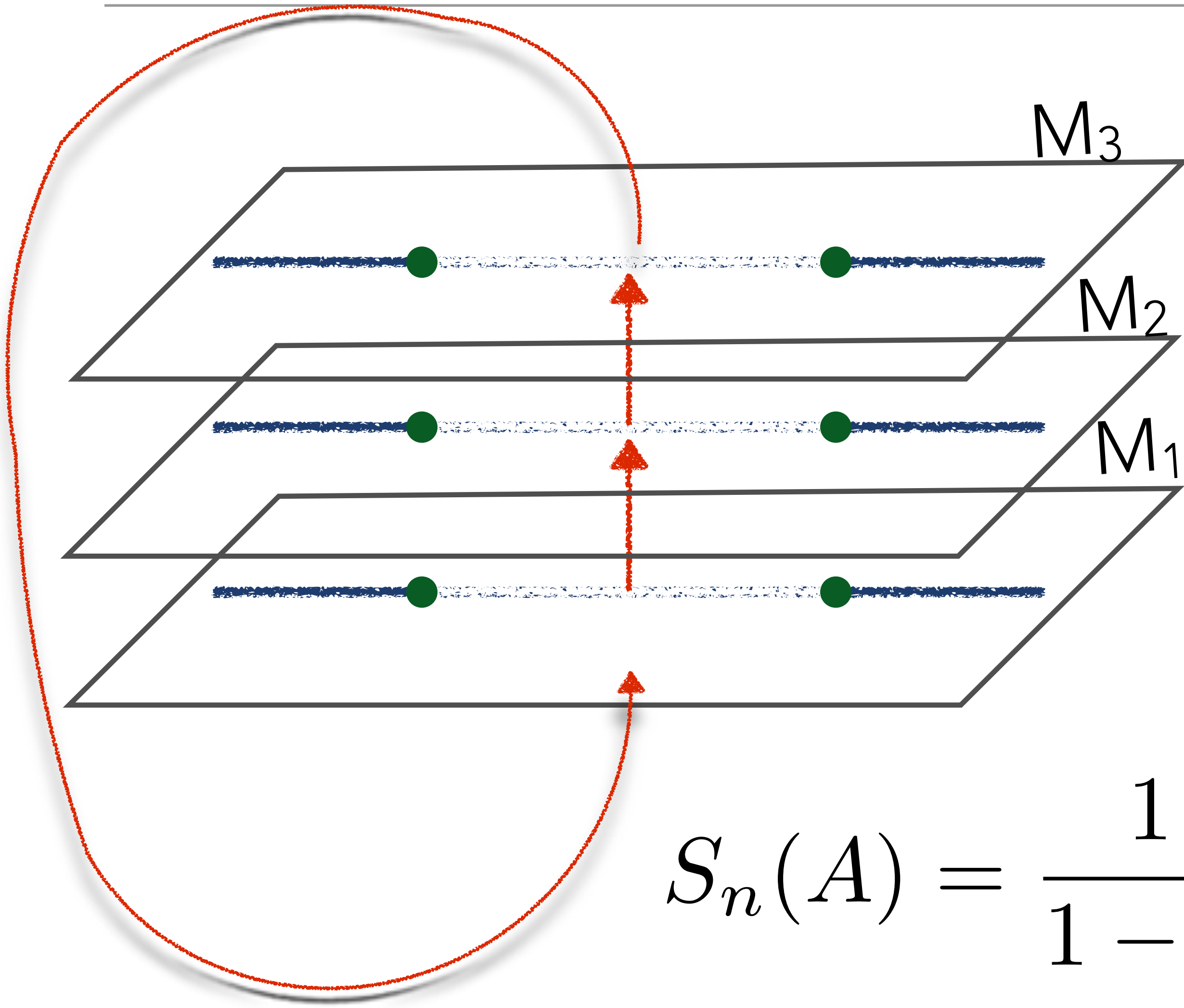
- Common technique: replica trick

$$S_n(A) = \frac{1}{1-n} \ln \mathrm{Tr}\rho_A^n$$

- Compute at integer n via path integral on ‘replicated space’, then continue to non-integer values: $S(A) = \lim_{n \to 1} S_n(A)$

- This talk - some theories are more controlled [Nishioka & Yaakov]: SUSY & localization & holography to get exact results in N=4 SYM

entanglement basics



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conical singularities

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- This talk - some theories are more controlled [Nishioka & Yaakov]: SUSY & localization & holography to get exact results in N=4 SYM

outline

1. entanglement & Rényi in N=4
   “a non-singular partition function & supersymmetric S_n”

2. exact computation (all orders in N and λ)
   “supersymmetric partition function of N=4 on ellipsoid”

3. gravity dual
   “IIB strings & hyperbolic BPS black holes”

4. conclusions and outlook
   “observations, questions, extensions”

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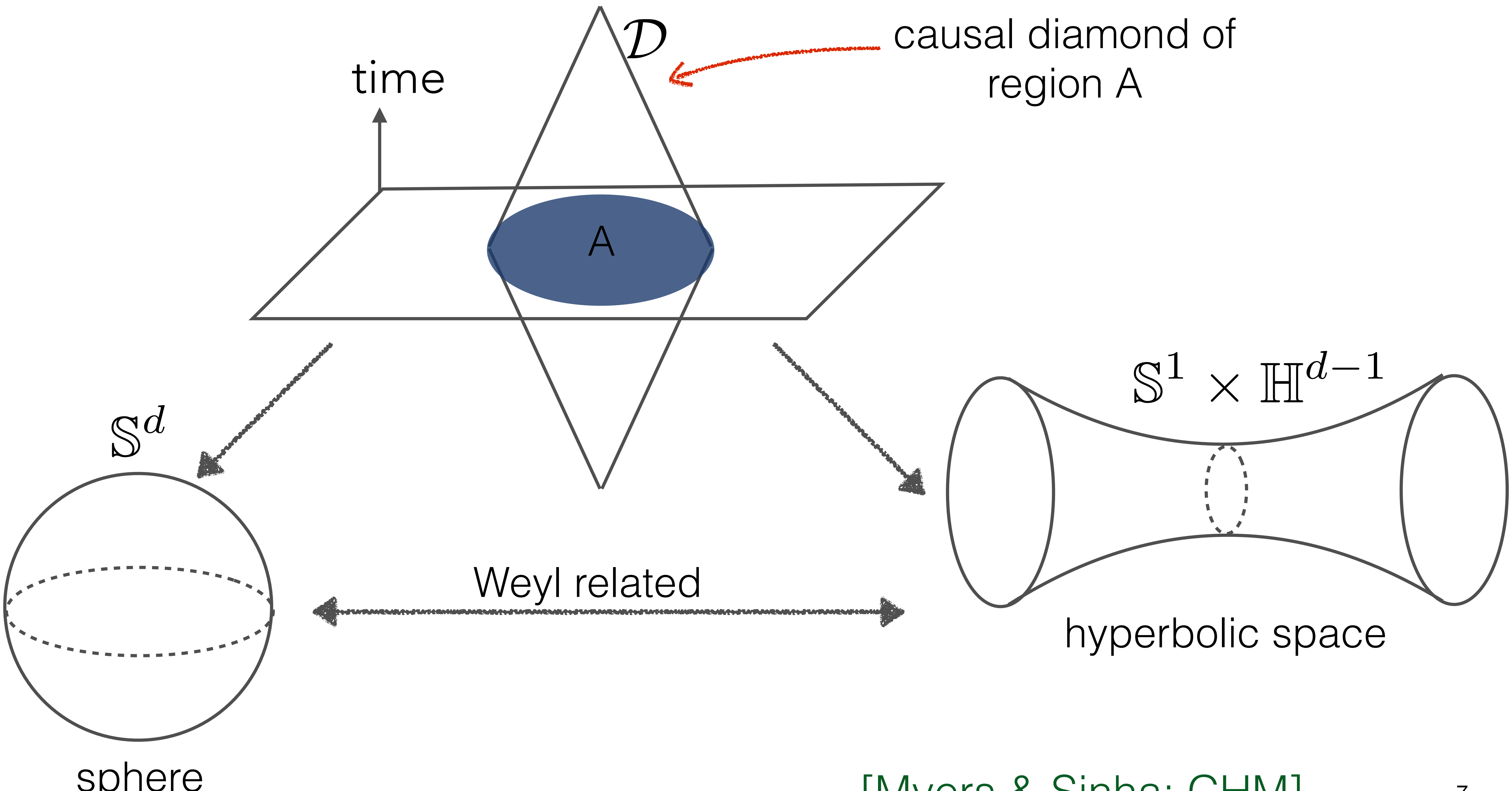
ERE under conformal maps

- Entanglement-Rényi entropy (ERE) for ball-shaped region simplifies



[Myers & Sinha; CHM]

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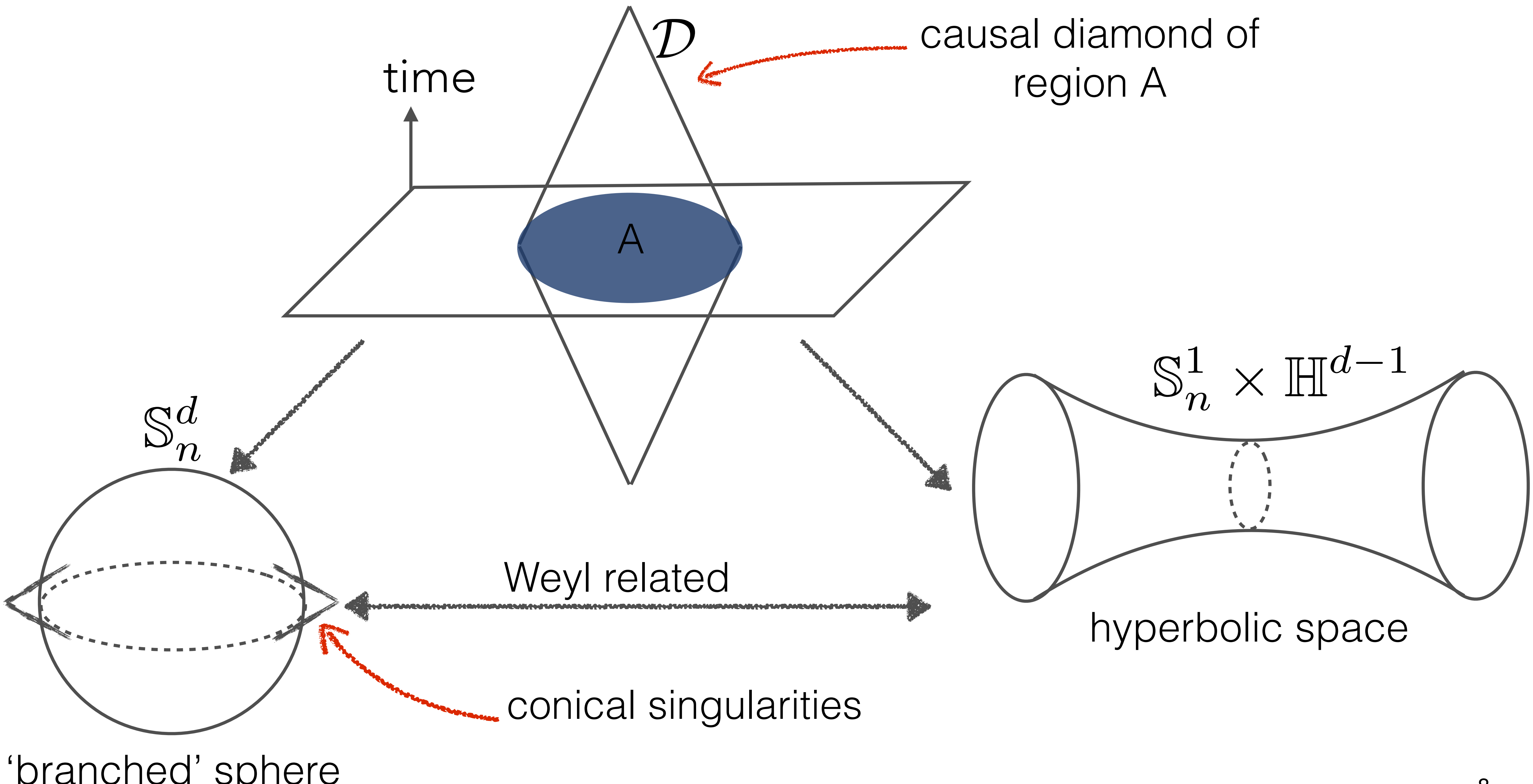
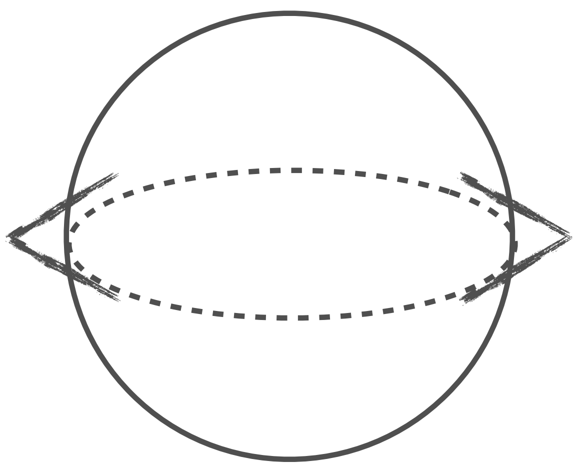


geometry: branched spheres

$\mathbb{S}^d_n$



'branched' sphere

$$ds^2_n = \left(d\theta^2 + n^2\cos^2(\theta)d\tau^2 + \sin^2(\theta)d\Omega^2_{d-2}\right)$$

$$\theta \in [0, \frac{\pi}{2}] \quad , \quad \tau \in [0, 2\pi)$$

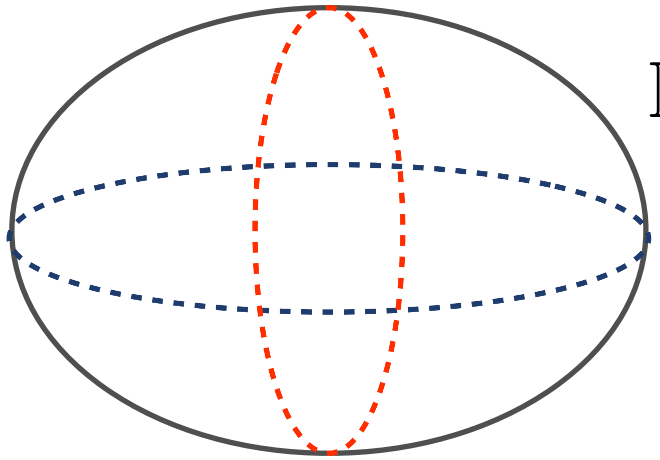
$f_\epsilon(\theta)$ $\Big\downarrow$ smoothing function

$$ds^2_{\epsilon,n} = \left(f_\epsilon(\theta)d\theta^2 + n^2\cos^2(\theta)d\tau^2 + \sin^2(\theta)d\Omega^2_{d-2}\right)$$

for the choice

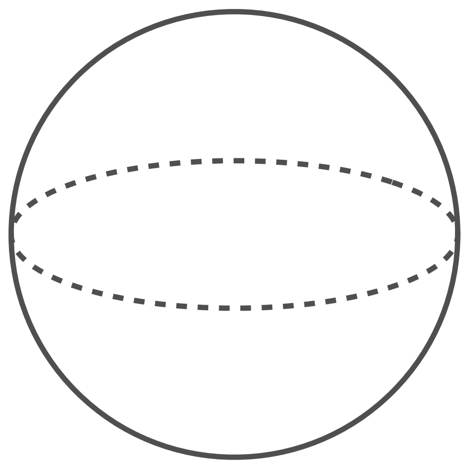
$$f(\theta) = n^2\sin^2(\theta) + \cos^2(\theta)$$

we get a **smooth** ellipsoid



$\mathbb{E}^d(n,1)$

SUSY on curved space



Define N=4 (N=2*) on S^{4}: derive corrections order by order (& use localization to get <W>) [Pestun]

Systematics: obtain rigid SUSY on curved spaces by coupling to off-shell SUGRA [Festuccia & Seiberg]

$$\mathcal{L}_{\mathbb{R}^4} \to \mathcal{L}_{\mathcal{M}}\left[\eta \to g\right] + \delta\mathcal{L}_{\mathcal{M}}$$

Here: couple to N=2 off-shell SUGRA

Solve KSE for background fields

Was done for ellipsoid by [Hama & Hosmichi]: must couple to scalar (M), SU(2) current (V), (anti-) self-dual tensor (T,$\overline{T}$)

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the partition function

using this [Hama & Hosomichi] worked* out Z(ellipsoid) for a general N=2 theory

$$Z = \int da e^{-\frac{16\pi^2}{g_{YM}^2}\mathrm{Tr}(a^2)} |Z_{\mathrm{inst}}|^2 \prod_{\alpha\in\Delta^+} \frac{\Upsilon(ia\cdot\alpha)\Upsilon(-ia\cdot\alpha)}{\Upsilon(ia\cdot\alpha+\frac{Q}{2})^2}$$

Cartan of SU(N)

Nekrasov partition function

$$Q = \sqrt{n} + \frac{1}{\sqrt{n}}$$

$$\Upsilon(x) = \prod_{p,q\geq 0}\left(p\sqrt{n}+\frac{q}{\sqrt{n}}+Q-x\right)\left(p\sqrt{n}+\frac{q}{\sqrt{n}}+x\right)$$

*localization: add Q-exact operator: path integral is exactly given by saddle-point + fluctuation determinants → finite matrix integral

Wigner comes to the aid of Rényi

At large N, saddle point & continuum limit → integral equation for eigenvalue density ρ(x)

The solution is the famous Wigner semi-circle distribution

$$\rho(x) = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2}, \qquad \mu = \frac{\sqrt{\lambda}Q}{4\pi}$$

SUGRA limit of super Rényi entropy

$$F = -N^2\frac{Q^2}{4}\log\left(R/\delta\right) - \frac{N^2}{8}Q^2\ln\lambda + \ldots$$

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universal term

finite part (depends on scheme)

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Universal part of ERE λ- independent ~ Weyl anomaly on ellipsoid
(previously known how to relate EE to anomaly, but not ERE - SUSY helps)

summary

1. map ERE[n] of N=4 to Z(branched sphere[n])

2. add background fields to preserve (subset of) SUSY
   → Z(branched sphere [n]) ≃ Z(ellipsoid[n])

3. use localization to evaluate Z(ellipsoid[n])
   → universal part a[n] for all N,λ

4. use Wigner to find SUGRA limit in a particular scheme

WANTED dual supergravity solution ◆ must have 4 supercharges ◆ must be a solution to type IIB ◆ possibly has hyperbolic horizon **WANTED**

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strategy in gravity

1. find dual solution for any n
2. use gauge-gravity to compute Z_n
3. $S_n(A) = \frac{1}{1-n} \ln \frac{Z_n}{Z_1^n}$
4. extract universal part (log divergence)
5. match scheme and compare finite part

the black hole solution

hints:
branched sphere: tensor backgrounds $T, \overline{T}$ vanish, rest nontrivial
but: space is singular → focus on hyperbolic 'Weyl frame'

↔ look for BH with hyperbolic horizon & SU(2) gauge field

hyperbolic black hole in Romans N=4⁺ theory

$$ds_5^2 = -\frac{f(r)}{H(r)^{4/3}}dt^2 + H(r)^{2/3}\left(\frac{dr^2}{f(r)} + r^2 ds^2(\mathbb{H}^3)\right)$$

$$A^{I=3} = \left[i\sqrt{2(1-\tfrac{m}{q})}(1-H^{-1}) + \mu\right] dt$$

$$H(r) = 1 + \frac{q}{r^2}$$

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BPS → m = 0

gravity computation of partition function

have fixed (a) gravity solution with the right
- (super)symmetry
- R-symmetry
- tunable temperature: get any Rényi index we want

chemical potential / charge: compute Gibbs free energy F(β[n], μ[n])

$$F(\beta[n], \mu[n]) = I_E(\beta[n], \mu[n])$$

as in field theory, so in holography: regulate & renormalize:

$$I_E = \lim_{\Lambda \to \infty} \left[\int_{r_h}^{\Lambda} d^5x \sqrt{g_E} \mathcal{L}_E + I_{\mathrm{GH}}(\Lambda) + I_{\mathrm{ct}}(\Lambda) \right]$$

gravity computation of partition function

only contribution comes from IR end of integral (i.e. the horizon)

$$I_n = -\frac{1}{4} Q^2 \frac{\pi}{2g^3 G_N^{(5)}} \ln\left(2R/\delta_{\mathrm{SG}}\right)$$

∞ volume of $\mathbb{H}^3$

comment on RG scheme: relate UV cutoff in field theory and gravity by matching fixed physical quantity (e.g. quark mass)

$$I_n = -\frac{Q^2}{4} N^2 \ln\left(R/\delta\right) - \frac{Q^2}{8} N^2 \ln \lambda$$

from 'dictionary'

RG scheme

$R/\delta_{\mathrm{SG}} \sim R/\delta\sqrt{\lambda}$

comments, extensions

1. limit n → 1 recovers ‘usual EE’
2. universal part comes from anomaly: contrast with ‘usual ERE’
3. can add fundamental quark, localization still goes through, as does gravity calculation

$$\ln W_n = \frac{(n+1)}{2} \ln W_1$$

subtlety: limit n → 1 does not recover ‘usual EE’ for heavy quark, different ensemble!

can take λ derivative to recover correct ‘usual EE’ [Lewkowycz, Maldacena]
at large N, λ recover SUGRA calculation of [Jensen, Karch ; JS]

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conclusion & outlook

ERE typically hard to calculate, especially in strongly-coupled QFT

sERE is a related quantity that is exactly calculable
(and we did so in terms of ellipsoid partition function for N=4 SYM)

Observations:

1) parameter map

$$\frac{Q^2}{4} \longleftrightarrow 1 + M^2$$

gives result of massive N=2* theory on S^4, whose gravity dual can be constructed numerically [Bobev, Elvang, Freedman, Pufu]
→ extend to exact duality?

2) sEE is not in all cases equivalent to EE.
what does sERE compute?
can one always recover EE from sERE (in our case, yes)

conclusion & outlook

precise understanding of Weyl anomaly

N=2* theory on ellipsoid & its dual?

knowledge of Rényi entropy gives entanglement spectrum
most detailed information about state available from entanglement

→ can we do a similar thing using sERE?

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thank you for your attention