Functional determinants, Index theorems, and Exact quantum black hole entropy

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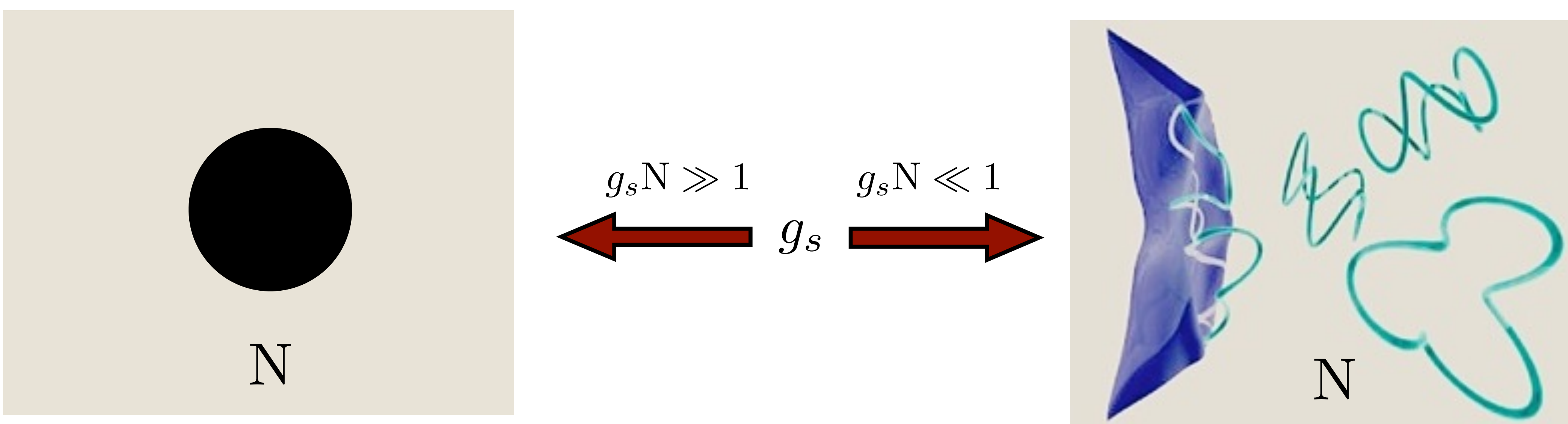
Indian Strings Meeting
Puri, Dec19, 2014

BPS black hole entropy is a detailed probe of aspects of quantum gravity

Black hole entropy $\Longrightarrow$ underlying ensemble of microscopic states

Macroscopic

Microscopic



Q: How many fundamental BH degrees of freedom?

Aim of talk

In a theory of

- 4d N=2 supergravity,
- coupled to vector multiplets and hyper multiplets,
- with generic higher derivative couplings,

compute the perturbatively exact formula for the quantum gravitational entropy of a 1/2 BPS black hole.

Based on:

S.M., V. Reys, in preparation, B. de Wit, S.M., V. Reys, in progress.

and on the work of:

Sen (Quantum entropy function program),
Banerjee, Cardoso, Dabholkar, David, de Wit, Denef, Gaiotto, Gomes, Gupta, Hama, Hosomichi, Jatkar, Lal, Mahapatra, Mandal, Mohaupt, Moore, Pestun, Pioline, Shih, Strominger, Vafa, Yin, …

Near-horizon region is an independent quantum system, fixed by charges

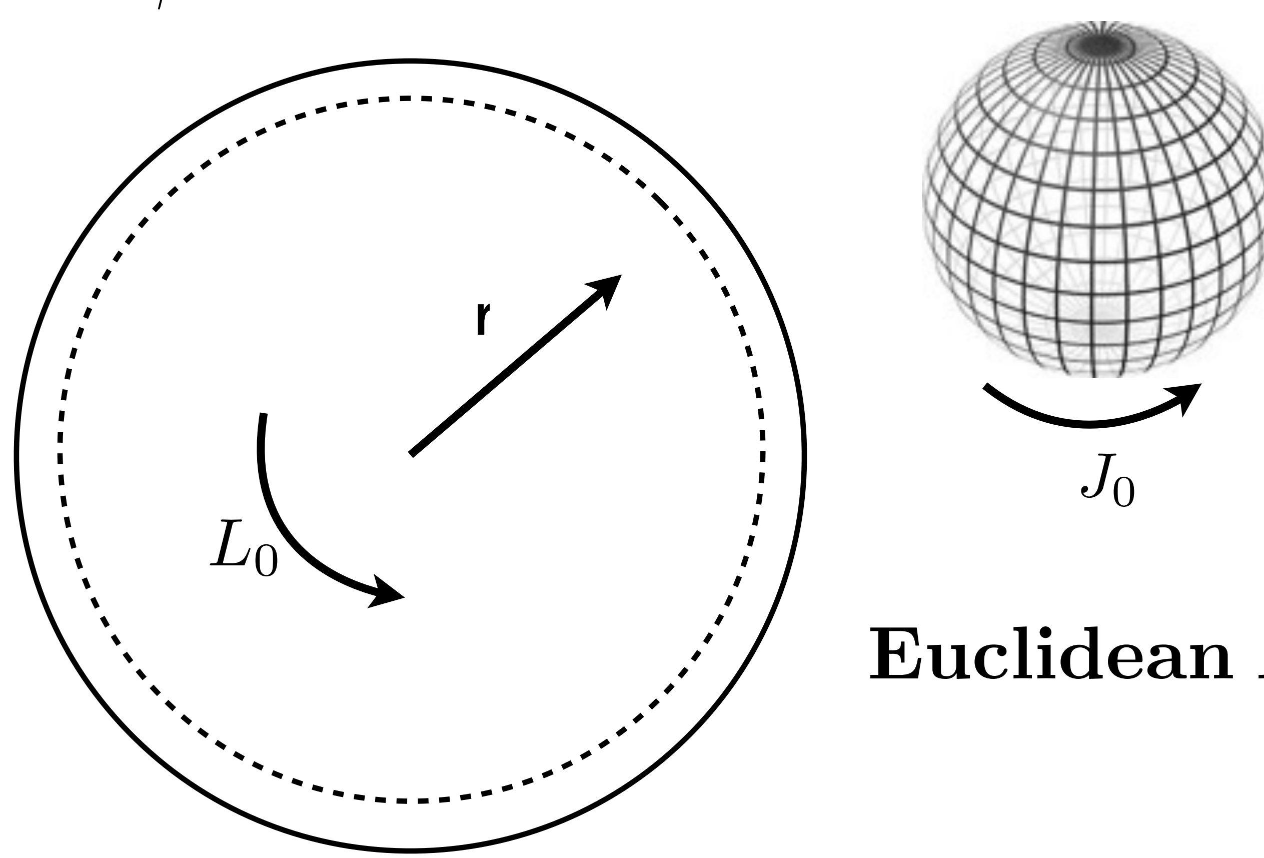
4d BPS black hole with charges (q_I, p^I)

$$ds^2 = v\left((r^2-1)\,d\theta^2 + \frac{dr^2}{r^2-1}\right) + v\left(d\psi^2 + \sin^2\psi\, d\phi^2\right)$$

$$F^I_{rt} = e^I_*\,, \quad F^I_{\psi\phi} = p^I \sin\psi$$

$$X^I = X^I_*$$

Near-horizon region:



$\mathbf{Euclidean\ AdS_2 \times S^2}$

Classical BPS black hole entropy is an AdS_2 extremization problem

(Sen '05)

4d theory with higher derivative couplings $\mathcal{L} = \mathcal{L}^{\text{eff}}(v, e_*^I, X_*^I)$

Equations of motion (Einstein equations + ...)

$$\frac{\partial \mathcal{L}^{\text{eff}}}{\partial v} = 0\,, \qquad \frac{\partial \mathcal{L}^{\text{eff}}}{\partial X_*^I} = 0\,, \qquad \frac{\partial \mathcal{L}^{\text{eff}}}{\partial e_*^I} = q_I$$

Attractor equations

Classical entropy (Bekenstein-Hawking-Wald)

$$S_{\text{BH}}^{\text{class}} = \pi \big(-q_I\, e_*^I - \mathcal{L}^{\text{eff}}\big|_{\text{attr}} \big) = \frac{A_{\text{H}}}{4} + \cdots$$

Attractor entropy

Quantum BPS black hole entropy is an AdS_2 functional integral

(Sen '08)

$$\exp(S^{\mathrm{qu}}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\left[-i\, q_I \oint A^I\right] \right\rangle^{\mathrm{reg}}_{\mathrm{AdS}_2}$$

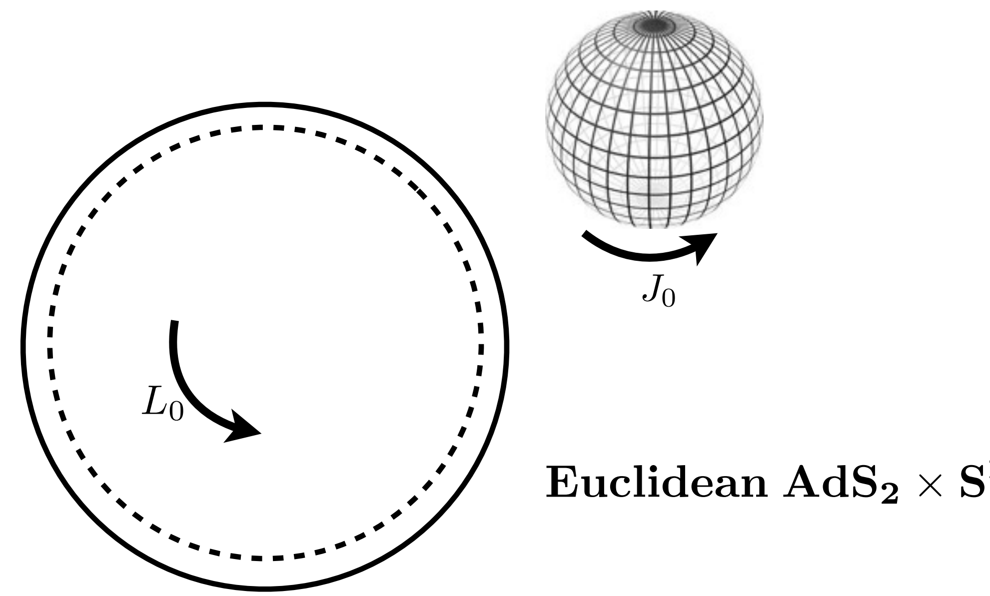
- Saddle point evaluation $\Longrightarrow$ $S^{\mathrm{qu}}_{\mathrm{BH}} = S^{\mathrm{class}}_{\mathrm{BH}} + \cdots$

- Leading logarithmic one-loop corrections systematized. New, improved, version available. (c.f. talks of Lal, Larsen)

(Sen + Banerjee^2, Gupta, Mandal, '10-'14, v2. Larsen, Keeler, Lisbão '14).

Set up for exact $\mathbf{AdS_2}$ functional integral

$$Z_{\mathrm{AdS}_2}(q_I) \equiv \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S}}$$



Supercharge Q with $Q^2 = L_0 - J_0$.

$\mathcal{M}$: Field space of supergravity.

$d\mu$: Measure on this field space.

$\mathcal{O}$: Wilson line.

$\mathcal{S}$: Action of graviton and other massless fields.

Supersymmetric functional integrals can be evaluated exactly using Localization

Duistermaat-Heckmann, Atiyah-Singer-Bott, Berligne-Vergne, Witten (1980s), Pestun '07

Off-shell symmetry algebra $Q^2 = H$ ⟵ Compact $U(1)$

$$I := \int_{\mathcal{M}} d\mu \, \mathcal{O} \, e^{-\mathcal{S}}$$

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$$I(\lambda) = \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S}+\lambda Q\mathcal{V}} \,, \qquad \mathcal{V} = \sum_{\psi} \int d^4x\, \overline{\psi}\, Q\, \psi$$

$$\frac{d}{d\lambda} I(\lambda) = \int_{\mathcal{M}} d\mu\, \mathcal{O}\, Q\mathcal{V}\, e^{-\mathcal{S}+\lambda Q\mathcal{V}} = 0$$

$$\Rightarrow I(0) = I(\infty) = \int_{\mathcal{M}_Q} d\mu_Q\, \mathcal{O}\, e^{-\mathcal{S}}\, Z_{\text{1-loop}}$$

The functional integral localizes onto the submanifold $\mathcal{M}_Q$ of solutions of the off-shell BPS equations $Q\,\Psi = 0$.

The steps

(A.Dabholkar, J.Gomes, S.M. ’10, ’11, ’14)

1. Formalism: 4d N=2 off-shell conformal supergravity coupled to vector multiplets. (de Wit, van Holten, Van Proeyen ’80)

2. Find all solutions of localization equations $Q\,\Psi = 0$, subject to $AdS_2 \times S^2$ boundary conditions. (R.Gupta, S.M. ’12)

3. Evaluate action on these solutions (including all higher derivative terms). Compute measure.

4. Only chiral-superspace integrals in the action contribute. (S.M., V.Reys, ’13)

   Governed by holomorphic prepotential $F(X^I)$ computable in string theory.

N=2 conformal supergravity (gauge choice)

- Local scale invariance in the off-shell theory.

$$X^I \to \lambda(x)\, X^I, \ (I = 0, \cdots, n_{\rm v}) \,, \quad g_{\mu\nu} \to \lambda(x)^{-2}\, g_{\mu\nu}$$

- $\Omega^2 \equiv \boxed{e^{-\mathcal{K}} = -i(X^I \bar{F}_I - \bar{X}^I F_I)}$ appears in the action as

$$\frac{1}{8\pi}\sqrt{-g}\left[-\frac{1}{6}\Omega^2\, R - g^{\mu\nu}\,\partial_\mu\Omega\,\partial_\nu\Omega\right]$$

(conformal compensator)

- Can choose gauge $e^{-\mathcal{K}} = 1$ or $\sqrt{g} = 1$

- In the large charge limit $(q_I, p^I) \to (\Lambda\, q_I, \Lambda\, p^I)$, $\Lambda \to \infty$,

$$e^{-\mathcal{K}} \sim \Lambda^2 \sim A_{\rm H}$$

Localization in N=2 supergravity

- In vector multiplet sector, scalar fields go off-shell:

$$X^I = X^I_* + \frac{C^I}{r}\,, \qquad \bar{X}^I = \bar{X}^I_* + \frac{C^I}{r}\,, \qquad Y_1^{I1} = -Y_2^{I2} = \frac{2C^I}{r^2}\,.$$

Localization manifold labelled by one parameter $\phi^I \equiv X^I(0)$ for each vector multiplet X^I.

(A.Dabholkar, J.Gomes, S.M. ’10)

- In the gravity multiplet sector, only solution is $AdS_2 \times S^2$.

(R.Gupta, S.M. ’12)

- Classical measure for scalar manifold $\prod_I dX^I\, M(X^I)$ has been computed. Does not scale with charges

(Cardoso, de Wit, Mahapatra ’12)

The exact N=2 BH entropy formula

$$Z^{\text{pert}}_{\text{AdS}_2}(q,p) = \int \prod_I d\phi^I \, M(\phi^I) \times$$
$$\times \, \exp\Big(-\pi \, q_I \, \phi^I + 4\pi \, \text{Im}\, \text{F}\big(\phi^I + ip^I\big)\Big) Z_{\text{1-loop}}(\phi^I)$$

Result (V. Reys, S.M., to appear)

$$Z_{\text{1-loop}}(\phi^I) = \exp\big(-a_0 \, \mathcal{K}(\phi^I)\big), \qquad a_0 = \sum_{\text{multiplets}} a_0^{\text{multiplet}}$$

Index theorem

In the large charge limit $S_{\text{BH}} = \frac{A_{\text{H}}}{4} + a_0 \log A_{\text{H}}$

Why is functional sdeterminant not trivial?

Hamiltonian H , Supercharge Q , $Q^2 = H$
Witten index $\mathrm{Tr}\,(-1)^F\, e^{-iHt}$ independent of *t.* ➡ Naive!

1. Background R-symmetry potential ⟹ no pairing (e.g. for hypermultiplets).

2. Supersymmetry does not commute with gauge fixing ⟹ pairing of states destroyed!

(in progress)

Gauge-fix by hand

(R. Gupta, I. Jeon)

BRST (BV) formalism

(S.M., V. Reys)

Gauge fixing and ghosts (vector multiplets)

- U(1) gauge symmetry $A_\mu \to A_\mu + \partial_\mu \theta$
- Introduce BRST complex, i.e. (b,c) ghosts, and Lagrange multiplier B
- Zero modes of gauge transformation present, introduce ghost-for-ghost c_0 .
- BRST operator Q_B , $Q_B\, A_\mu = \partial_\mu c\,, \ \ldots$, with $Q_B^2 = 0$
  Add gauge fixing term to action $Q_B \mathcal{V}_{\mathcal{B}}$

Gauge fixing and ghosts (vector multiplets)

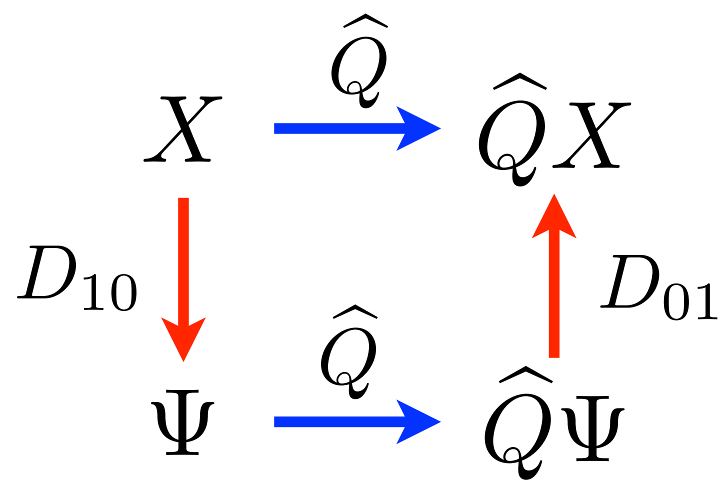
- New operator $\widehat{Q} = Q + Q_B$ with $\widehat{Q}^2 = H = L_0 - J_0$
- In supergravity, $Q^2 = L_0 - J_0 + \delta_{\rm gauge}$
- Algebra fixes the action of Q (and so of $\widehat{Q}$) on all the matter and ghost fields

All modes arranged in reps of $\widehat{Q}$

Physical action is deformed by $\widehat{Q}\,\widehat{\mathcal{V}}$, $\widehat{\mathcal{V}} = \mathcal{V} + \mathcal{V}_{\mathcal{B}}$

The sdeterminant is now an algebraic computation

Organize fields as:



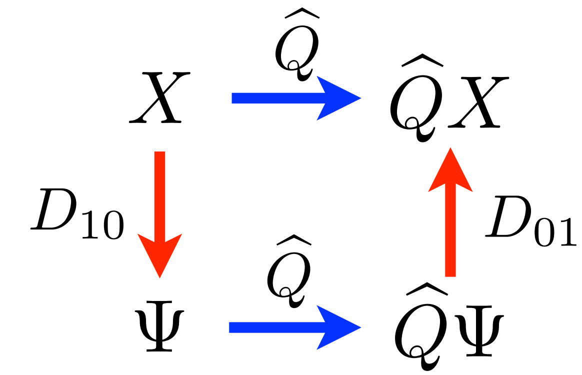
e.g.

$$\begin{matrix} \phi & \psi \\ \gamma\psi & F \end{matrix}$$

Want to compute $Z_{\text{1-loop}} = \left(\frac{\det K_f}{\det K_b} \right)^{\frac{1}{2}}$ for the deformation operator $\widehat{Q}\widehat{\mathcal{V}} = (X, K_b\, X) + (\Psi, K_f\, \Psi)$.

The sdeterminant is captured by an index

Linear algebra : $\dfrac{\det K_f}{\det K_b} = \dfrac{\det_\Psi H}{\det_X H} = \dfrac{\det_{\mathrm{Coker}\,D_{10}} H}{\det_{\mathrm{Ker}\,D_{10}} H}$



Write: $\dfrac{\det_\Psi H}{\det_{\mathrm{X}} H} = \prod_n \lambda_n^{a(n)}$

Eigenvalues encoded in index

$$\mathrm{ind}(D_{10})(t) := \mathrm{Tr}_{\mathrm{Ker}\,D_{10}}\, e^{-iHt} - \mathrm{Tr}_{\mathrm{Coker}\,D_{10}}\, e^{-iHt}$$

$$= \sum_n a(n)\, e^{-it\lambda_n}$$

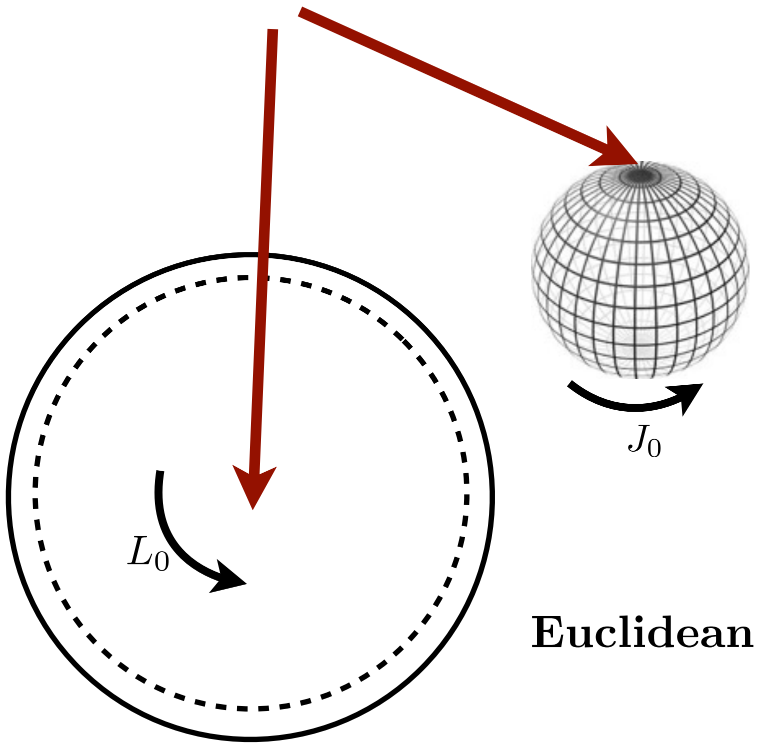
The index is captured by fixed points of U(1)

Atiyah-Bott-Singer index theorem

$$\mathrm{ind}(D_{10})(t) = \sum_{\mathrm{x=fixed\ pts\ of}\ H} \frac{\mathrm{Tr}_{X,\Psi}\,(-1)^F\, e^{-iHt}}{\det(1-\partial\widetilde{x}/\partial x)}\,.$$

$(\widetilde{x} = Hx)$

Fixed points of
$H = L_0 - J_0$



Euclidean $\mathbf{AdS_2 \times S^2}$

$$\Longrightarrow \quad \mathrm{ind}_{\mathrm{vec}}(D_{10}) = -4\sum_{n\geq 1} n\,\exp\left(-\frac{i\,t\,n}{\ell_{\mathrm{AdS}}}\right)$$

Eigenvalues $\lambda_n = n/\ell_{\mathrm{AdS}}$

Degeneracy $a(n) = -4\,n$

Regularize determinant using Zeta functions

(cf. Hawking '77)

$$\frac{\det_\Psi H}{\det_X H} = \prod_n \lambda_n^{a(n)} \quad = C\exp\big(-4\zeta_R(-1)\log \ell_{\mathrm{AdS}}\big)$$

$$\ell^2_{\mathrm{AdS}} = e^{-\mathcal{K}(\phi^I)}\,, \qquad \zeta_R(s) = \sum_{n\geq 1}\frac{1}{n^s}$$

$$\Rightarrow \quad \boxed{\frac{\det K_f}{\det K_b} = \frac{\det_\Psi H}{\det_X H} = C\exp\Big(-\frac{1}{6}\,\mathcal{K}(\phi^I)\Big)}$$

Main result for one-loop determinant

$$Z_{\text{1-loop}}(\phi^I) = \exp\big(-a_0\,\mathcal{K}(\phi^I)\big), \qquad a_0 = \sum_{\text{multiplets}} a_0^{\text{multiplet}}$$

Index theorem calculation ⇨ $a_0^{\text{vector}} = -\frac{1}{12}$

Similar calculation for hypers ⇨ $a_0^{\text{hyper}} = +\frac{1}{12}$

In the large charge limit $S_{\text{BH}} = \frac{A_{\text{H}}}{4} + a_0 \log A_{\text{H}}$

Consistency with large charge one-loop calculation

⇨ $a_0^{\text{grav}} = +2 \ , \ a_0^{3/2} = -\frac{11}{12}$

➡ Index theorem calculation underway

(B. de Wit, S.M., V. Reys)

1/8 BPS black holes in N=8 theory

(A.Dabholkar, J.Gomes, S.M. '10, '11)

- Truncation of N=8 to N=2 theory with 7 vector multiplets.
- AdS_2 path integral reduces to an 8-dimensional integral.
- Use measure of zero modes.

$$e^{S_{BH}^{\rm qu}}(N) = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \pi^2 N/4\sigma\right) = \widetilde{I}_{7/2}(\pi\sqrt{N})$$

A quantitative test

(A.Dabholkar, J.Gomes, S.M. '11)

N	$d_{\mathrm{micro}}(\mathrm{N})$	$\exp\big(S^{\mathrm{qu}}(\mathrm{N})\big)$	$\exp\big(S^{\mathrm{cl}}(\mathrm{N})\big)$
3	8	7.97	230.76
4	12	12.2	535.49
7	39	38.99	4071.93
8	56	55.72	7228.35
11	152	152.04	33506.14
12	208	208.45	53252.29
15	513	512.96	192400.81
...	...	...	...
10^5	exp(295.7)	exp(295.7)	exp(314.2)

$$d_{\mathrm{micro}}(\Delta) = e^{S^{\mathrm{qu}}_{BH}(\Delta)}\big(1 + O(e^{-\pi\sqrt{\Delta}/2})\big)$$

Why does the truncation work?

We kept: zero modes of 8 vector multiplets $\quad -\frac{1}{2} \times 8$

We threw away:

- Non-zero modes of 8 vector multiplets $\quad +\frac{5}{12} \times 8$
- (15-8) vector multiplets $\quad -\frac{1}{12} \times (15-8)$
- 10 hyper multiplets $\quad +\frac{1}{12} \times 10$
- 6 gravitini multiplets $\quad -\frac{11}{12} \times 6$
- 1 gravity multiplet $\quad +2$

Total 0

1/2 BPS black holes in N=2 theory

$$Z^{\rm pert}_{{\rm AdS}_2}(q,p) = \int \prod_{I=0}^{n_{\rm v}} d\phi^I \, (\phi^0)^{2-\frac{\chi}{12}} \, e^{-\mathcal{K}(\phi)} \times$$

$$\times \exp\Big(-\pi \, q_I \, \phi^I + 4\pi \, {\rm Im} F(\phi^I + i p^I) \Big)$$

$$\chi = 2(n_{\rm v} + 1 - n_{\rm h})$$

(*cf*. Ooguri-Strominger-Vafa '04, Denef-Moore '07, Sen, 1108.3842)

Note:

- Still need to complete graviton calculation
- This formula implies a surprising cancellation of states in N=2 string theory.