Thermalisation with chemical potentials, GGE and higher spin black holes

Gautam Mandal, Ritam Sinha and Nilakash Sorokhaibam

Department of Theoretical Physics
Tata Institute of Fundamental Research, Mumbai

Indian Strings Meetings, 2014

Introduction

Understanding thermalization in interacting quantum systems has been an important quest in physics. With the advent of holography, the issue of thermalization in strongly coupled QFTs has one fold increase in its importance, because of its correspondence with the problem of black hole formation.
In our present work, we are concerned with 'thermalisation' in 2D CFTs after a quantum quench and compare our results with black holes quasinormal frequencies.

History

- Thermalisation of point functions of a pure state: Cardy and Calabrese found the thermalisation of correlation functions in 2D CFT starting from certain pure states.
- Their initial state is a boundary state euclidean-time evolved by $\beta/4$ where β turns out to be the temperature of the stationary state. So, the initial state is $|\Psi_0\rangle = e^{-\beta H/4}|B\rangle$ and $\langle\Psi_0| = \langle B|e^{-\beta H/4}$.
- The calculation relies on the technical assumption that the asymptotic behaviour in the euclidean region(large imaginary times) may simply be analytically continued to find the behaviour at large times. It is shown in their paper to work for the lattice free bosons and Ising-XY chain.
- Simple quasiparticle picture

Geometry, conformal transformation

- The CFT lives in a strip of width $\beta/2$ with coordinates $w = \sigma + i\tau$ and $\bar{w} = \sigma - i\tau$, which can be transformed to upper half plane(UHP) using the conformal map $z = ie^{2\pi w/\beta}$ and $\bar{z} = -ie^{2\pi \bar{w}/\beta}$.

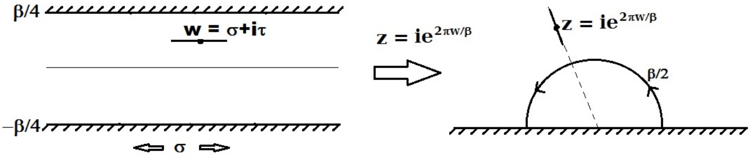

Figure: Strip to UHP transformation

- This same map also takes a cylinder, with periodic side β, to the full complex plane $\mathbb{C}$.

Analytic continuation

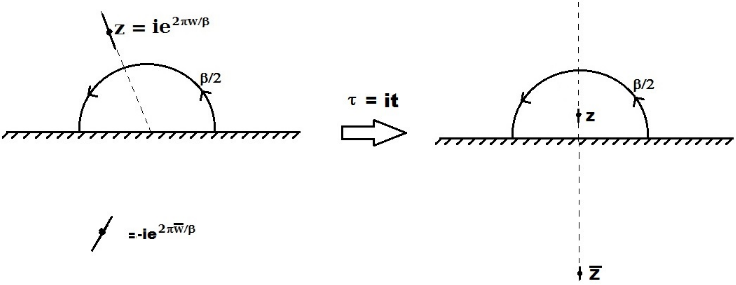

Figure: Wick rotation

As $t \to \infty$, the point z get fixed on the imaginary axis and moves towards 0, and $\bar{z}$ goes to $-i\infty$. Moreover the interval length shrinks to zero as $t \to \infty$.

Correlation functions

Correlation functions can be calculated by transforming to the UHP and using method of images. One point function of a primary operator is a one-point function $\phi_{h,\bar{h}}(w,\bar{w})$ if the operator is only holomorphic(or anti-holo) or a two point function if it contains both holo and anti-holo components, for the later case with $h = \bar{h}$ at $w = 0 + i\tau$ and $\bar{w} = 0 - i\tau$,

$$\begin{aligned}\langle\phi_{h,\bar{h}}(w,\bar{w})\rangle_{strip} &= \Big(\frac{dz}{dw}\Big)^h\Big(\frac{d\bar{z}}{d\bar{w}}\Big)^{\bar{h}}\langle\phi_{h,\bar{h}}(z,\bar{z})\rangle_{UHP} \\ &= \Big(\frac{dz}{dw}\Big)^h\Big(\frac{d\bar{z}}{d\bar{w}}\Big)^{\bar{h}}\langle\phi_h(z)\phi_{\bar{h}}(z'=\bar{z})\rangle_{\mathbb{C}} \\ &= C[i\cos(2\pi\tau/\beta)]^{-2h} \xrightarrow[t\to\infty]{\tau\to it} i^{-2h}Ce^{-4\pi ht/\beta}\end{aligned}$$

Similarly 2-point functions would give 2-point or 3-point or 4-point functions depending on the field content of the operators.

Thermalization function $I(t)$

The thermalization function for a spatial region A is defined as

$$
\begin{aligned}
I(t) &= \frac{\hat{Z}_{St,Cyl}(A)}{[\hat{Z}_{St,St}(A)\hat{Z}_{Cy,Cy}(A)]^{\frac{1}{2}}} \\
&= \frac{\mathrm{Tr}(\rho_{dyn,A}(t)\rho_{eqm,A}(\beta,\mu_i))}{[\mathrm{Tr}(\rho_{dyn,A}(t)^2)\mathrm{Tr}(\rho_{eqm,A}(\beta,\mu_i)^2)]^{1/2}}
\end{aligned}
$$

where 'Cy' means cylinder and "St' is strip, the dynamical reduced density matrix of 'A' is

$$\rho_{dyn,A}(t) = \mathrm{Tr}_{\bar{A}}\ \rho_{dyn}(t),\ \rho_{dyn}(t) \equiv (\exp[-iHt]|\psi_0\rangle\langle\psi_0|\exp[iHt])$$

and the "equilibrium reduced density matrix" is

$$\rho_{eqm,A}(\beta,\mu_i) = \mathrm{Tr}_{\bar{A}}\ \rho_{eqm} = \mathrm{Tr}_{\bar{A}}\ \exp[-\beta H]$$

and the quenched initial state is $|\psi_0\rangle = \exp[(-\beta H)/4]|Bd\rangle$

Short interval expansion

Consider a CFT living on two different reimann surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$ joined together at interval/spatial region A, short interval expansion gives:

$$\hat{Z}_{\mathcal{R}_1,\mathcal{R}_2}(A) = \frac{Z_{\mathcal{R}_1,\mathcal{R}_2}(A)}{Z_{\mathcal{R}_1} Z_{\mathcal{R}_2}} = \sum_{k_1,k_2} C_{k_1,k_2} \langle \phi_{k_1}(w_1, \bar{w}_1) \rangle_{\mathcal{R}_1} \langle \phi_{k_2}(w_2, \bar{w}_2) \rangle_{\mathcal{R}_2}$$

where $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ are points on the interval A in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively; k_1,k_2 label a complete basis of quasiprimary operators of the CFT Hilbert space. The coefficients C_{k_1,k_2} are calculated from the case when $\mathcal{R}_1$, $\mathcal{R}_2$ are complex planes, using uniformization map. The expansion is in the limit when l, the length of the interval A is small compare to any other length scales in the theories.

Calculation of $I(t) \xrightarrow{t\to\infty} 1$

Using the short interval expansion,
$\hat{Z}_{St,Cyl}(A)$,$\hat{Z}_{St,St}(A)$,$\hat{Z}_{Cy,Cy}(A)$ can be written as

$$\hat{Z}_{St,Cy} = \sum_{k_1,k_2} C_{k_1,k_2} \langle \phi_{k_1}(w_1,\bar{w}_1)\rangle_{St} \langle \phi_{k_2}(w_2,\bar{w}_2)\rangle_{Cy}$$

$$\hat{Z}_{St,St} = \sum_{k_1,k_2} C_{k_1,k_2} \langle \phi_{k_1}(w_1,\bar{w}_1)\rangle_{St} \langle \phi_{k_2}(w_2,\bar{w}_2)\rangle_{St}$$

$$\hat{Z}_{Cy,Cy} = \sum_{k_1,k_2} C_{k_1,k_2} \langle \phi_{k_1}(w_1,\bar{w}_1)\rangle_{Cy} \langle \phi_{k_2}(w_2,\bar{w}_2)\rangle_{Cy}$$

The only one-that point functions on the strip that survive in the limit $t \to \infty$ are ones for which $\langle \phi_k \rangle_{St} = \langle \phi_k \rangle_{Cy}$. All other one point functions on the strip dies off exponentially as we have seen.

So, in the limit $t \to \infty$,

$$Z_{St,St} = Z_{St,Cy} = Z_{Cy,Cy} \Rightarrow I(t \to \infty) = 1 - \alpha(l)e^{-8\pi ht}$$

where $\alpha(l) =$. It can be argued that going to from the strip to UHP in the limit $\beta << l << t \to \infty$, the l dependence exponentiate and gives

$$I(t \to \infty) = 1 - \alpha' e^{-8\pi h(t-l/2)/\beta}$$

Bulk-dual of $|\psi_0\rangle$

The bulk dual is a BTZ black hole with an end-of-world brane behind the horizon.

Turning on a single chemical potential μ_n

The effect of turning on a single chemical potential μ_n on the calculation of I(t) is that

$$\langle \phi_k(w,\bar{w})\rangle_{St} \to \langle \phi_k(w,\bar{w})\rangle^{\mu}_{St} \equiv \frac{\langle e^{-\frac{\mu_n}{4}W_n}\phi_{k1}(w,\bar{w})e^{-\frac{\mu_n}{4}W_n}\rangle_{str}}{\langle e^{-\frac{\mu_n}{2}W_n}\rangle_{St}}$$

$$\langle \phi_{k1}(w,\bar{w})\rangle_{Cy} \to \langle \phi_{k1}(w,\bar{w})\rangle^{\mu}_{Cy} \equiv \frac{\langle e^{-\mu_n W_n}\phi_{k1}(w,\bar{w})\rangle_{Cy}}{\langle e^{-\mu_n W_n}\rangle_{Cy}}$$

This can be calculated perturbatively using

$$\begin{aligned} W_n &= \frac{1}{2\pi}\int_\Gamma W_{\tau\tau\ldots\tau}d\sigma = \frac{1}{2\pi}\int_\Gamma \Big(i^n dw_1\, \mathcal{W}_n(w_1) + (-i)^n \text{antiholo} \\ &= \frac{1}{2\pi}\left(\frac{2\pi}{\beta}\right)^{n-1}\Big[i^n \int_\Gamma dz_1 \Big(z_1^{n-1}\mathcal{W}_n(z_1) \\ &\quad + \sum_{m=1}^{n/2} a_{n,n-2m} z_1^{n-2m-1}\mathcal{W}_{n-2m}(z_1)\Big) + (\text{antihol})\Big] \end{aligned}$$

Explicit μ_n^1 order calculation

W_n commutes with H so the z_1 and $\bar{z}_1$ integration contours are appropriately before or after the ϕ_k insertions. And we have

$$\begin{aligned} & \langle \mathcal{W}_n(z_1)\phi_k(z)\phi_k(z')\rangle_{\mathbb{C}} \\ = \; & q_n(z_1-z)^{-n}(z_1-z')^{-n}(z-z')^n\langle\phi_k(z)\phi_k(z')\rangle_{conn} \end{aligned}$$

Then doing the z_1 integral explicitly, the time dependence correction of the μ_n^1 order is $\tilde{\mu}_n(2\pi \mathrm{Q}_n t/\beta + r_1)$ where $Q_n = \frac{1}{(2\pi)^{n-2}}(i^n q_n + (-i)^n \bar{q}_n)$ and $\mathrm{q}_n = q_n + \sum_{m=1}^{n/2-1} a_{n,n-2m} q_{n-2m}$.

Alternative calculation for order μ_n^1 and generalization to higher orders

For every correlators $\langle \mathcal{W}_n(z_1)\phi_k(z)\phi_{\bar{k}}(z')\rangle_{St}$ there is a corresponding $\langle \phi_k(z)\phi_{\bar{k}}(z')\mathcal{W}_n(z_2)\rangle_{St}$ so looking for log dependence $\ln(z_1 - z)$ or $\ln(z_1 - z')$, the integration contour can be closed up or down, and there is a relative $-ve$ sign between these two ways of closing the integral due to the difference in the sense of rotation.

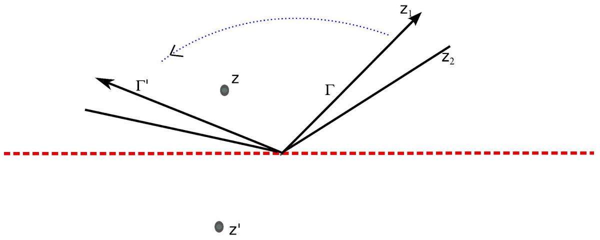

Figure: The contour deformation argument.

Then the $\ln(z_1 - z)$ or $\ln(z_1 - z')$ term is given by the residue.

$$\begin{aligned} J_n(z_1, z) &\equiv \int dz_1 g(z_1)(z_1 - z)^{-n} \\ &= R_n(z)\ln(z_1 - z) + \text{meromorphic in } (z_1 - z) \text{ where} \\ R_n(z) &= \text{Residue}_{z_1 = z}[g(z_1)(z_1 - z)^{-n}] \\ &= \frac{1}{2\pi i}\oint_z dz_1\, [g(z_1)(z_1 - z)^{-n}] \end{aligned}$$

Note: It is not necessary to consider the spatial direction to be compactified and take the compactified length $L \to \infty$, the closing of contour in the preceding slide was just an exercise to calculate the log terms.

Now the closed contour can be shrink to arbitrary size around the pole at z or z'. So the dominant contribution is from the leading term in the OPE $\mathcal{W}_n(z_1)\phi_k(z) = q_n\phi_k(z)/(z_1 - z)^n$. So $\mathcal{W}_n(z_1)$, $\bar{\mathcal{W}}_n(\bar{z}_1)$ contracting with $\phi_k(z)$, $\phi_k(z')$ gives $2\pi \mathrm{Q}_n t/\beta$.

This way of calculation can be easily generalized for higher order μ_n, with proper normalization of μ as done before $\tilde{\mu} = \frac{\mu_n}{\beta^{n-1}}$. μ_n^2 correction is $\frac{1}{2}\mu_n^2(\mathrm{Q}_n 2\pi t/\beta)^2$ and so the mu_n expansion sums up to

$$\langle\phi_k(w,\bar{w})\rangle^{\mu}_{str} = \langle\phi_k(w,\bar{w})\rangle_{str} \times \alpha(\mu)\exp[-\mu_n \mathrm{Q}_n 2\pi t/\beta + O(\mu^2))]$$

1-point function on the cylinder and thermalization

Expanding for the cylinder case, we get

$$\begin{aligned} & \langle e^{-\mu_n W_n} \phi_{k1}(w,\bar{w}) \rangle_{Cy} / \langle e^{-\mu_n W_n} \rangle_{Cy} \\ = \ & \langle \phi_{k1}(w,\bar{w}) \rangle_{Cy} - \mu(\langle W_n \phi_{k1}(w,\bar{w}) \rangle_{Cy} - \langle W_n \rangle_{Cy} \langle \phi_{k1}(w,\bar{w}) \rangle_{Cy}) \end{aligned}$$

As before, in $t \to \infty$ limit,

$$\langle phi_k(w,\bar{w}) \rangle^{\mu}_{St} = \langle phi_k(w,\bar{w}) \rangle^{\mu}_{Cy} \tag{1}$$

$$hboxso, Z_{St,Cy}(\beta,\mu_n) = Z_{St,St}(\beta,\mu_n) = Z_{Cy,Cy}(\beta,\mu_n) \tag{2}$$

Hence we get,

$$I(t,\beta,\mu_n) \xrightarrow{t\to\infty} 1 \tag{3}$$

Thermalization with arbitrary number of chemical potentials

Turning on other higher spin currents, $W_3, W_4, W_5, \ldots..$, with $\hat{t} = 2\pi t/\beta$, the 1-point function on the strip becomes

$$\langle \phi(w, \bar{w}) \rangle_{St}^{\vec{\mu}} = \exp(-2(\vec{\tilde{\mu}}.\vec{q})\hat{t})(1 + O(\mu \hat{t}) + O(\mu^2 \hat{t}^2) + \ldots) \quad (4)$$

So,

$$I(t) = 1 - \alpha(l) \exp(-4(\vec{\tilde{\mu}}.\vec{q})\hat{t})$$

where α is the same as in case without higher spins. Also, in the limit $\beta << l << t \to \infty$,

$$I(t) = 1 - \alpha(l) \exp(-4(\vec{\tilde{\mu}}.\vec{q})(t - l/2)/\beta)$$

Higher spin black holes

(1) It has been shown by that the equilibrium ensemble , in the context of W_n identified with W_∞ charges, correspond to a higher spin bulk dual which is a black hole.
(2) Our result implies, therefore, that after the thermalization time, bulk scalars in a local region effectively start perceiving a higher spin black hole geometry.
(3)Small enough geodesics start seeing black hole geometry

(4) Our result implies

$$\hat{\rho}_{dyn,A}(t) = \left[1 - \frac{1}{2}\frac{C_{k,k}}{C_{0,0}}\left(\langle\phi_k(u,\bar{u})\rangle_{str}\right)^2\right]\hat{\rho}_{eqm,A} + \hat{Q}f(t)$$
$$f(t) \propto \langle\phi_k(u,\bar{u})\rangle_{str} \qquad (5)$$

This means that the dynamical reduced density matrix differs from the equilibrium reduced density matrix by an amount proportional to the expectation value of the field ϕ_k. This has the bulk interpretation that so far as observables in a small local region are concerned, they perceive the geometry as that of a black hole perturbed by a normalizable mode of the bulk scalar dual to ϕ_k. The decay of ϕ_k therefore corresponds to QNM of the scalar field.

(5) Narain et al indeed show that

$$\omega = -i(1 + \lambda + \text{constant } \mu_3 q_3) \qquad (6)$$

We can show that $1 + \lambda = 2h$, which agrees with our result (and with Cardy's result).

Discussion

- Local quenches
- Quasinormal modes
- Caldeira leggett model
- Dynamical Entanglement Entropy