PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

On Spacetime Entanglement

with E. Bianchi, M. Headrick, J. Rao, S. Sugishita & J. Wien

Black Hole Entropy:

- Bekenstein and Hawking: “black holes carry entropy!”

geometry

$$S_{BH} = \frac{k_B\, c^3}{\hbar}\ \frac{\mathcal{A}}{4G}$$

- “horizons carry entropy!”: de Sitter space and Rindler wedge

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• “horizons carry entropy!”: de Sitter space and Rindler wedge

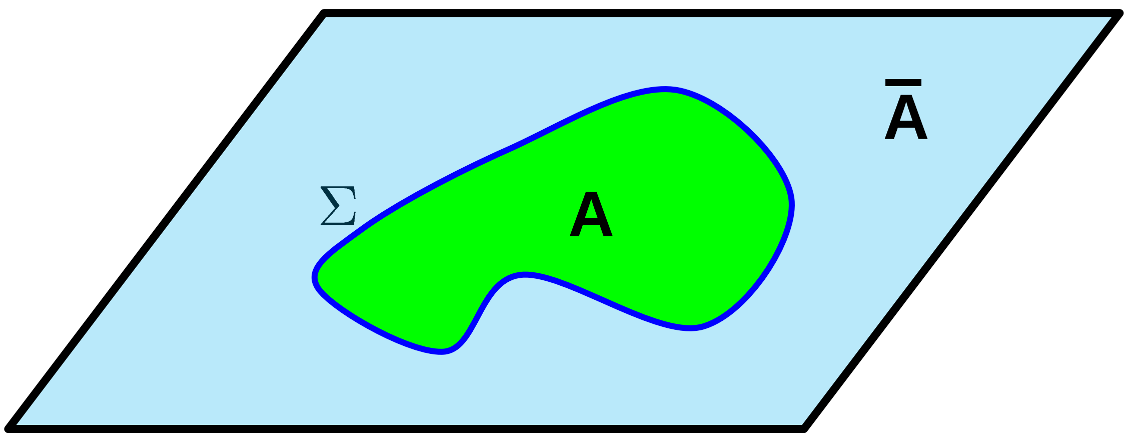
• window into quantum gravity?!?

Spacetime Entanglement Conjecture

(Bianchi & RM)

- in a theory of quantum gravity, for any sufficiently large region A in a smooth background, consider entanglement entropy between dof describing A and $\bar{A}$; contribution describing short-range entanglement is finite and described in terms of geometry of entangling surface with leading term:

$$S_{\mathrm{EE}} = \frac{\mathcal{A}_\Sigma}{4G_N} + \cdots$$

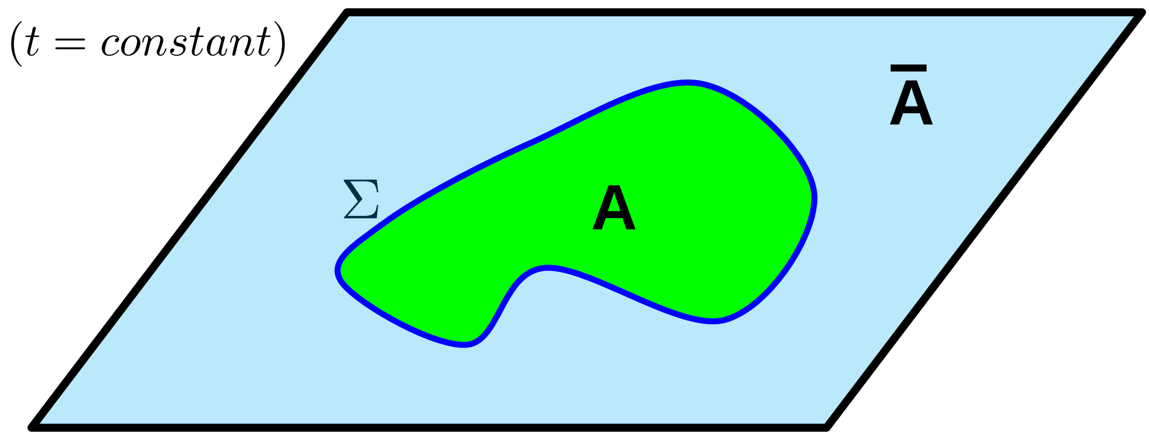


- higher order terms controlled by higher curvature gravitational couplings, similar to Wald entropy (RM, Pourhasan & Smolkin)

Entanglement Entropy in QFT

- entanglement entropy: general diagnostic to give a quantitative measure of entanglement using entropy to detect correlations between two subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix ρ_A

⟶ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



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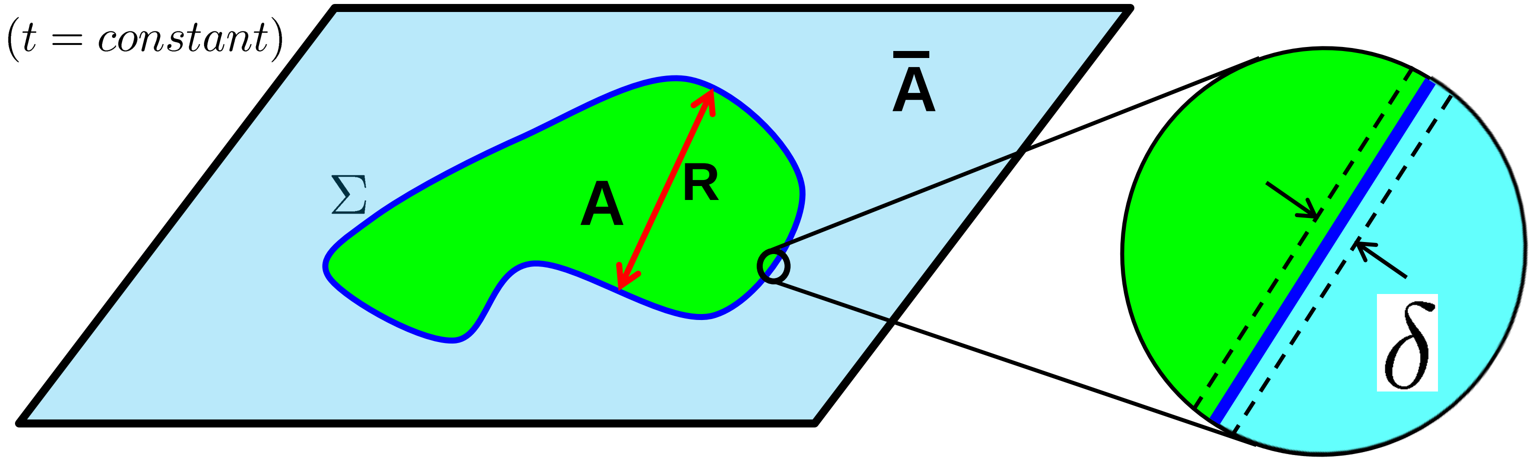

- result is UV divergent!
- must regulate calculation: δ = short-distance cut-off

$$S \;=\; c_0\,\frac{R^{d-2}}{\delta^{d-2}} \;+\; c_2\,\frac{R^{d-4}}{\delta^{d-4}} \;+\; \cdots$$

d = spacetime dimension

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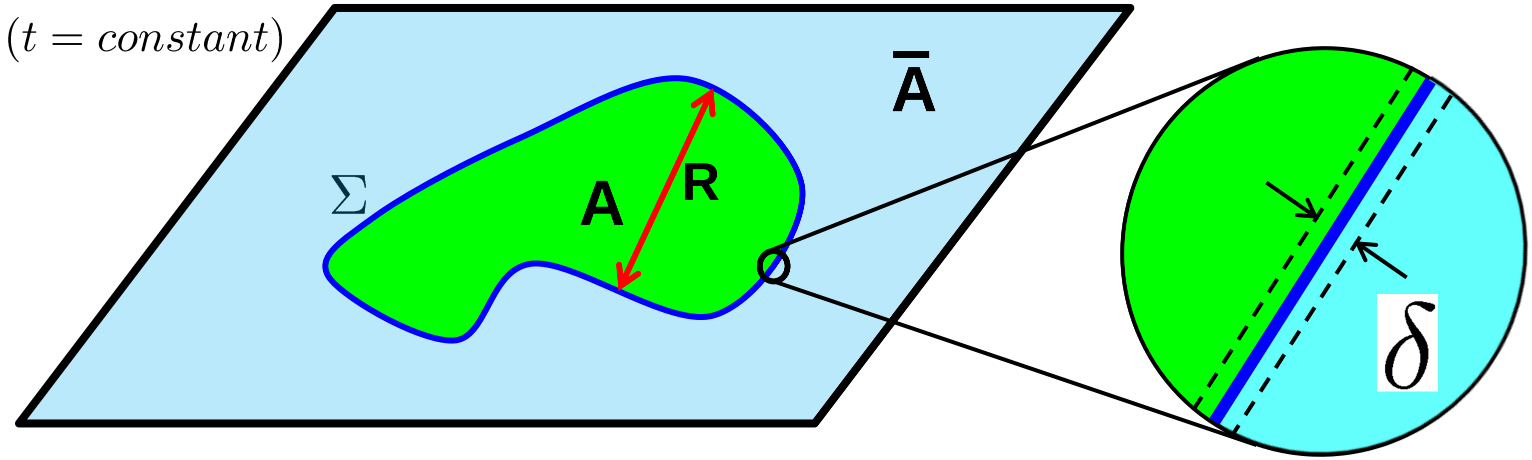

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$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$ d = spacetime dimension

⟶ geometric structure, eg, $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \tilde{c}_2 \frac{\oint_\Sigma \text{“}curvature\text{”}}{\delta^{d-4}} + \cdots$

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- recall Sorkin's suggestion for origin of black hole entropy:

$$S_{EE}\;=\;\tilde{c}_0\,\frac{\mathcal{A}_\Sigma}{\delta^{d-2}}+\cdots \qquad \text{vs} \qquad S_{BH}=2\pi\,\frac{\mathcal{A}_{horizon}}{\ell_P^{\,d-2}}$$

$\longrightarrow$ "area law" suggestive of BH formula if $\delta\simeq\ell_P$

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- problems?: leading QFT singularities are regulator dependent

→ represent renormalization of gravity couplings (Susskind & Uglum; ...)

- active topic in 90's but various "technical issues" left unresolved, eg, contributions of vectors or non-minimally coupled scalars

- interest in $S_{BH} = S_{EE}$ had some revival recently

(Jacobson & Satz; Cooperman & Luty; Donnelly & Wall; . . .)

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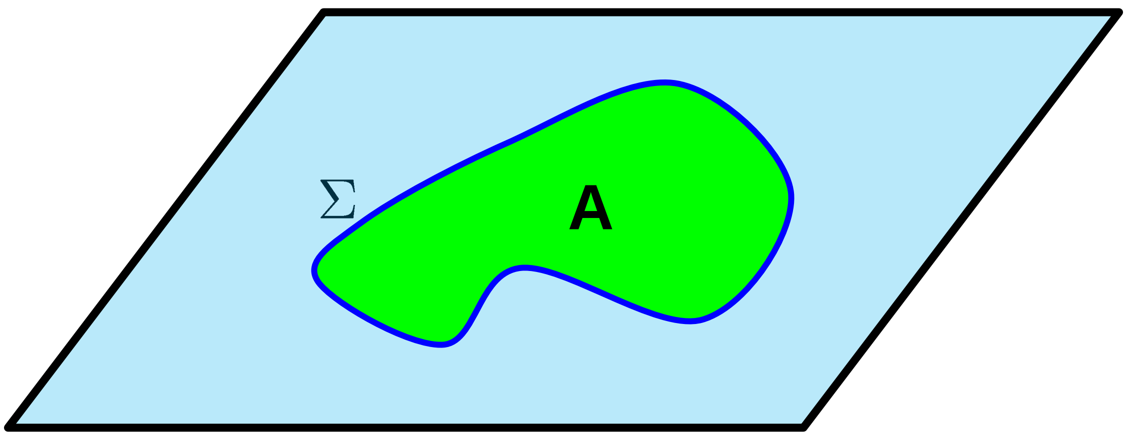


• **same renormalization works for general entangling surface!!**

(Bianchi & RM)

AdS/CFT Correspondence:

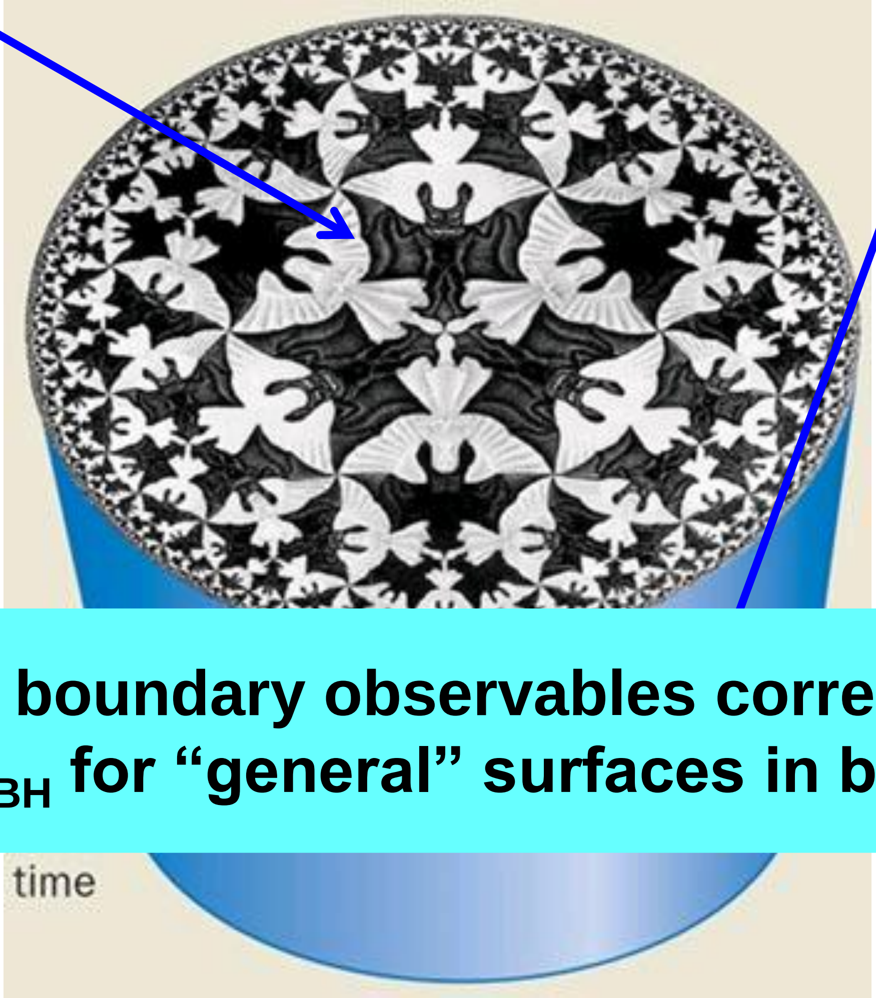
Bulk: gravity with negative Λ in **d+1** dimensions

“holography”

Boundary: quantum field theory in **d** dimensions

anti-de Sitter space

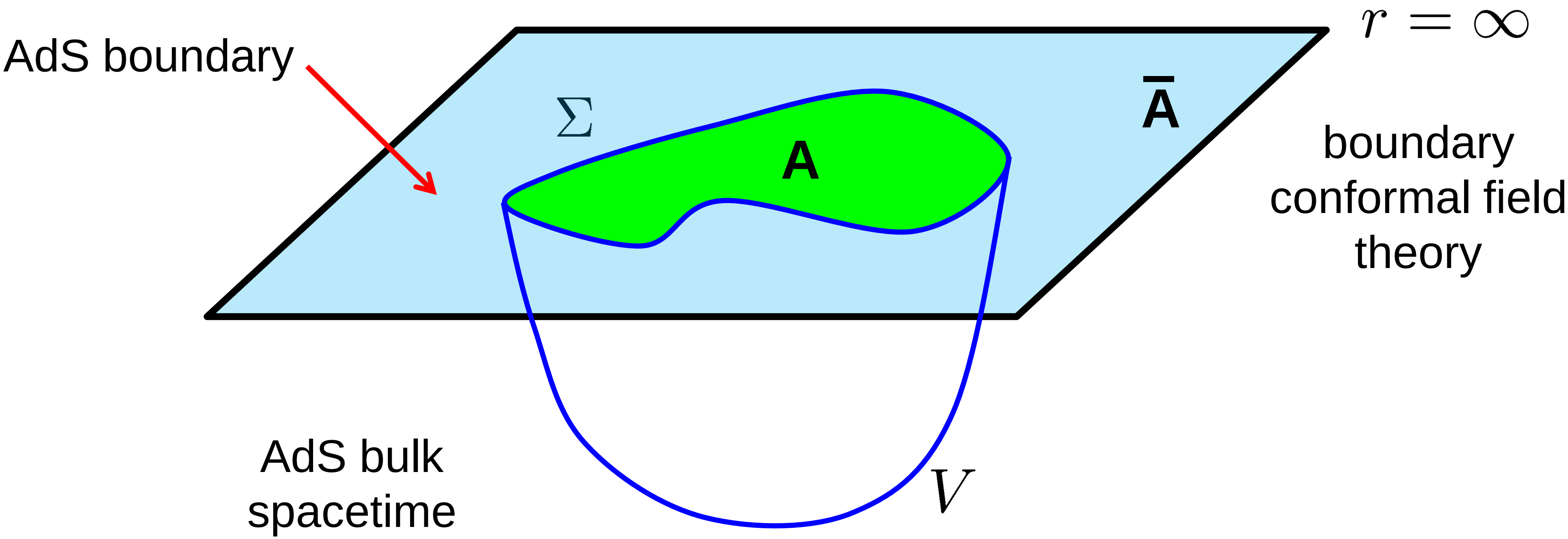
conformal field theory



Are there boundary observables corresponding to S_{BH} for “general” surfaces in bulk?

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

S_{BH} applied in unusual circumstances

Lessons from Holographic EE:

(entanglement entropy)$_{\text{boundary}}$ = (entropy of extremal surface)$_{\text{bulk}}$

- R&T prescription assigns gravitational entropy $S_{BH} = \mathcal{A}/(4G_N)$ to "unconventional" bulk surfaces/regions :

not black hole! not horizon! not boundary of causal domain!

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- S_{BH} on other surfaces speculated to give new entropic measures of entanglement in boundary theory

  $\longrightarrow$ causal holographic information
  (Hubeny & Rangamani; H, R & Tonni; Freivogel & Mosk; Kelly & Wall; **. . .**)

  $\longrightarrow$ entanglement between high and low scales
  (Balasubramanian, McDermott & van Raamsdonk)

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$\longrightarrow$ hole-ographic spacetime

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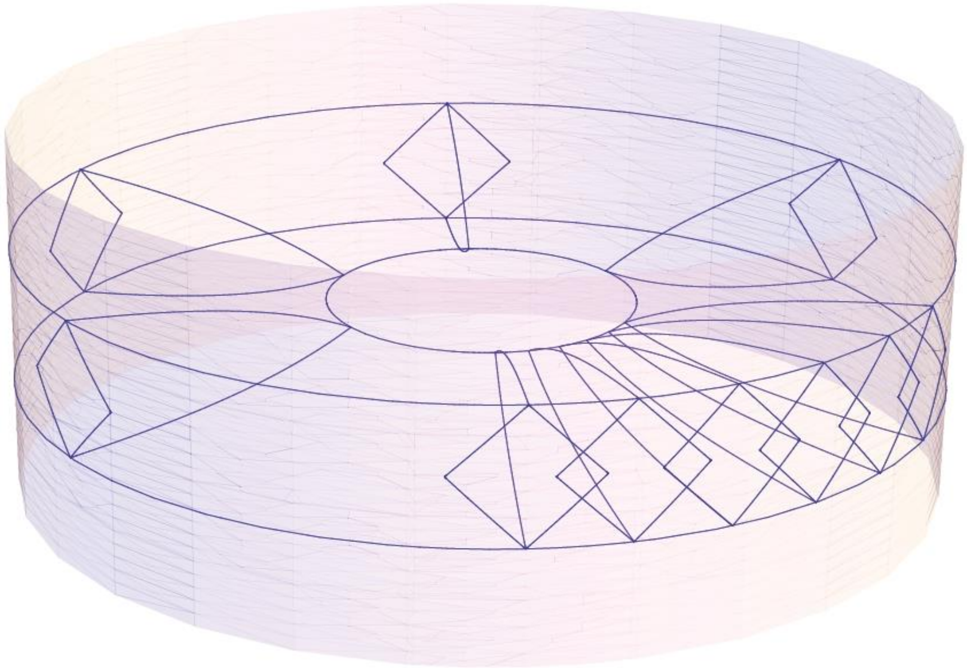
arXiv:1310.4204 [hep-th]

"hole-ographic spacetime":

two new ideas:

- **residual entropy**: collective uncertainty associated with family of observers confined to finite time strip; maximum entropy of global density matrix consistent with density matrices of subsy's



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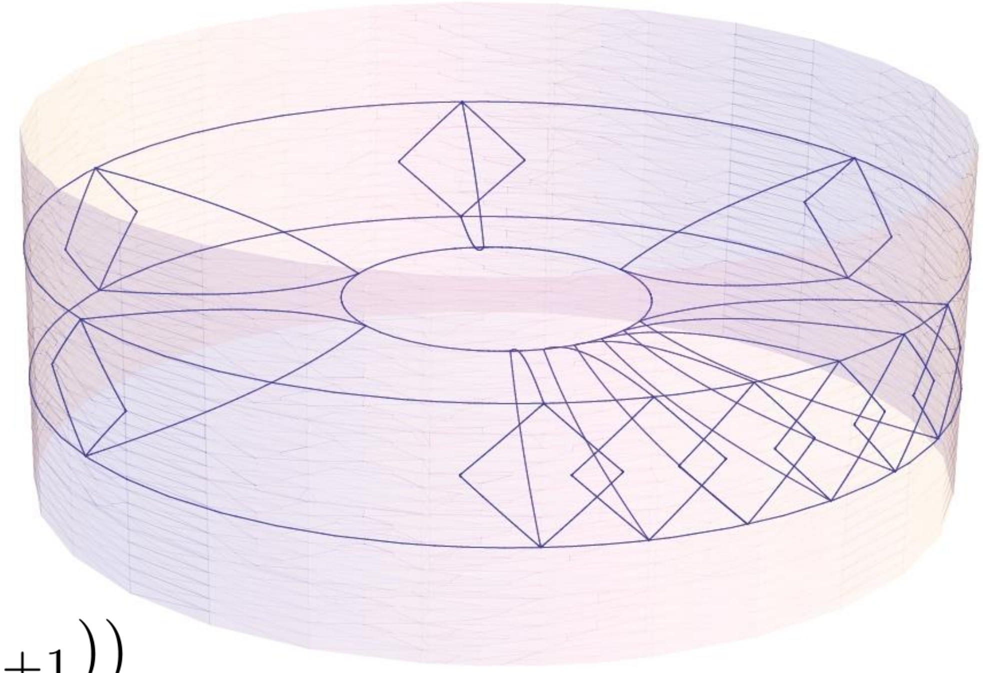

- **differential entropy**:

$$E = \sum (S(I_j) - S(I_j \cap I_{j+1}))$$

boundary observables which yield gravitational entropy of closed curves inside of d=3 AdS space with certain continuum limit

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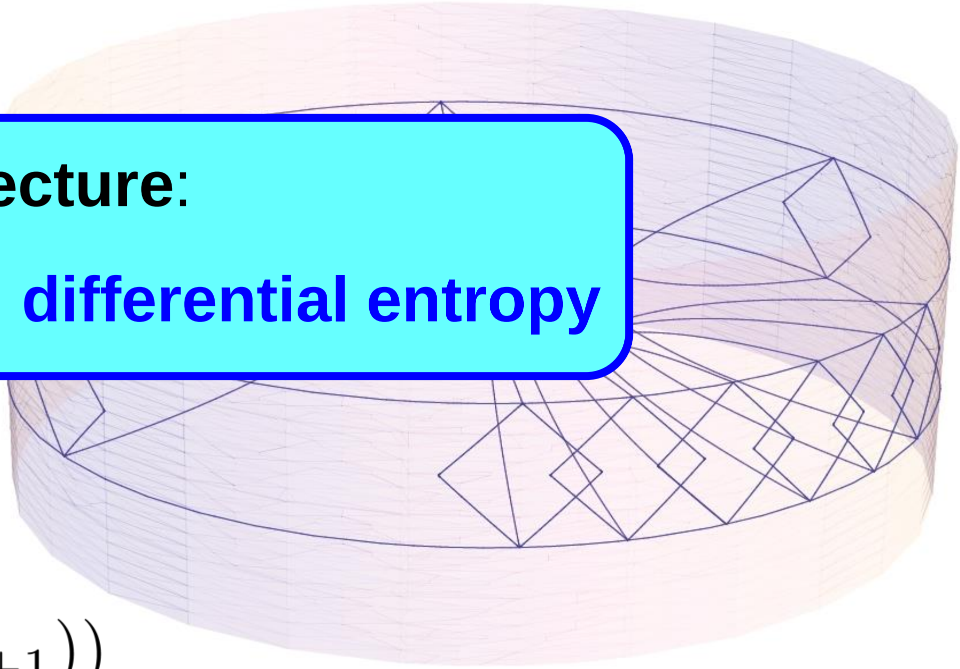
Conjecture:

residual entropy = differential entropy



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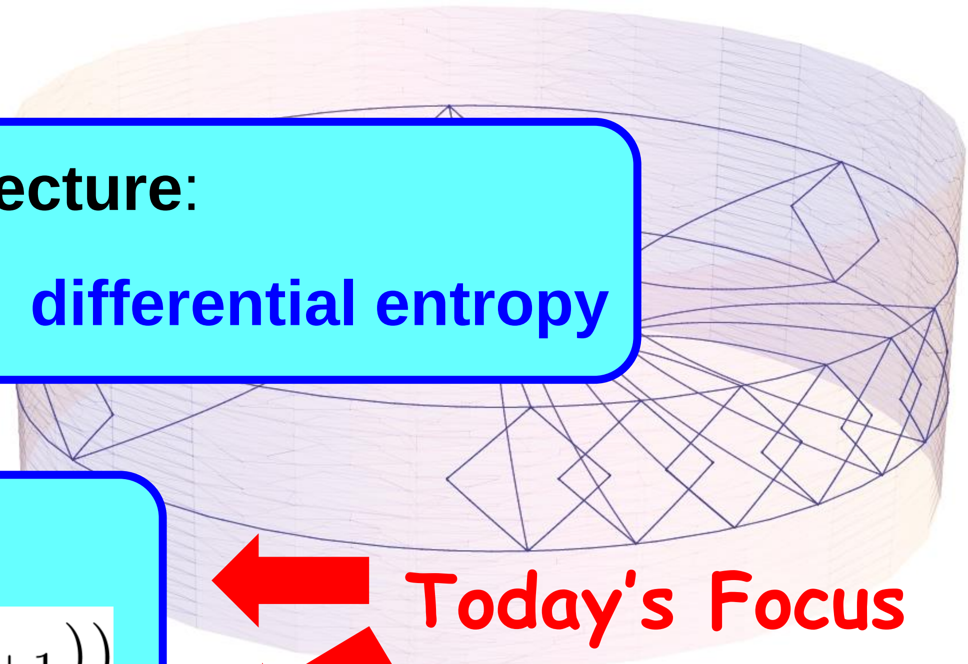

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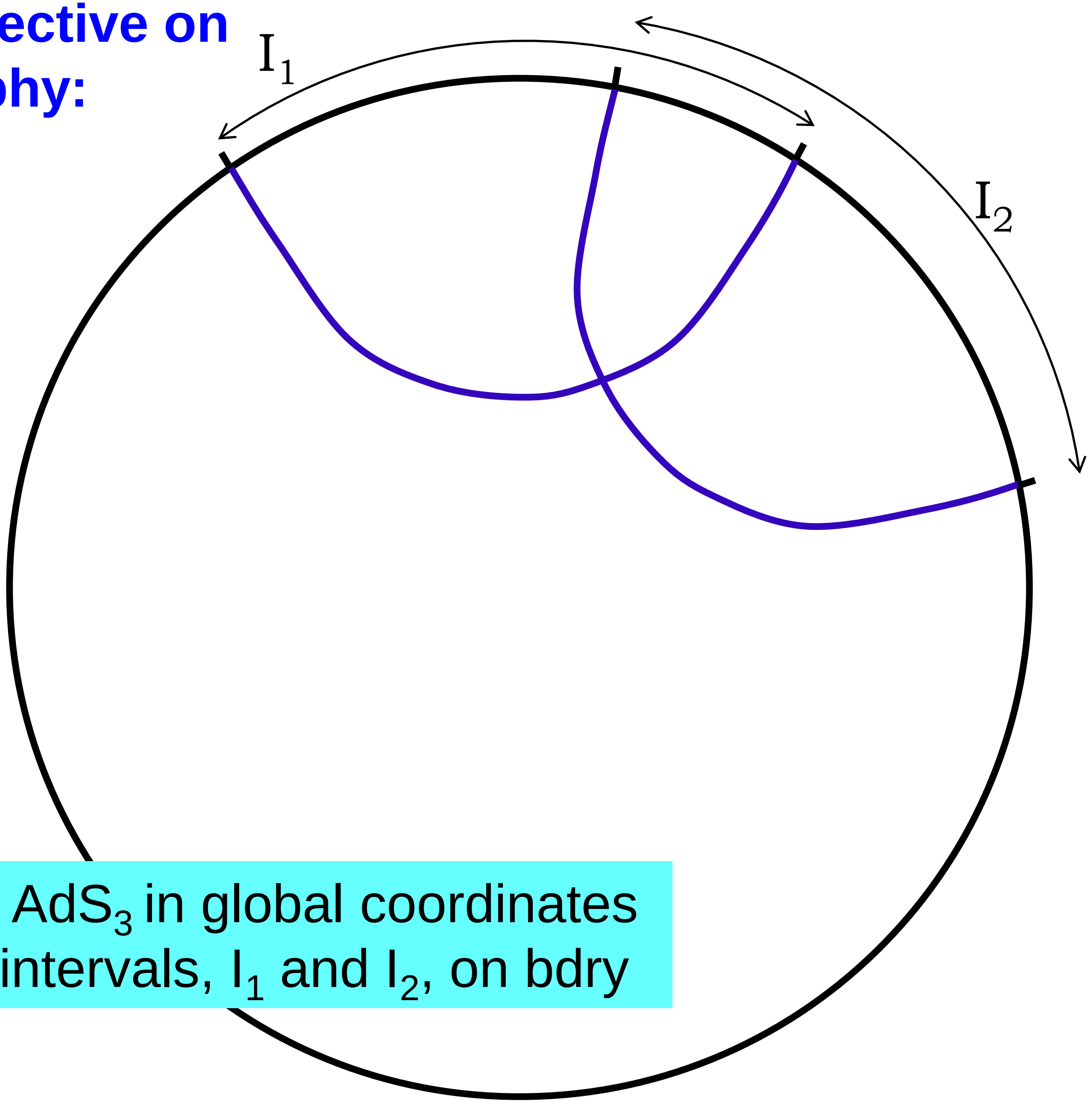
Today’s Focus

boundary observables which yield gravitational entropy of closed curves inside of d=3 AdS space with certain continuum limit

One perspective on hole-ography:



consider AdS_3 in global coordinates with two intervals, I_1 and I_2, on bdry

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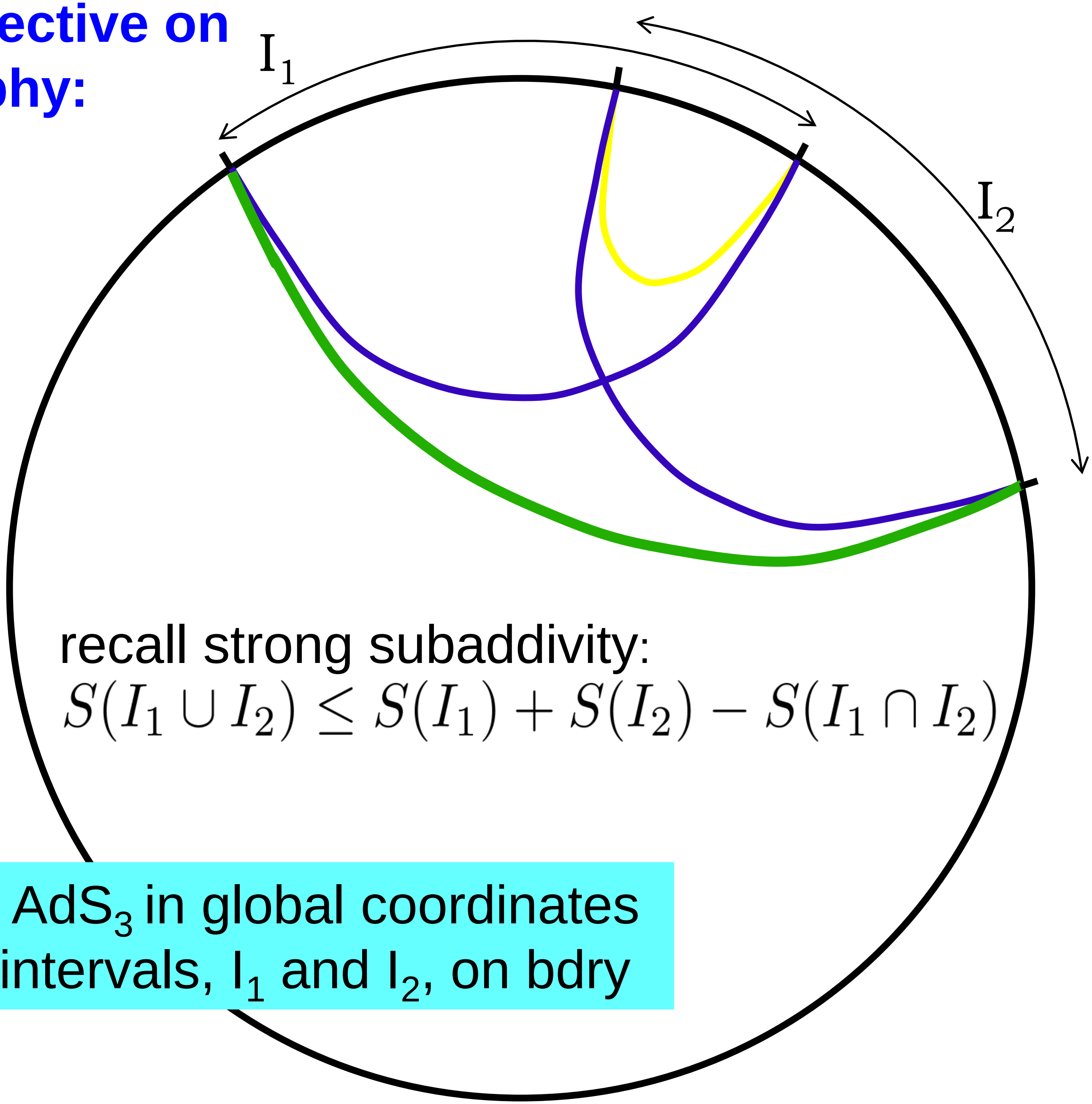
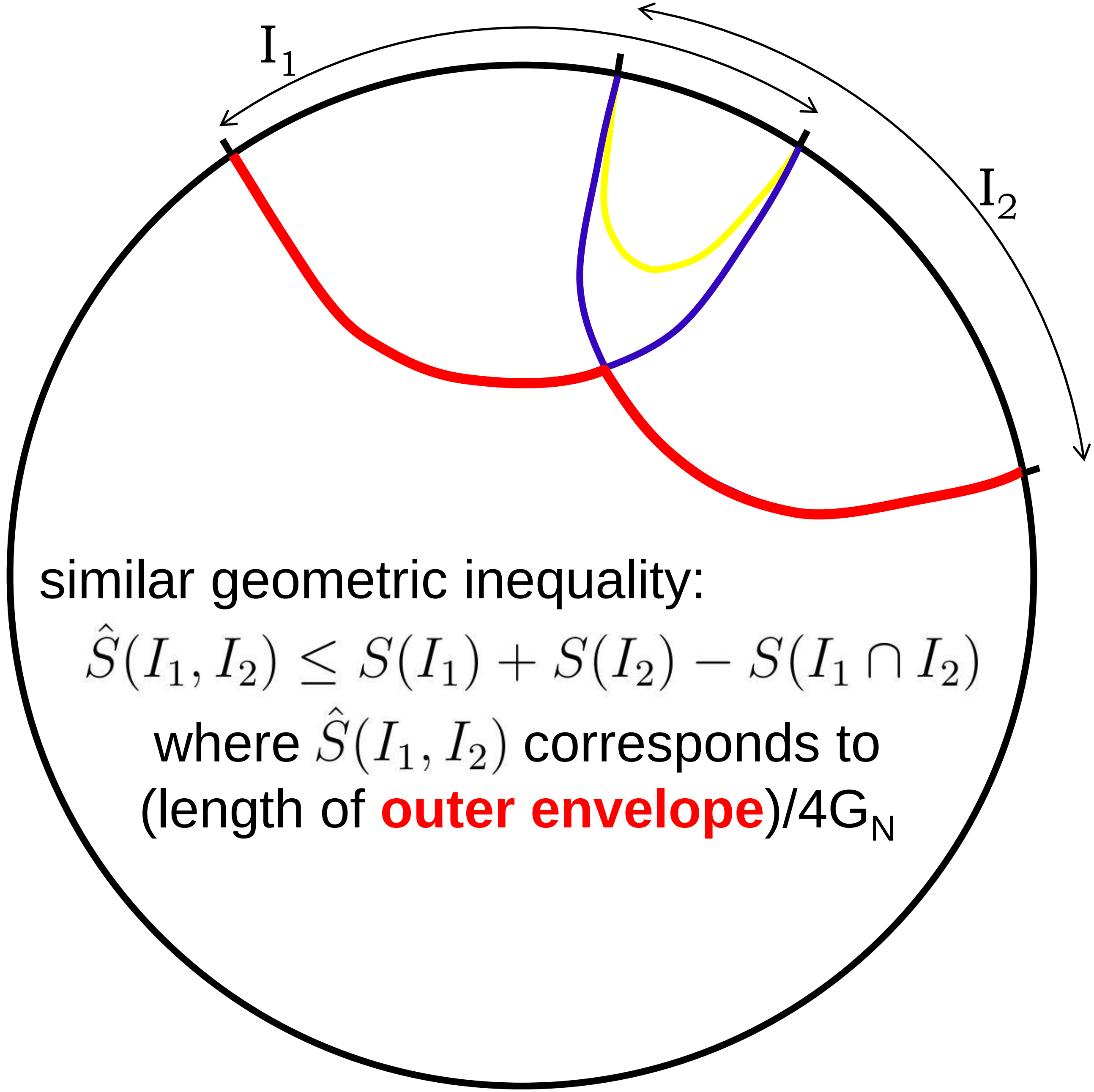


recall strong subadditivity:

$$S(I_1 \cup I_2) \leq S(I_1) + S(I_2) - S(I_1 \cap I_2)$$

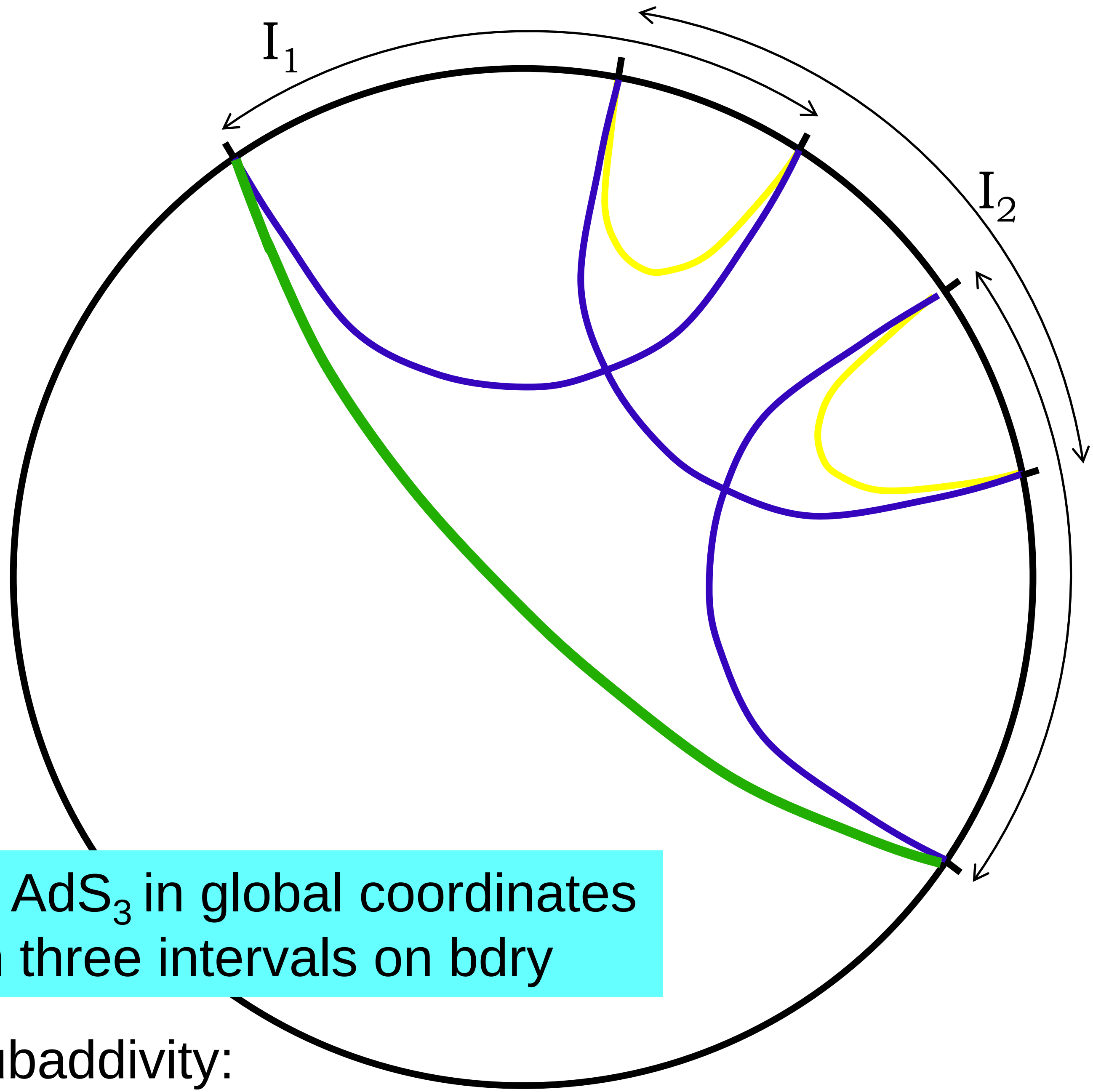
consider AdS_3 in global coordinates with two intervals, I_1 and I_2, on bdry



similar geometric inequality:

$$\hat{S}(I_1, I_2) \leq S(I_1) + S(I_2) - S(I_1 \cap I_2)$$

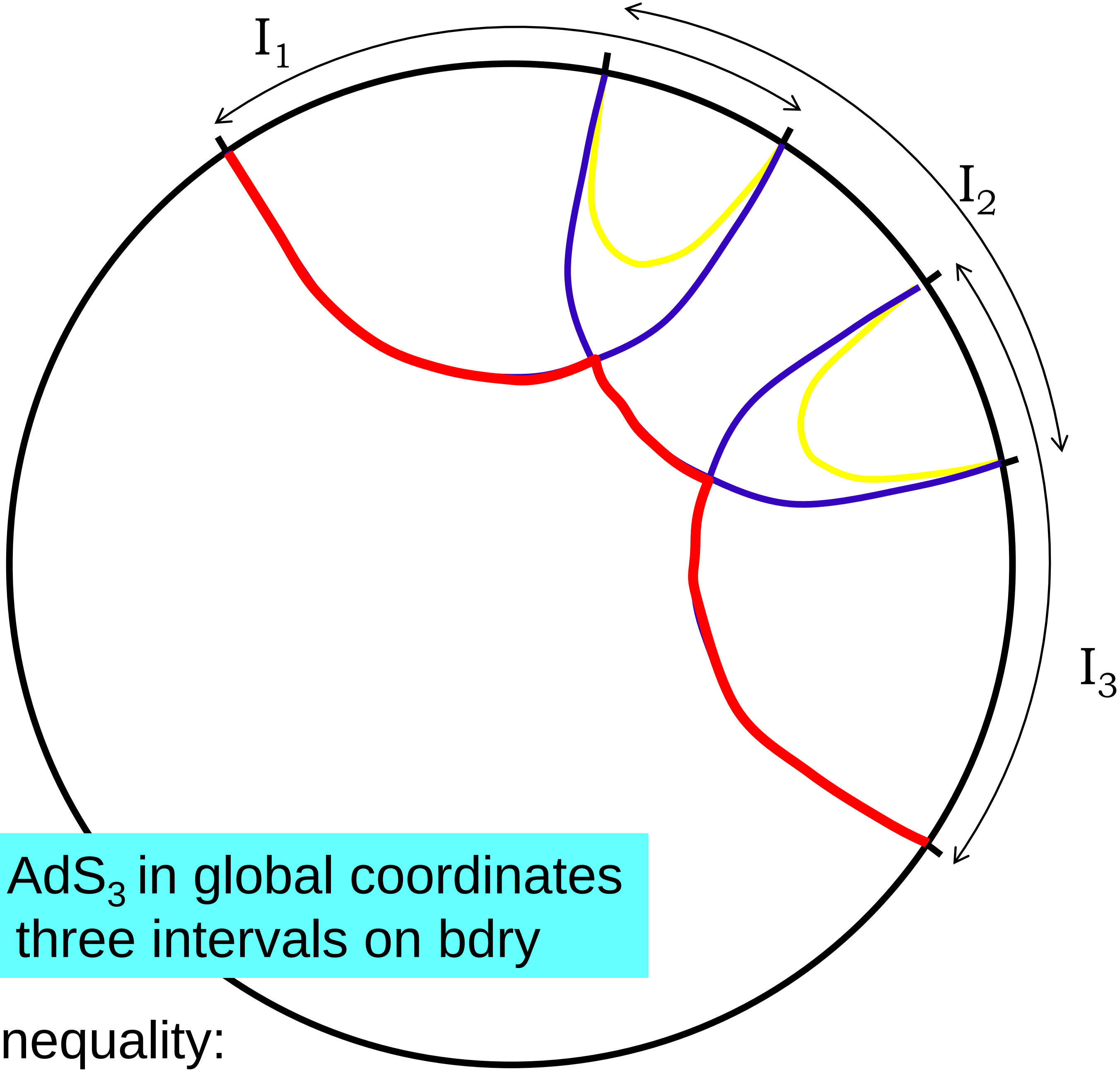
where $\hat{S}(I_1, I_2)$ corresponds to (length of **outer envelope**)/$4G_N$



consider AdS_3 in global coordinates
now with three intervals on bdry

strong subaddivity:

$$S(I_1 \cup I_2 \cup I_3) \leq S(I_1) + S(I_2) + S(I_3) - S(I_1 \cap I_2) - S(I_2 \cap I_3)$$



consider AdS$_3$ in global coordinates
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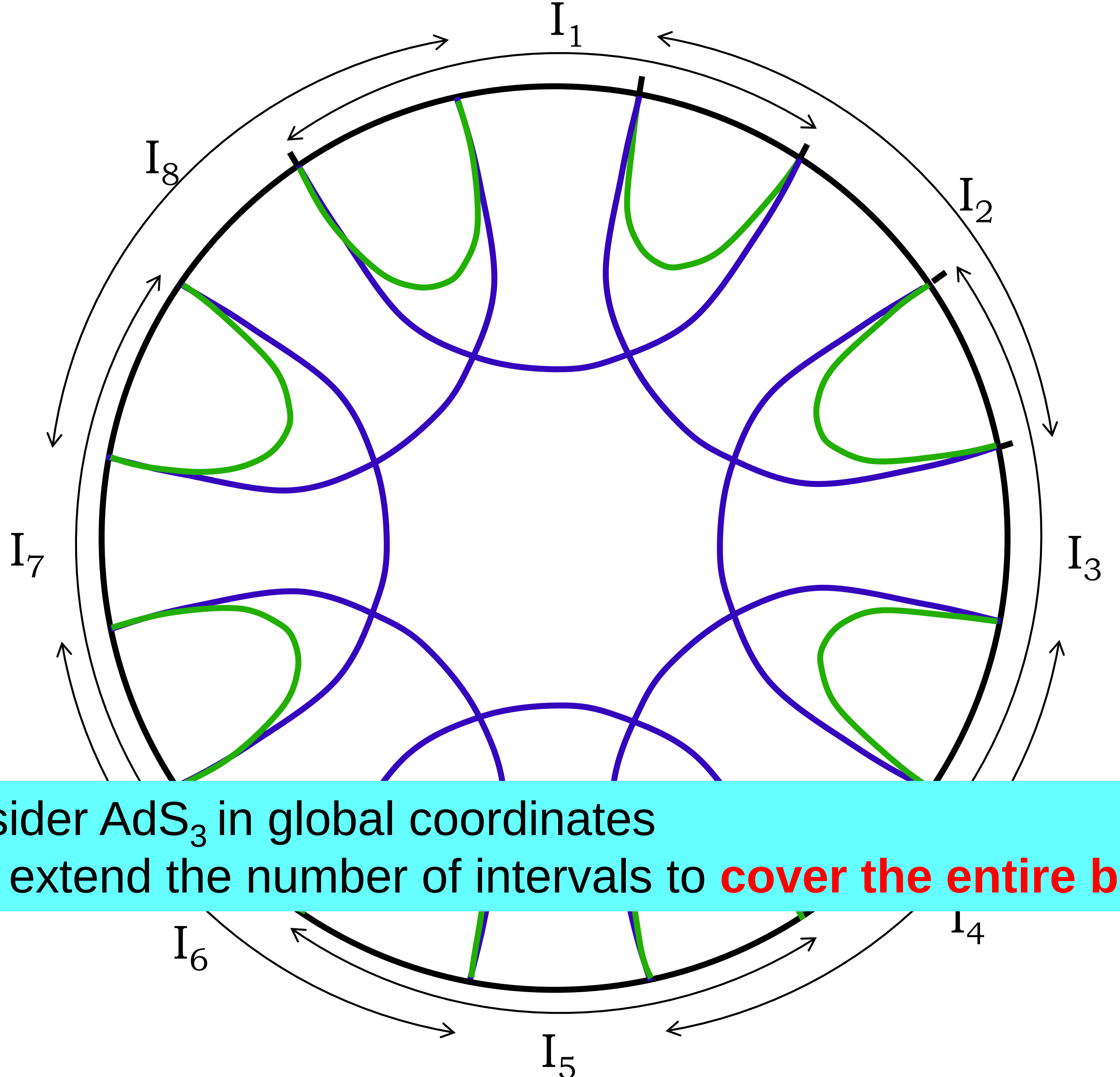
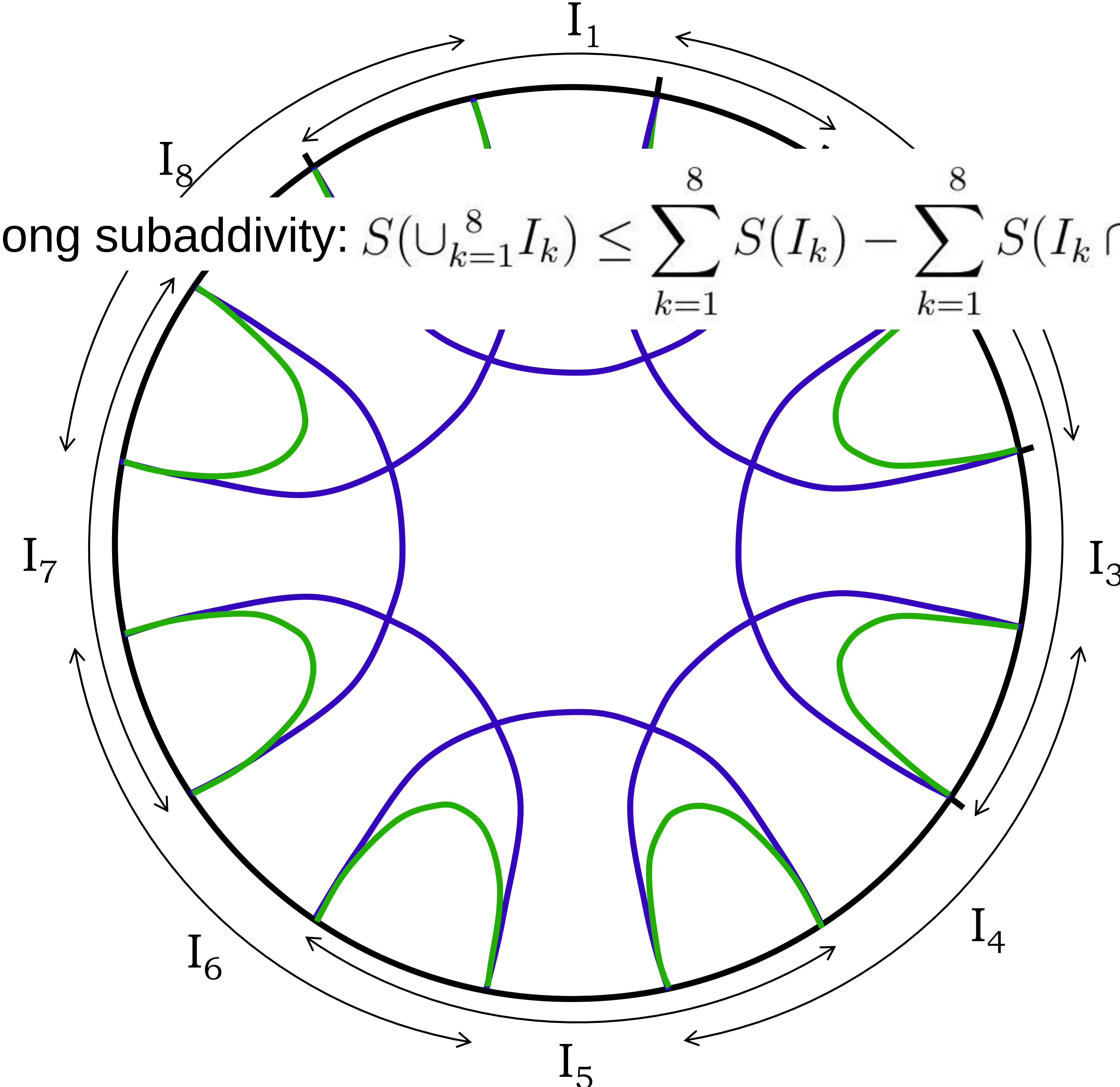
geometric inequality:

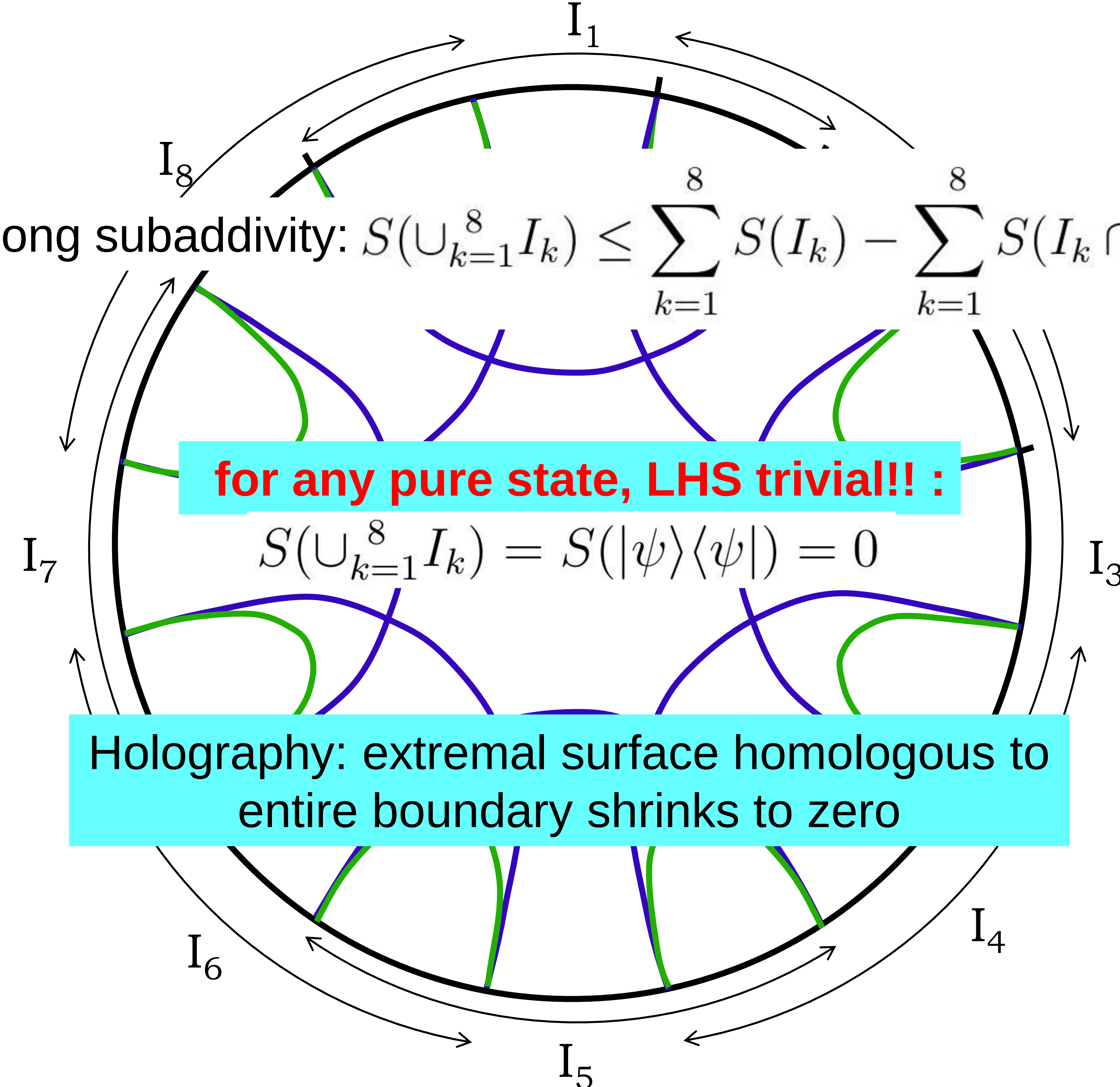
$$\hat{S}(I_1, I_2, I_3) \leq S(I_1) + S(I_2) + S(I_3) - S(I_1 \cap I_2) - S(I_2 \cap I_3)$$



consider AdS_3 in global coordinates
now extend the number of intervals to **cover the entire bdry**



strong subaddivity: $S(\cup_{k=1}^{8} I_k) \leq \sum_{k=1}^{8} S(I_k) - \sum_{k=1}^{8} S(I_k \cap I_{k+1})$

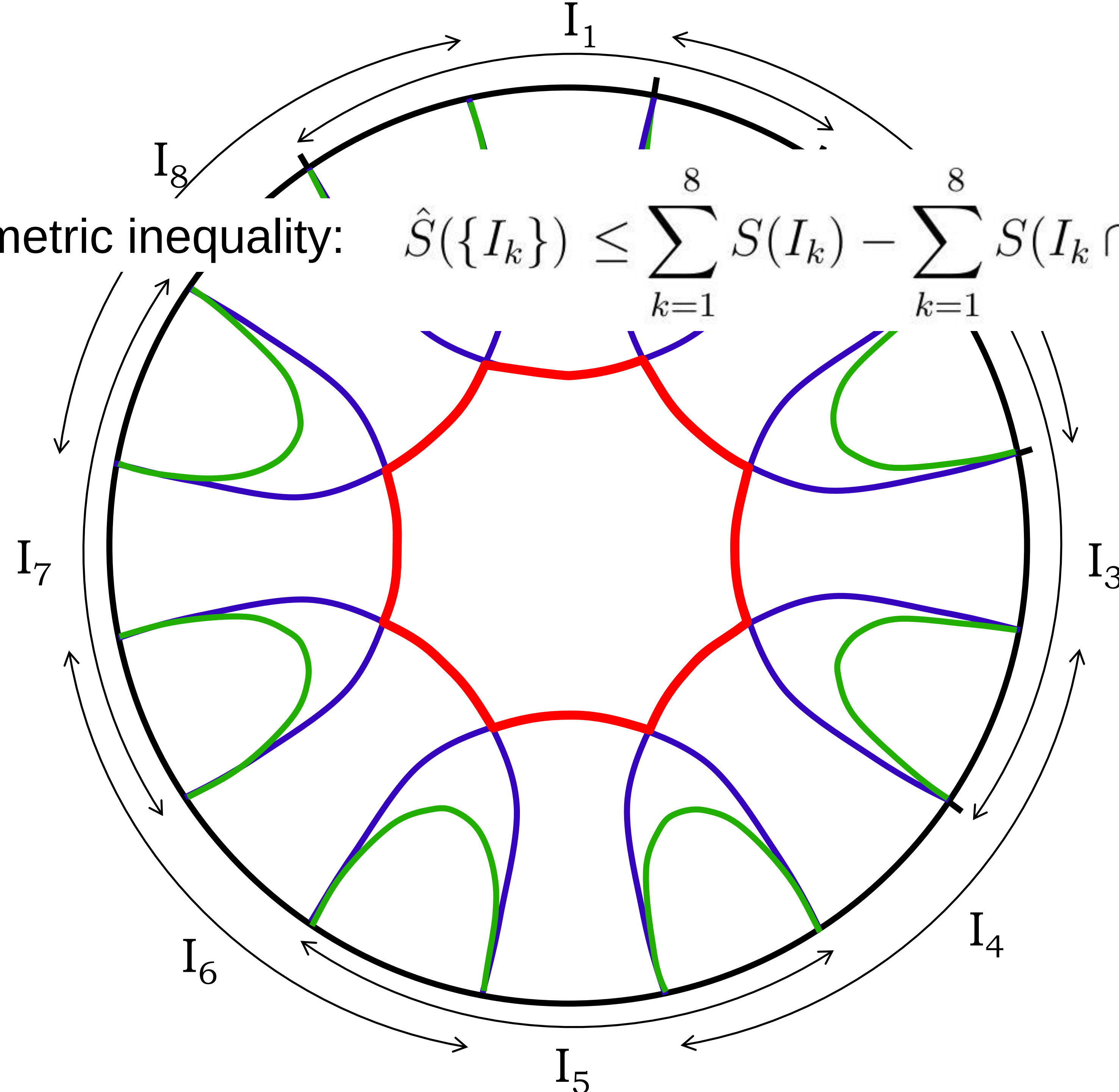


strong subaddivity: $S(\cup_{k=1}^{8} I_k) \leq \sum_{k=1}^{8} S(I_k) - \sum_{k=1}^{8} S(I_k \cap I_{k+1})$

for any pure state, LHS trivial!! :

$$S(\cup_{k=1}^{8} I_k) = S(|\psi\rangle\langle\psi|) = 0$$

Holography: extremal surface homologous to entire boundary shrinks to zero



geometric inequality: $\hat{S}(\{I_k\}) \leq \sum_{k=1}^{8} S(I_k) - \sum_{k=1}^{8} S(I_k \cap I_{k+1})$

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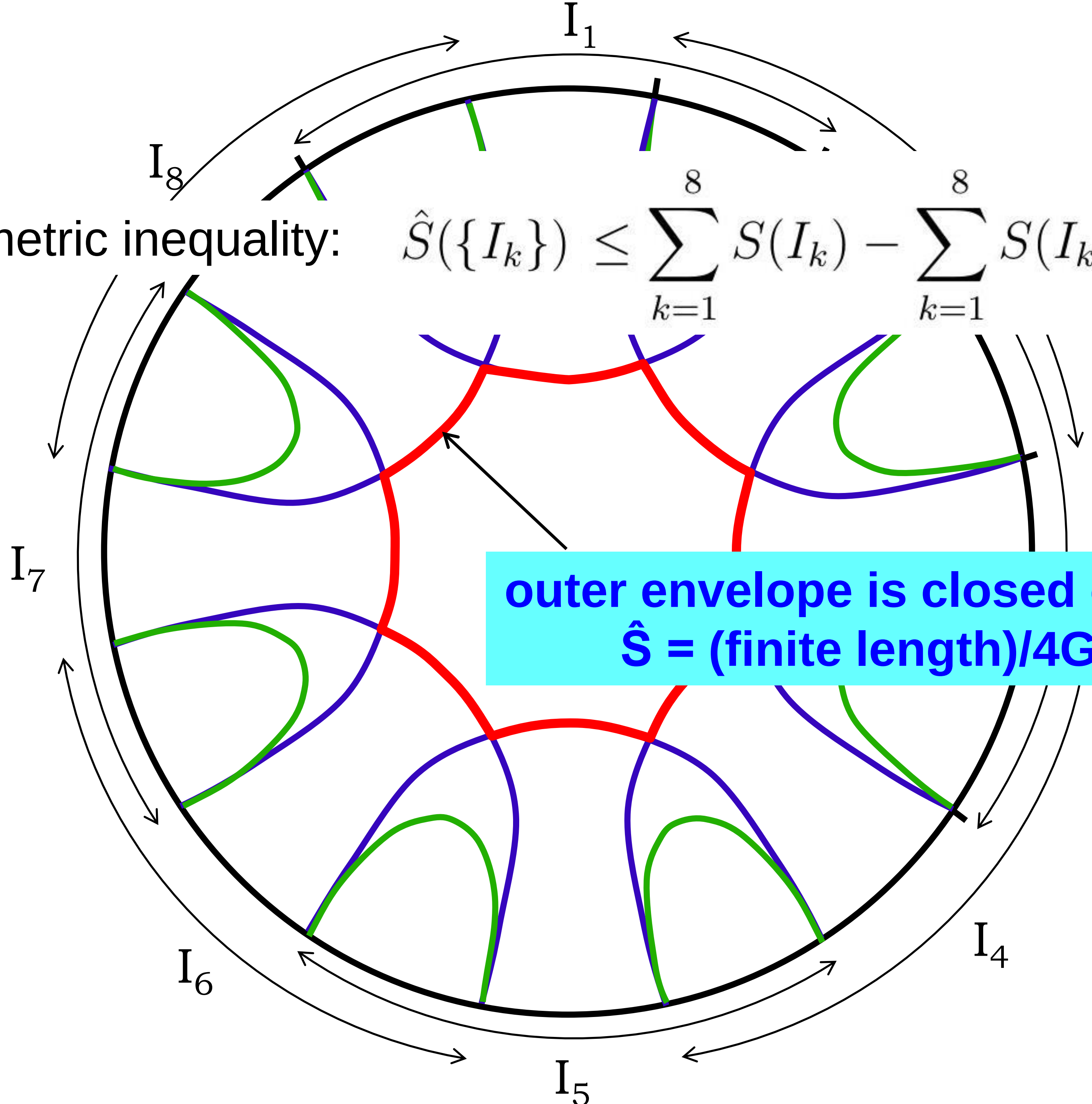
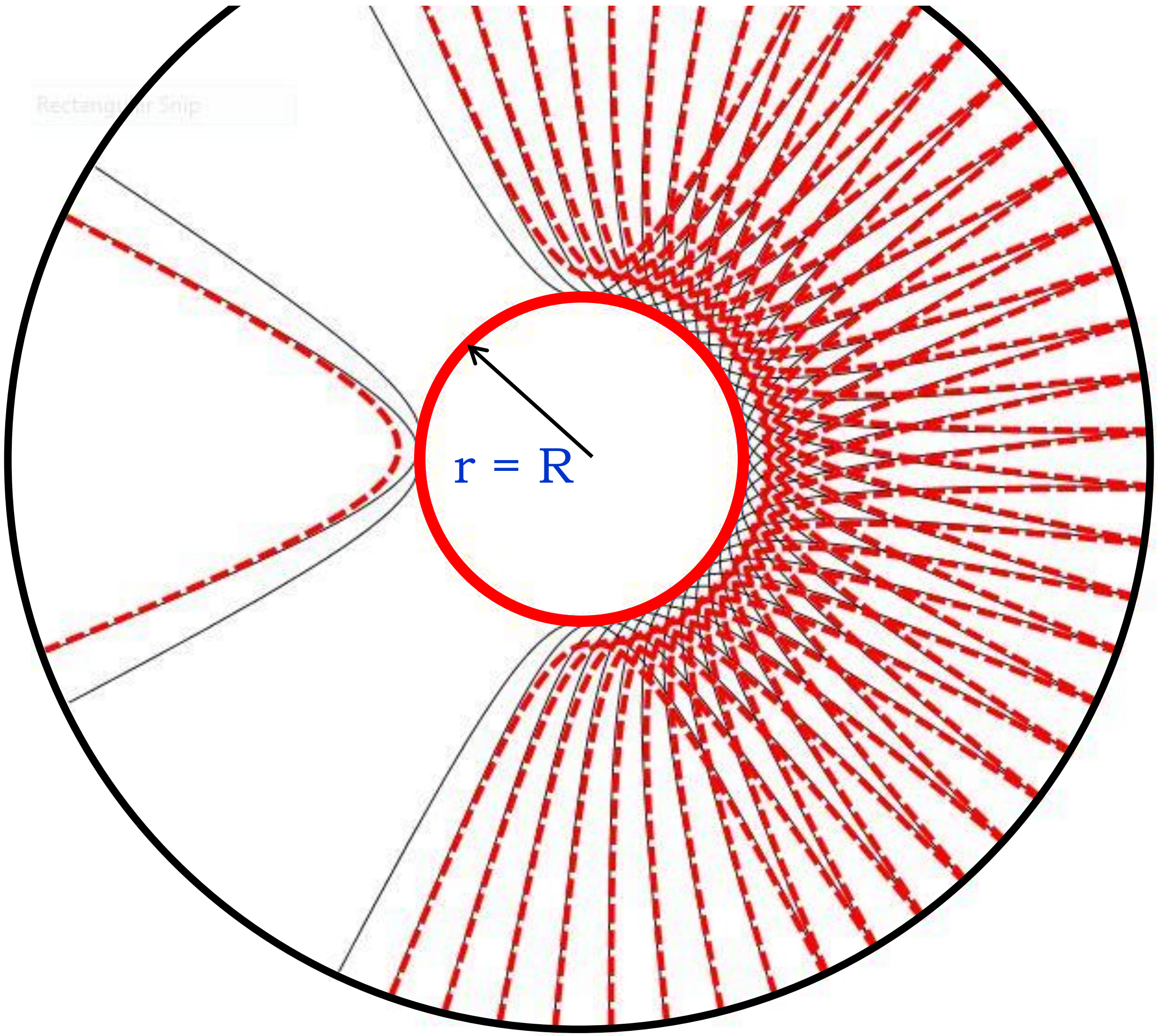


outer envelope is closed curve!
$\hat{S}$ = (finite length)/$4G_N$

(Balasubramanian, Chowdhury, Czech, de Boer & Heller)

- keep length of intervals is fixed but take number of intervals to infinity
- outer envelope becomes a smooth circle of constant radius



geometric inequality: $\hat{S}(\{I_k\}) \leq \Sigma_{k=1}^{\infty} S(I_k) - \Sigma_{k=1}^{\infty} S(I_k \cap I_{k+1})$

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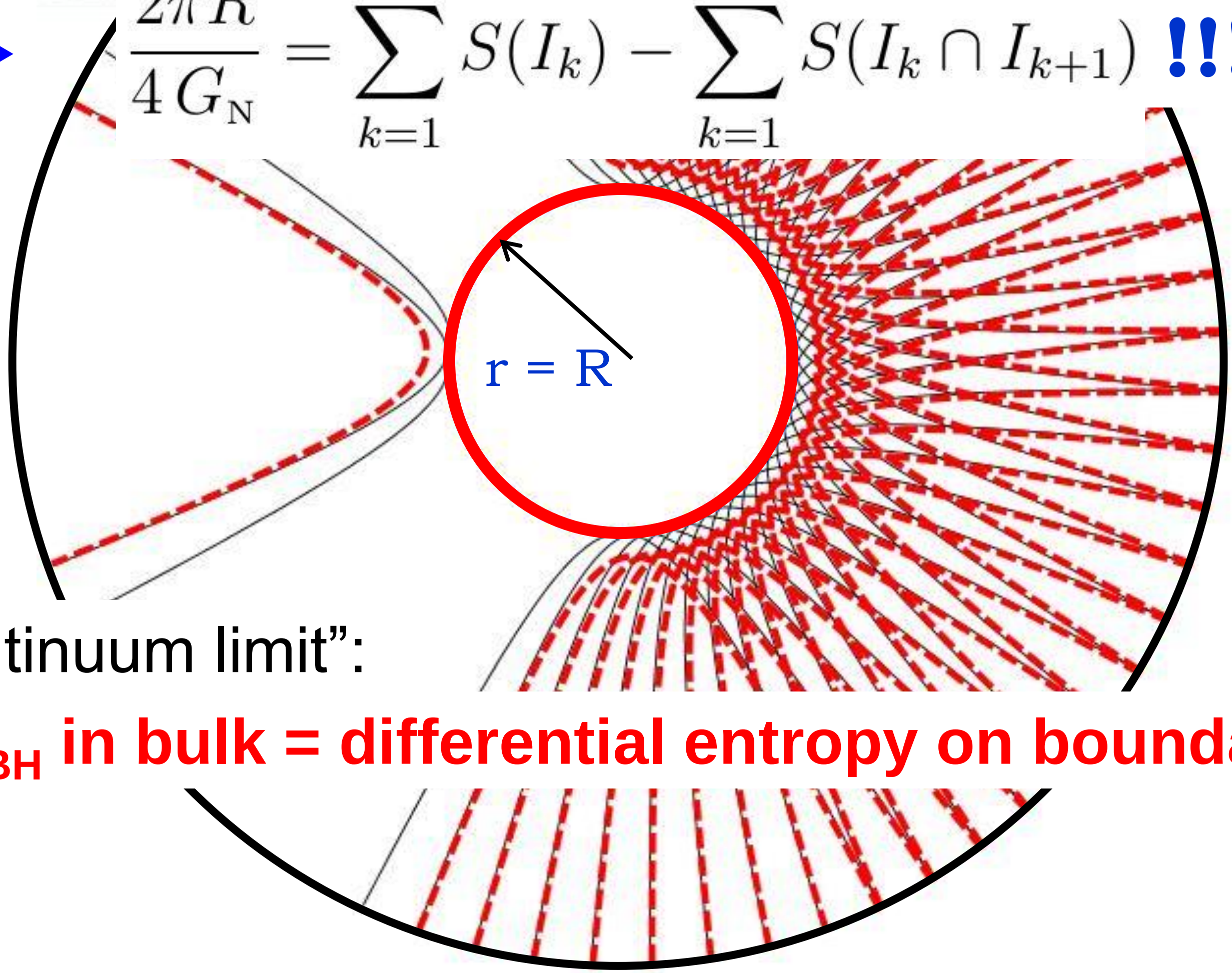
- keep length of intervals is fixed but take number of intervals to infinity
- outer envelope becomes a smooth circle of constant radius
- **surprise is that the geometric inequality is precisely saturated!**

$$\frac{2\pi R}{4\,G_N} = \sum_{k=1}^{\infty} S(I_k) - \sum_{k=1}^{\infty} S(I_k \cap I_{k+1}) \quad \textbf{!!!}$$



in “continuum limit”:

S_{BH} in bulk = differential entropy on boundary

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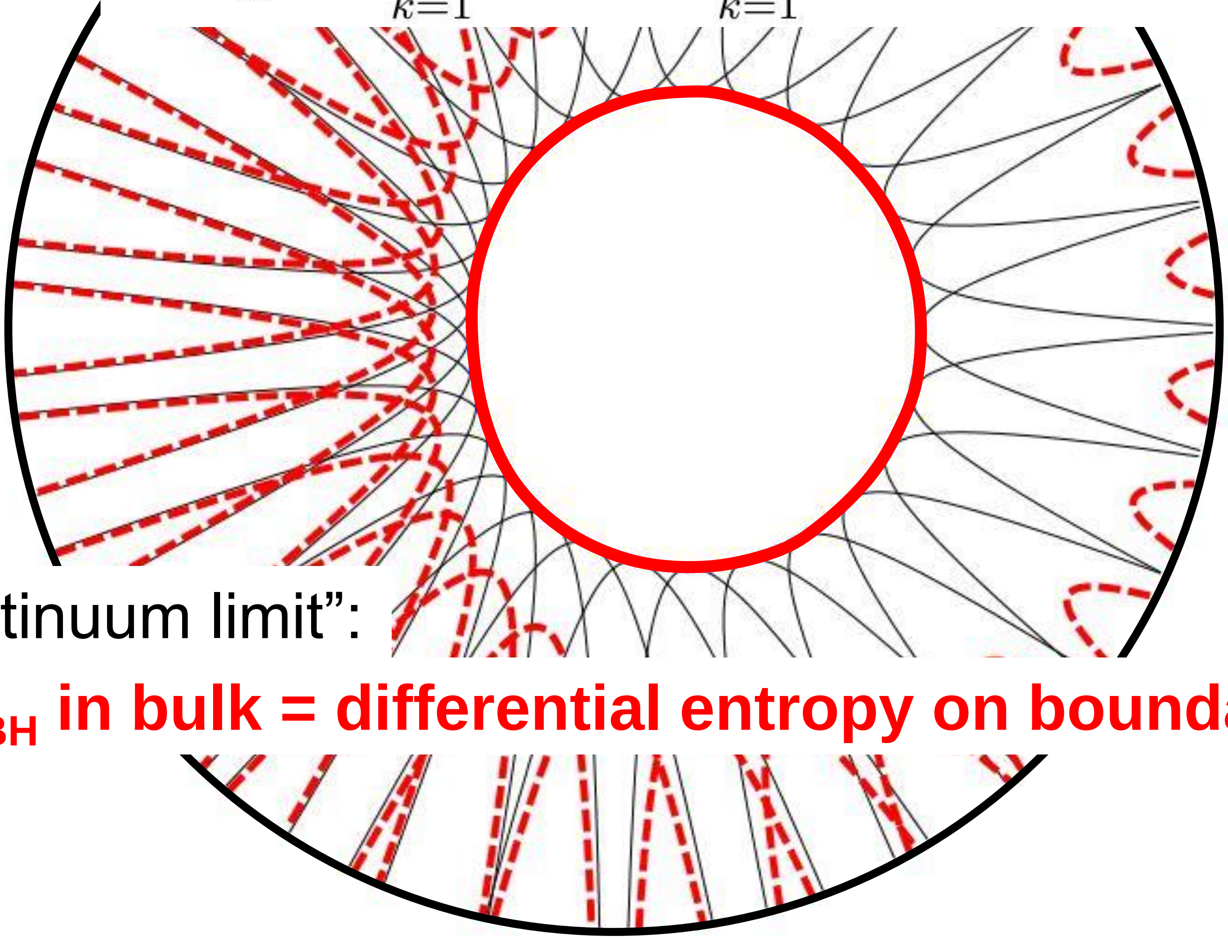
- prescription extends to general curves in the bulk with $\hat{S} \propto$ length of curve
- **geometric inequality is again saturated!**

$$\longrightarrow \quad \frac{\text{length}}{4\,G_{\text{N}}} = \sum_{k=1}^{\infty} S(I_k) - \sum_{k=1}^{\infty} S(I_k \cap I_{k+1}) \quad \textbf{!!!}$$



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(Balasubramanian, Chowdhury, Czech, de Boer & Heller: arXiv:1310.4204)

"hole-ographic spacetime" & differential entropy:

- calculate "gravitational entropy" of general closed curves in bulk of d=3 AdS space, in terms of entanglement entropies of regions covering time slice on boundary

$$\frac{\mathcal{A}}{4\,G_N} \le \sum \left(S(I_j) - S(I_j \cap I_{j+1})\right)$$

(RM, Rao & Sugishita: arXiv:1403.3416)

(Headrick, RM & Wien: arXiv:1408.4770)

$\longrightarrow$ extends to higher dimensions

$\longrightarrow$ extends to more general holographic backgrounds

$\longrightarrow$ extends to certain higher curvature theories in the bulk (ie, Lovelock gravity)

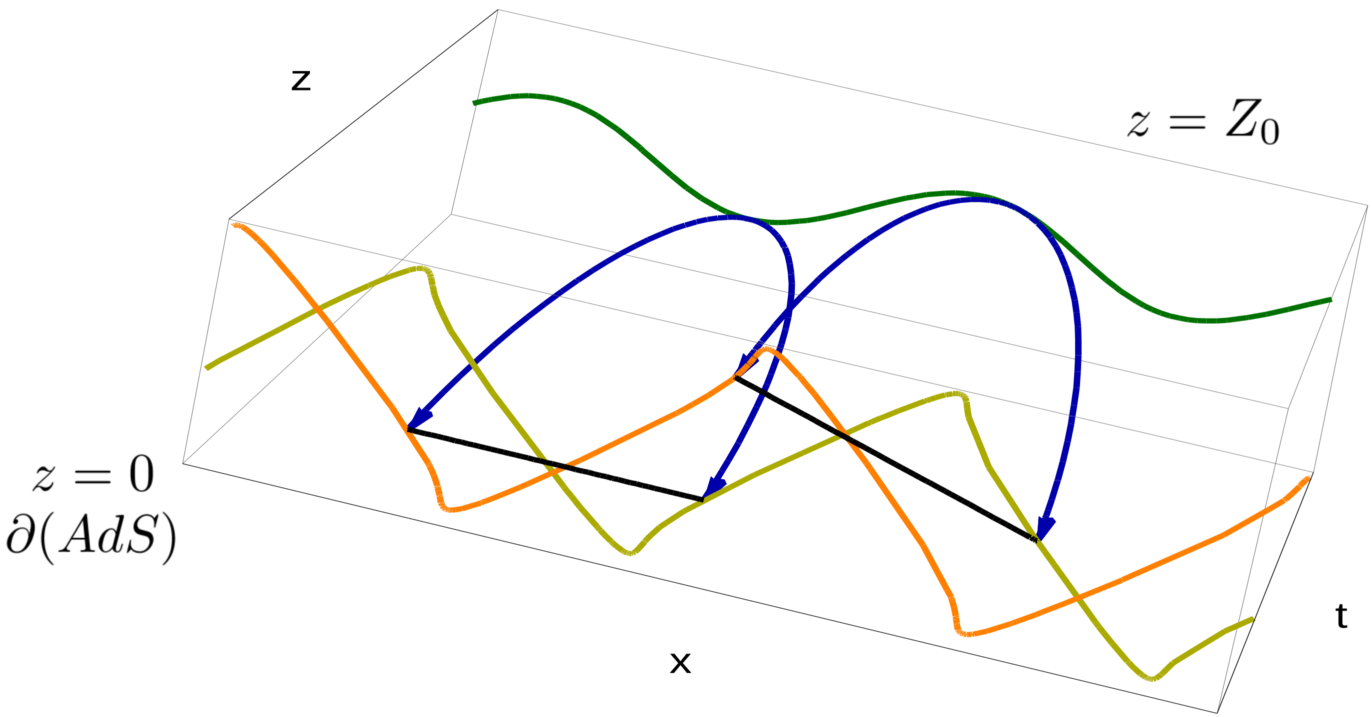
$\longrightarrow$ extends to bulk surfaces that vary in space and time

(RM, Rao & Sugishita: arXiv:1403.3416)

(Headrick, RM & Wien: arXiv:1408.4770)

Salient lessons:

- HEE evaluated by extremizing an entropy functional in bulk
- boundary data: two "independent" surfaces defining family of intervals: $\vec{\gamma}_L(\lambda) = \{t_L(\lambda), x_R(\lambda)\}$; $\vec{\gamma}_R(\lambda) = \{t_R(\lambda), x_R(\lambda)\}$
- define boundary intervals by finding extremal HEE surface which is tangent to bulk surface at each point



(Headrick, RM & Wien: arXiv:1408.4770)

- general "hole-ographic" construction can be packaged very simply using lessons from **classical mechanics**
- consider on-shell action: $S_{on} = \int_{s_i, q_i^a}^{s_f, q_f^a} ds\ \mathcal{L}(q^a, \partial_s q^a)$
- varying boundary conditions:

$$\delta S_{on} = p_f^a\, \delta q_f^a - E_f\, \delta s_f - p_i^a\, \delta q_i^a + E_i\, \delta s_i + \int ds [\, \cancel{eom} \cdot \delta q\,]$$

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- consider family of boundary conditions: $\{s_i(\lambda), q^a_i(\lambda)\},\ \{s_f(\lambda), q^a_f(\lambda)\}$

$$\partial_\lambda S_{on} = p^a_f\, \partial_\lambda q^a_f - H_f\, \partial_\lambda s_f - p^a_i\, \partial_\lambda q^a_i + H_i\, \partial_\lambda s_i$$

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periodic

- consider family of boundary conditions: $\{s_i(\lambda), q_i^a(\lambda)\},\ \{s_f(\lambda), q_f^a(\lambda)\}$

$$0 = \int_0^1 d\lambda \left[p_f^a\, \partial_\lambda q_f^a - H_f\, \partial_\lambda s_f - p_i^a\, \partial_\lambda q_i^a + H_i\, \partial_\lambda s_i \right]$$

(Headrick, RM & Wien: arXiv:1408.4770)

- general "hole-ographic" construction can be packaged very simply using lessons from **classical mechanics**

- consider on-shell action: $S_{on} = \int_{s_i, q_i^a}^{s_f, q_f^a} ds\ \mathcal{L}(q^a, \partial_s q^a)$

- varying boundary conditions:

$$\delta S_{on} = p_f^a\,\delta q_f^a - E_f\,\delta s_f - p_i^a\,\delta q_i^a + E_i\,\delta s_i + \int ds[\,\cancel{eom}\cdot\delta q\,]$$

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- further require reparametrization invariance: $s \;\to\; \tilde{s}(s)$

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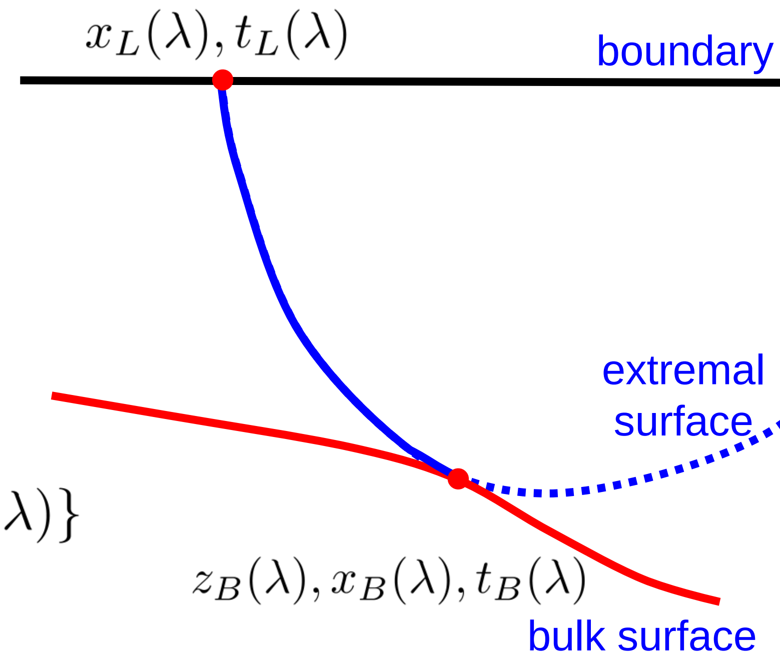
classical mechanics lemma:

- apply lemma to entropy problem with end-point data:

$$\{s_i(\lambda), q_i^a(\lambda)\} = \{s = 0, z = 0, x_L(\lambda), t_L(\lambda)\}$$

$$\{s_f(\lambda), q_f^a(\lambda)\} = \{s_{tang}(\lambda), z_B(\lambda), x_B(\lambda), t_B(\lambda)\}$$



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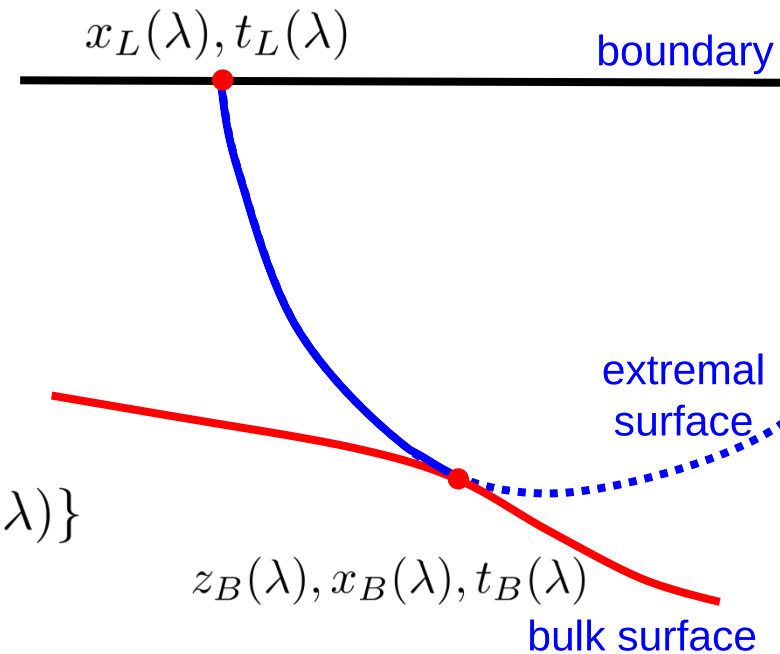

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$$\longrightarrow \underbrace{\int_0^1 d\lambda\, \mathcal{L}(q_B^a, \partial_\lambda q_B^a)}_{\text{Gravitational Entropy}} = \underbrace{-\int_0^1 d\lambda\, \frac{dq_L^a}{d\lambda}\frac{dS_{EE}}{dq_L^a}}_{\text{Differential Entropy}}$$

for general surfaces in general backgrounds

(with generalized planar symmetry)

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"hole-ography": robust entry in holographic dictionary

eg, extends from AdS_3 to higher dimensions, higher curvatures, general holographic backgrounds

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- **generalized planar symmetry:**

$$ds^2 = \underbrace{g_{ij}(x)\,dx^i\,dx^j}_{x^i = \{t,x,z\}} + \underbrace{g_{ab}(x,y)\,dy^a\,dy^b}_{y^a = d-2}$$

"planar" coord's

along with $\det[\,g_{ab}(x,y)\,] \;=\; f(x)\,h(y)$

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- **generalized planar symmetry:**

includes z, t & x!!

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along with $\det[\, g_{ab}(x, y)\,] \ = \ f(x)\, h(y)$

- very broad class of backgrounds including: Dp-brane throats, Lifshitz and Schrödinger geometries, black holes in previous backgrounds, AdS-Vaidya metrics, . . .

Question:

- "differential entropy" defined a boundary observable where input is a family of intervals in boundary geometry
- have bulk-to-boundary construction: start with bulk surface and construct corresponding boundary data, ie, $\vec{\gamma}_L(\lambda)$ and $\vec{\gamma}_R(\lambda)$
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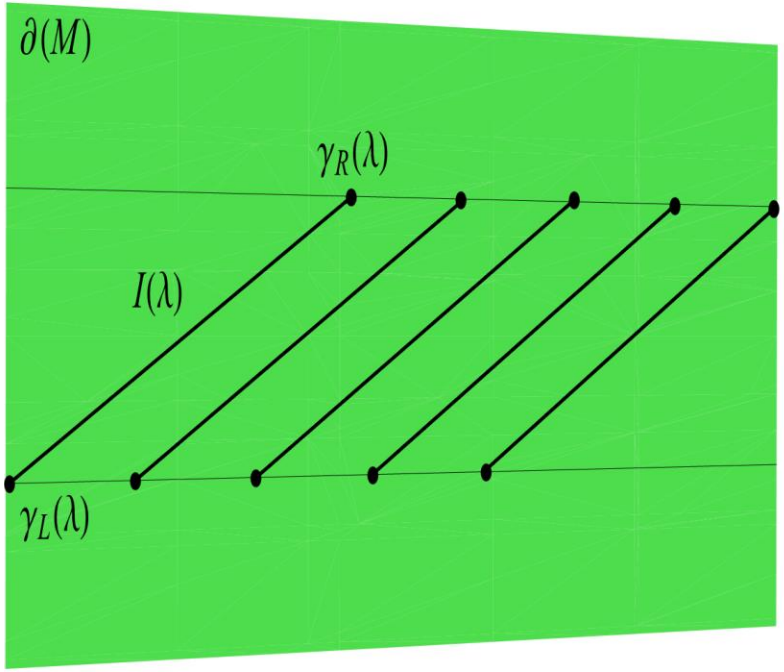

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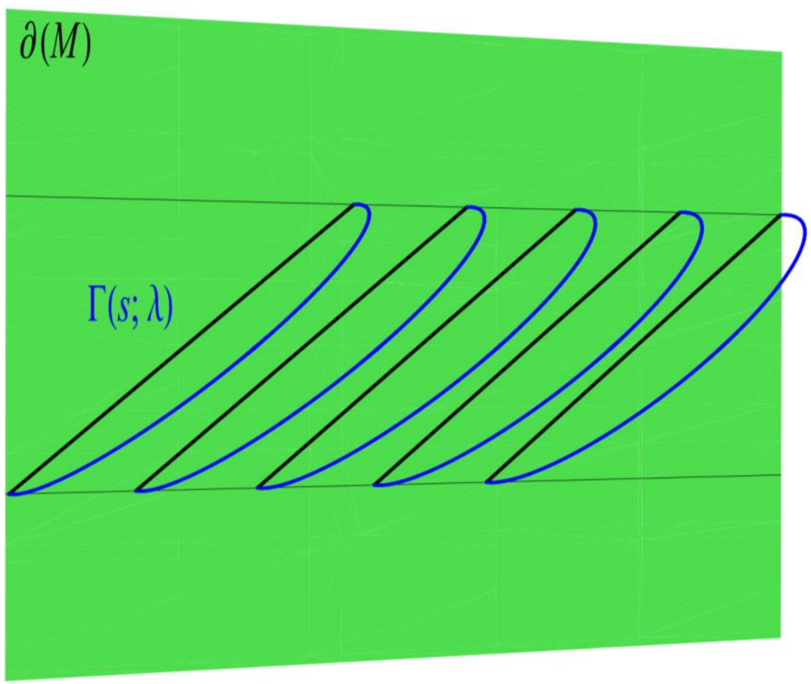

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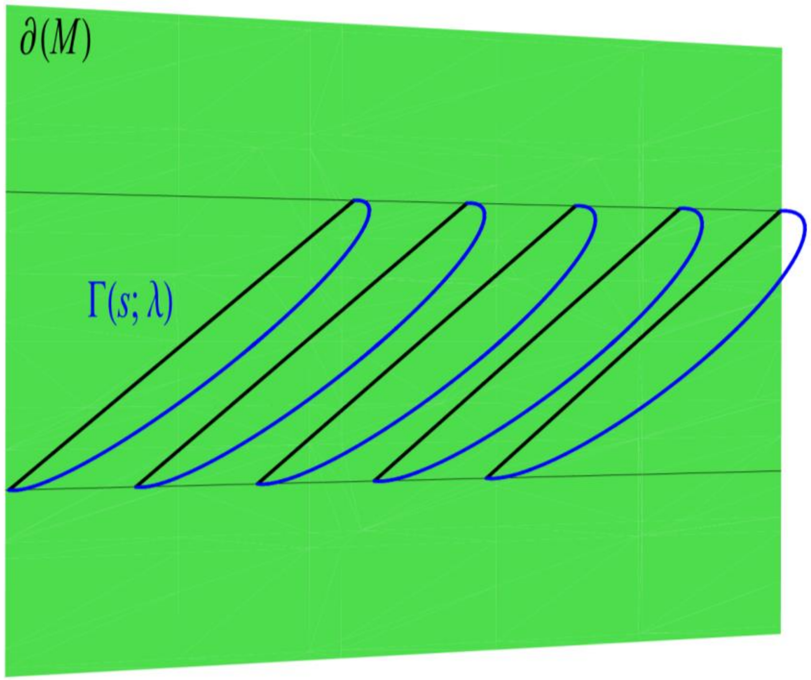
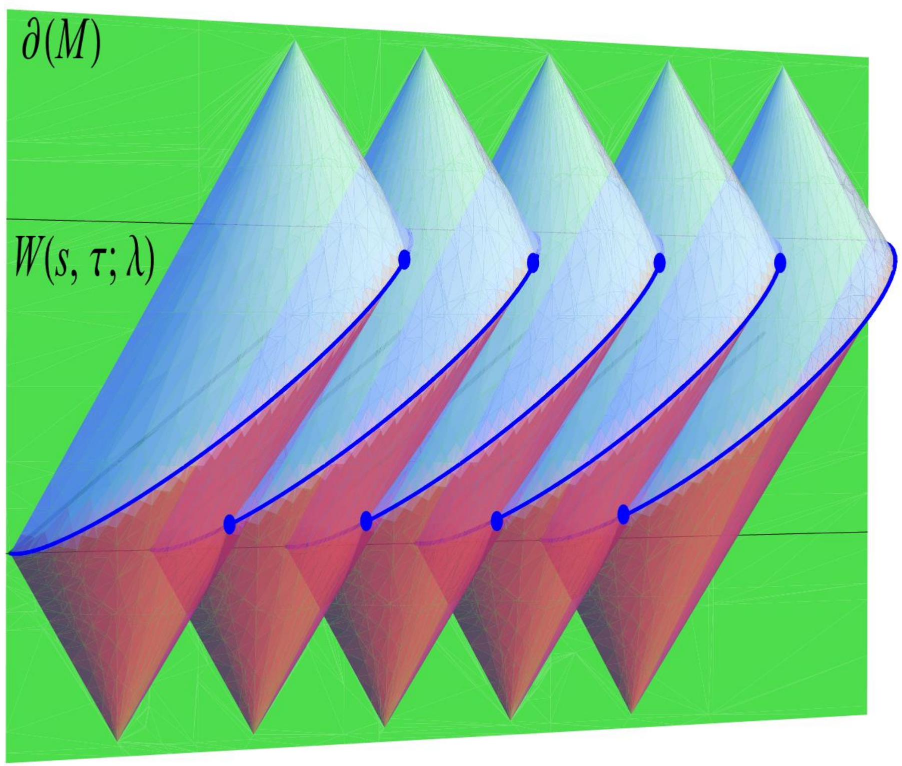


→ are these intervals still associated with some bulk surface??

Question:

- boundary-to-bulk construction?? can we reverse engineer bulk surface from boundary data??
- look at "guts" of proof of bulk-to-boundary construction*

→ sufficient for curves to intersect light sheet emerging from neighbour: $\frac{\partial_\lambda x_B^i}{|\partial_\lambda x_B|} = \frac{\partial_s x_B^i}{|\partial_s x_B|} + k^i$ with $\begin{cases} k \cdot k = 0 \\ k \cdot \partial_s x_B^i = 0 \end{cases}$



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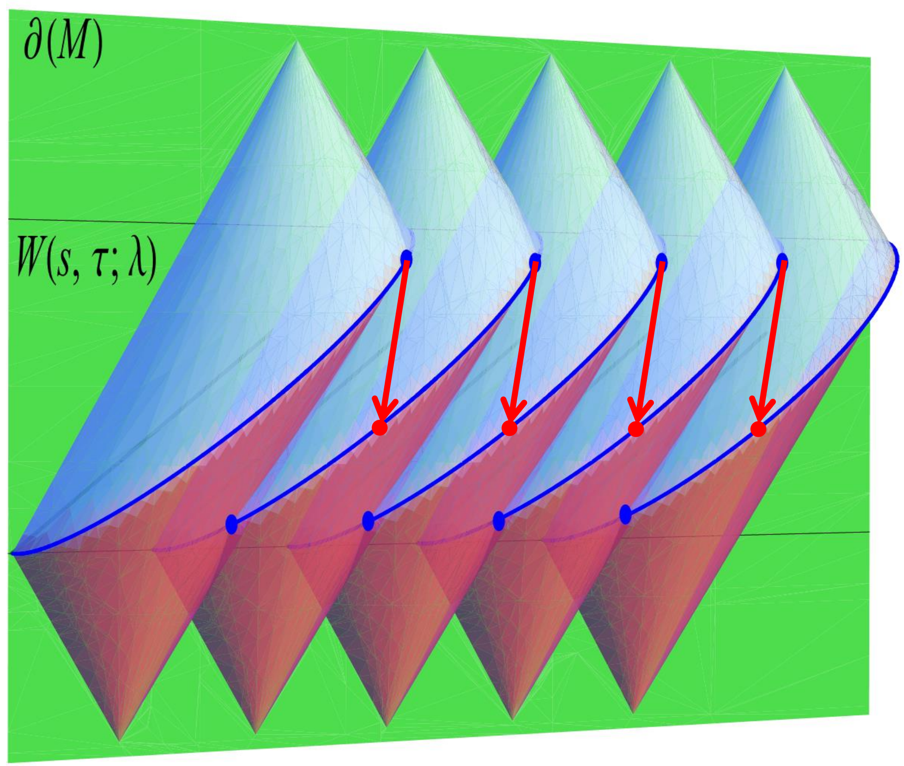

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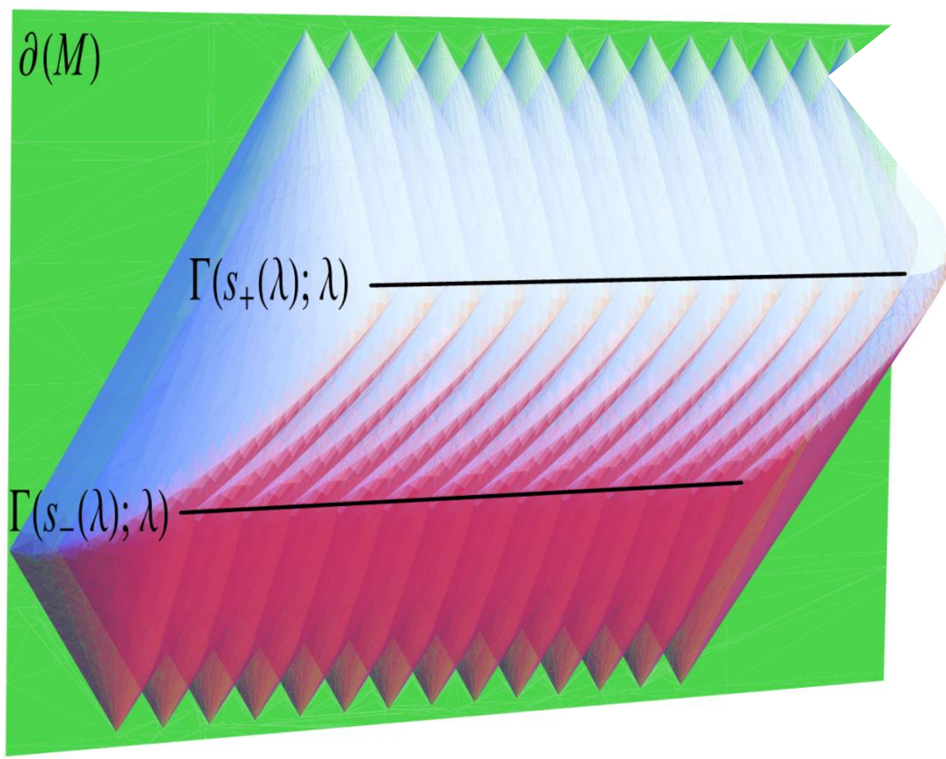

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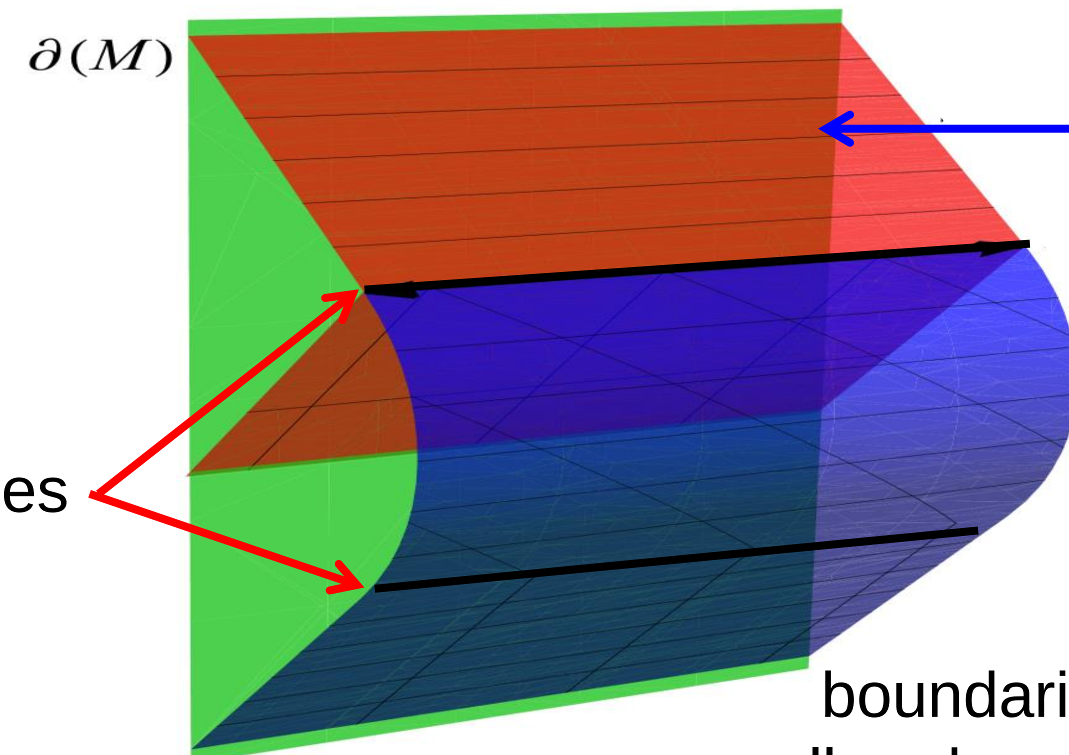


"enveloping surface"

two surfaces

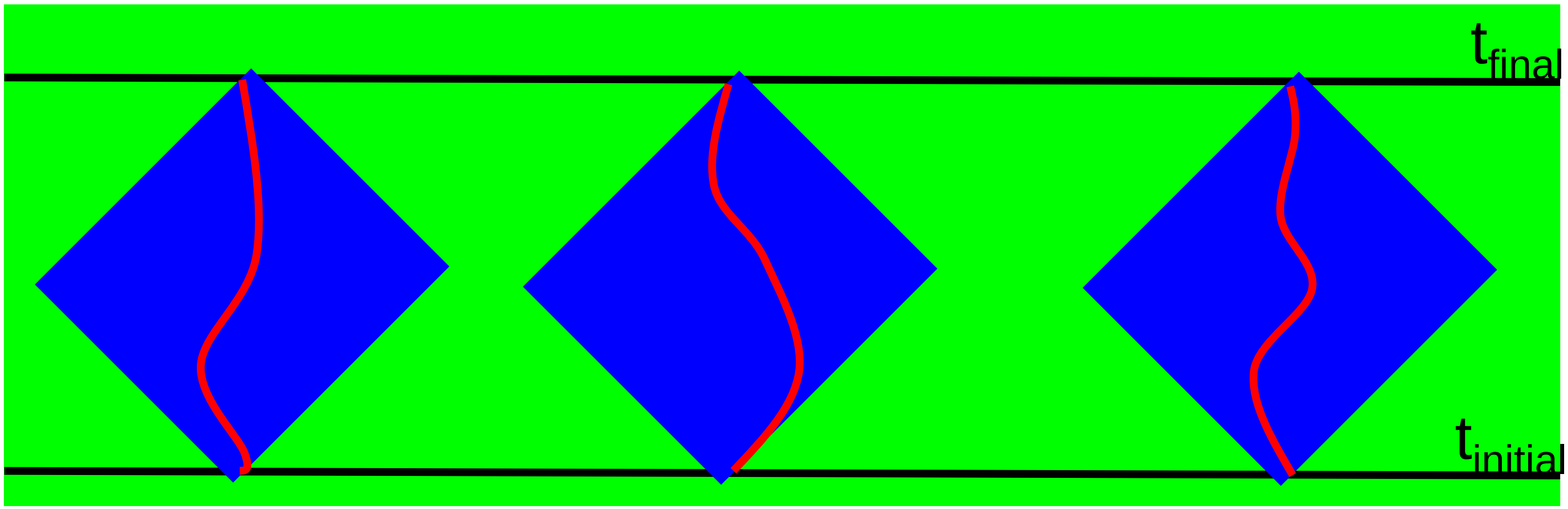
boundaries between null and space-like normal

* rest only applies for Einstein gravity in bulk and $S_{BH}=A/4G$

Time strips:

(Balasubramanian, Chowdhury, Czech, de Boer & Heller)

- observations limited to be within a finite “time strip”



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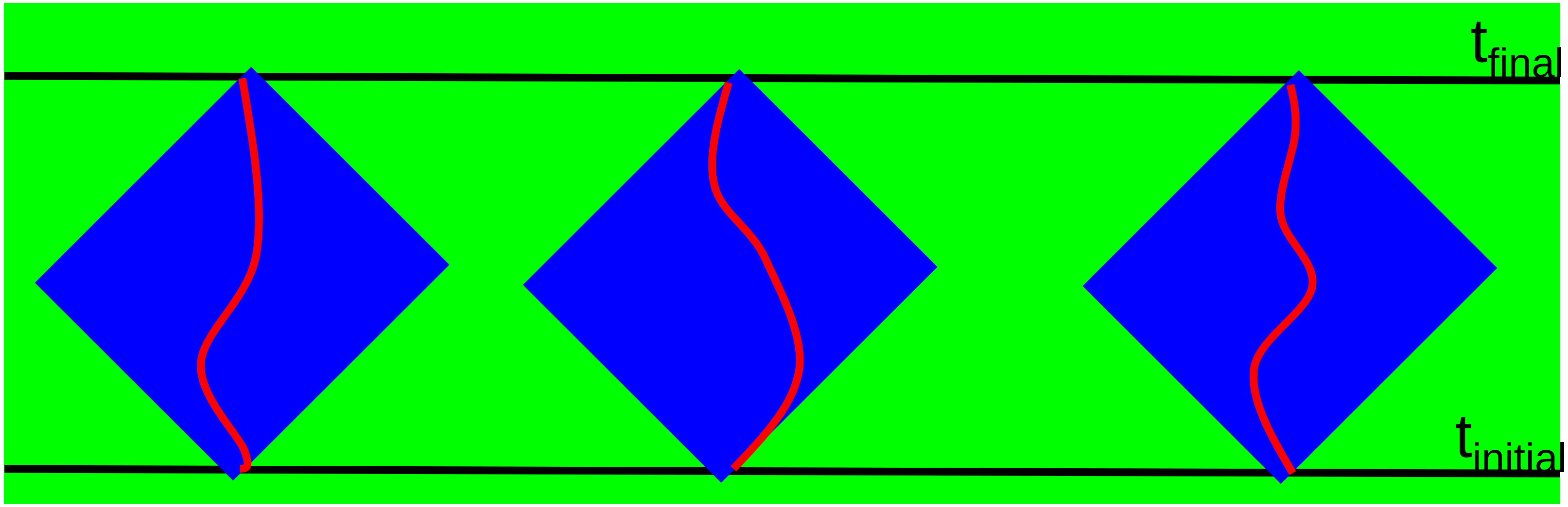
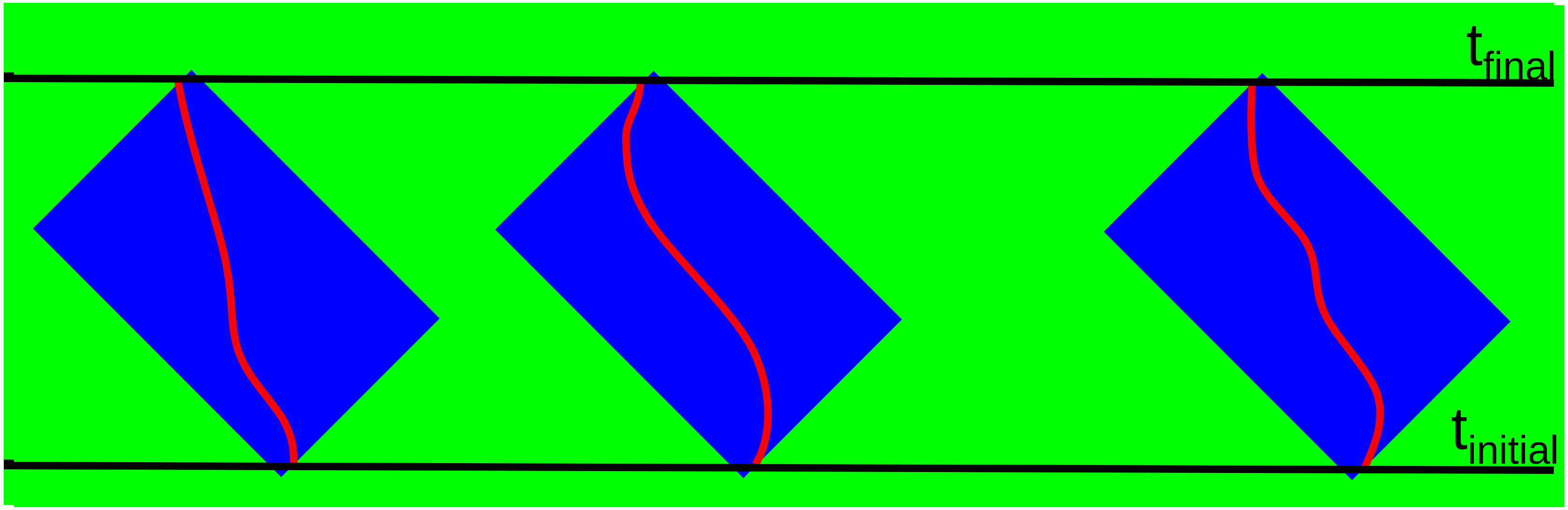


- time strip alone does not fix “protocol” & differential entropy



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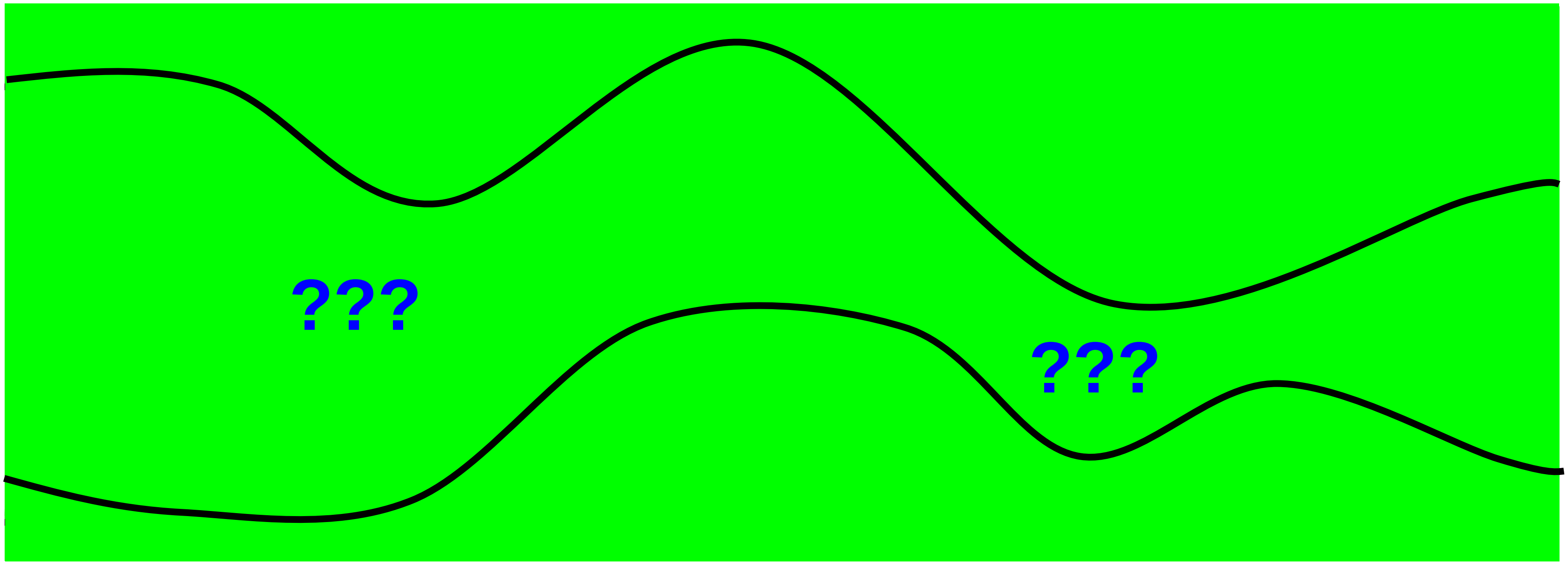

- many families of observers for the same time strip
- **Question**: what is most effective protocol to minimize the differential entropy for a given time strip?*

(* Hint: not maximum proper time protocol)

More questions:

- **Beyond generalized planar symmetry**

  → improved by approach of Czech, Dong & Sully

strategy is to foliate bulk surface with codimension one “loops” and use as b.c. (like alignment of tangent vectors) to construct extremal surfaces and corresponding “loops” in boundary theory

- how general is construction?

More questions:

- **Beyond generalized planar symmetry**

  → improved by approach of Czech, Dong & Sully

- how general is construction?
- covariant formulation of differential entropy?
- tiling boundary with finite regions?

Conclusions:

- holographic S_{EE} suggests new perspectives

→ quantum information & entanglement may yield important insights to fundamental issues in quantum gravity

- spacetime entanglement: S_{BH} applies for generic large regions
- "hole-ography" (ie, gravitational entropy = differential entropy) points to a precise definition in AdS/CFT context
- "differential operators": new insights on quantum gravity in AdS

Lots to explore!