Should entropy really be proportional to area ?

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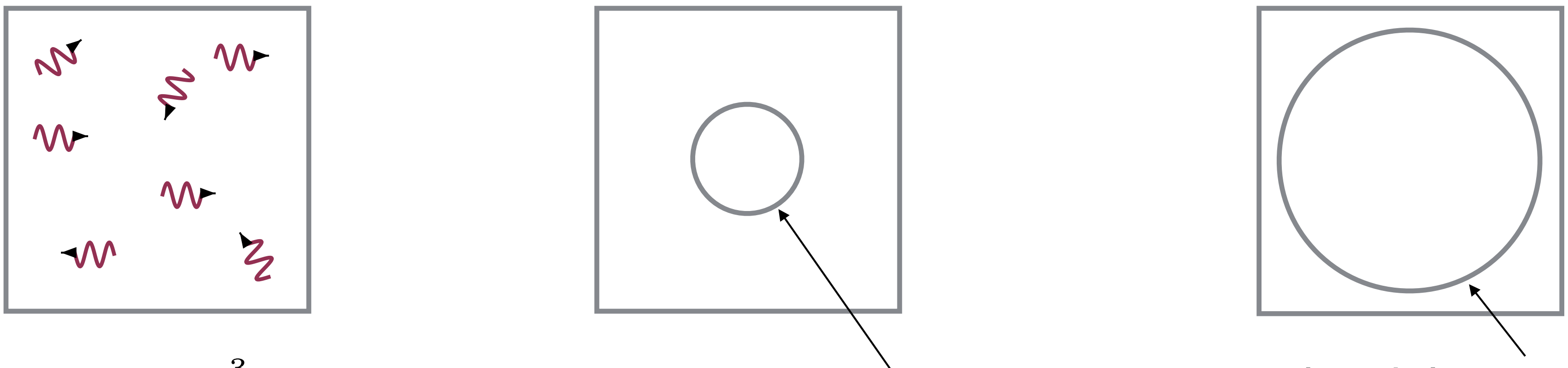
(Work in collaboration with Ali Masoumi, arXiv:1406.5798, arXiv:1412.2618)

See also Fischler + Susskind, Banks + Fischler, Brustein+Veneziano, Sasakura, E. Verlinde …

(A) Entropy in a box

How much entropy can we put in a box of volume V ?

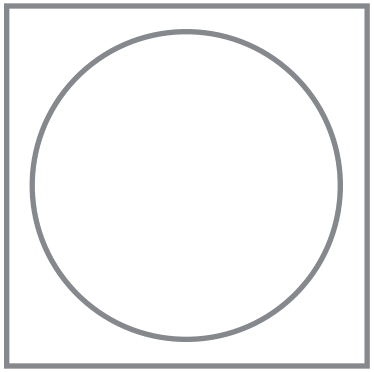


$$S \sim E^{\frac{3}{4}}$$

Can we put even more entropy in the box ?

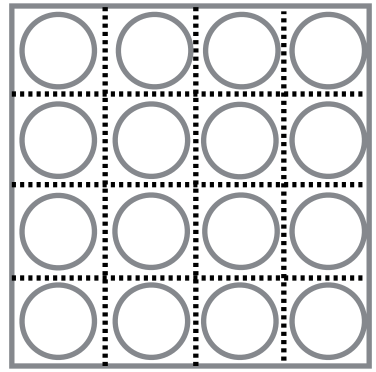
E_1

S_1



E_2

S_2



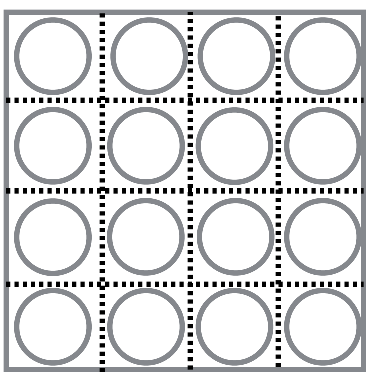
much more entropy !

$S_2 > S_1$

We have the area of the walls of the box, as well as the area of all the partitions

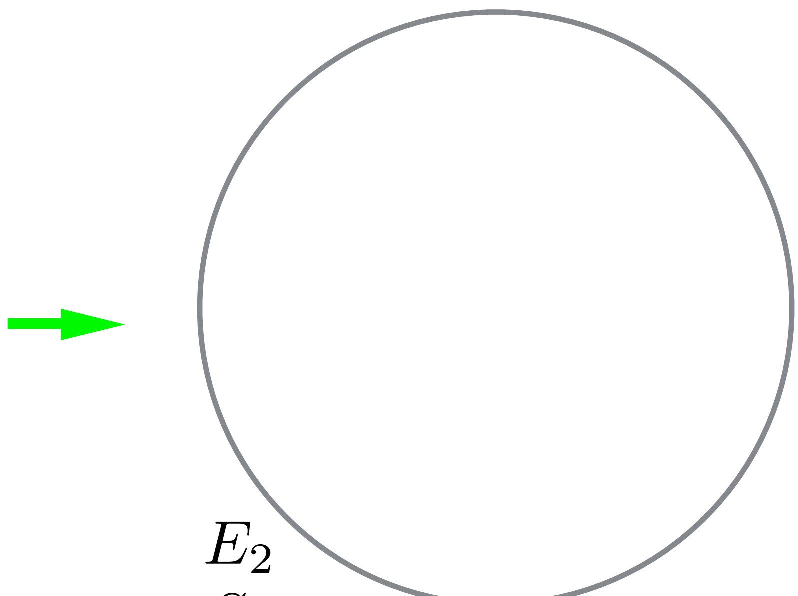
$E_2 > E_1$

So we are not saying that the energy E_1 will give more entropy when broken into smaller holes



E_2

S_2



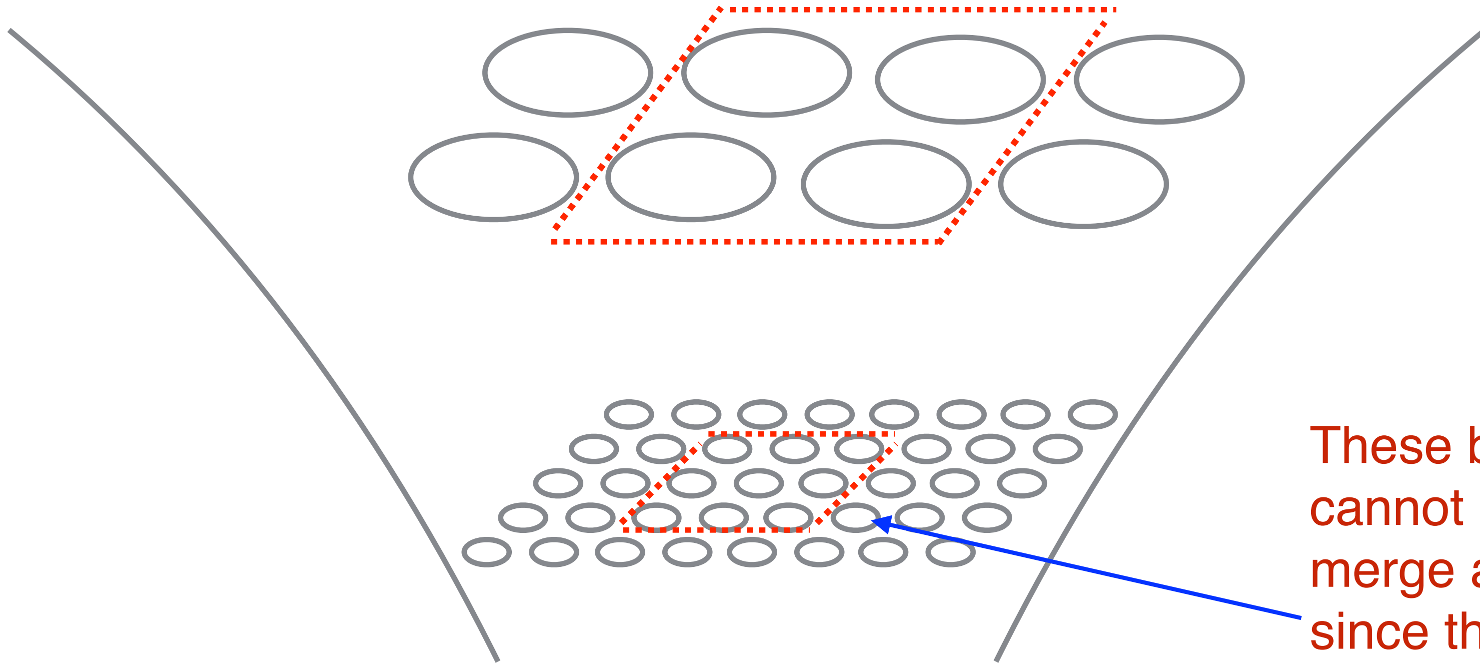
E_2

S_3

If we let the small holes merge into one hole, it will have a size much bigger than the box

$S_3 > S_2 > S_1$

What is the entropy in a cosmological situation like the early universe ?



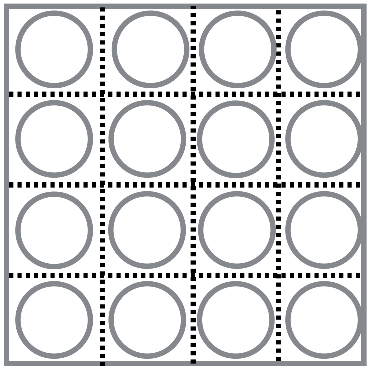
These black holes cannot immediately merge and expand, since the space around them is not “empty”

The tendency to merge will create a pressure

As the universe expands, the holes can merge to make larger holes …

(Banks+Fischler: “The black hole gas”)

How much entropy do we get ?



Let the box have volume V

Let the energy in the box be E

Let the radius of each hole be R

The number of holes is $N \sim \frac{V}{R^3}$

The entropy of each hole is $S_{hole} \sim \frac{R^2}{G}$

Thus the total entropy is $S \sim N\, S_{hole} \sim \frac{V}{R\,G}$

The energy of each hole is $E_{hole} \sim \frac{R}{G}$

Thus the total energy is $E \sim N\, E_{hole} \sim \frac{V}{R^2 G}$

Thus $$S \sim \sqrt{\frac{E\,V}{G}}$$

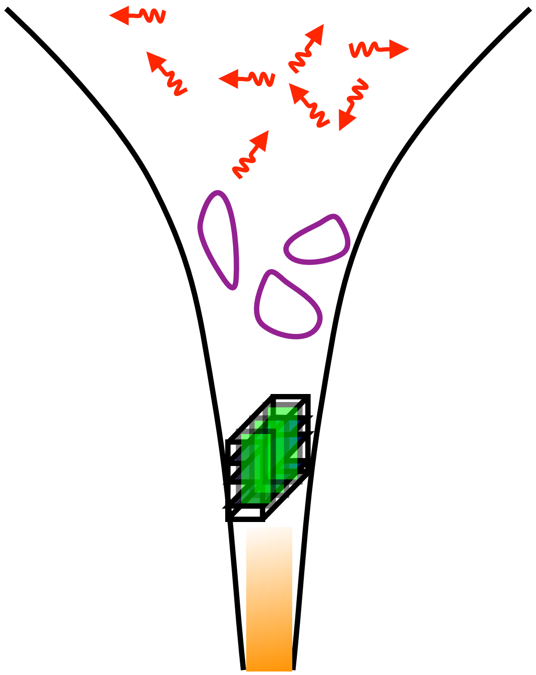
Writing $\rho = \frac{E}{V}$

we see that S is extensive

$$S \sim \sqrt{\frac{\rho}{G}}\, V$$

(Same expression in all dimensions)

What does string theory tell us near the initial singularity of cosmology ?



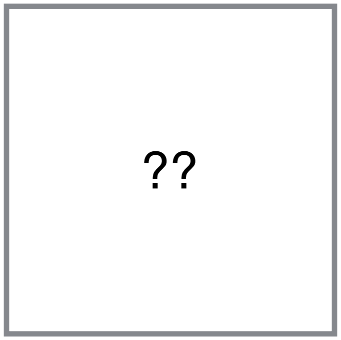
Conjecture $$S \sim \sqrt{\frac{\rho}{G}}\, V$$

We can write this as $$s \sim \sqrt{\frac{\rho}{G}}$$

or (introducing a dimensionless constant of order unity) $$s = K\sqrt{\frac{\rho}{G}}$$

Another derivation of the equation of state $S \sim \sqrt{\frac{E\,V}{G}}$

(Masoumi + SDM arXiv:1406.5798)



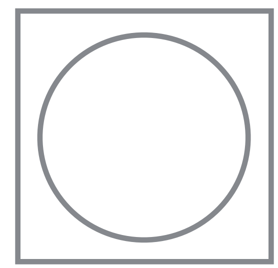
Take a box of volume V

In this box put an energy E

What is S in the limit $E \to \infty$?

Let us assume : (1) The expression for S in the limit $E \to \infty$ is invariant under S,T dualities

(2) For $E \sim E_{bh}$, we get $S \sim S_{bh} \sim \frac{A}{G}$



$$\rho_{bh} = \frac{E_{bh}}{V}$$

Then we get $$S \sim \sqrt{\frac{E\,V}{G}}$$

for all energies such that $$\rho_{bh} \lesssim \rho \lesssim \rho_p$$

Checking these conditions:

We take a torus T^9 in 9+1 dimensional string theory

(1) T-duality: Under a T-duality in the direction, x^1 we have

$$\frac{EV}{G} = \frac{EL_1L_2\ldots L_9}{8\pi^6 g^2 l_s^8} \rightarrow \frac{E(\frac{(2\pi l_s)^2}{L_1})L_2\ldots L_9}{8\pi^6 g^2(\frac{(2\pi l_s)^2}{L_1^2})l_s^8} = \frac{EV}{G}$$

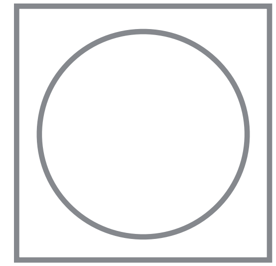
S-duality: We can write the quantity $\frac{EV}{G}$ purely in planck units

$$\frac{EV}{G} = (El_p)(Vl_p^{-9}) \qquad (G = l_p^8)$$

Thus it will be invariant under S-duality

Thus any function $f\left(\frac{EV}{G}\right)$ is invariant under T,S dualities

(2) Matching onto the Area entropy of black holes:



Let the torus have all sides equal to L

We have $$E_{bh} \sim \frac{L^7}{G}$$

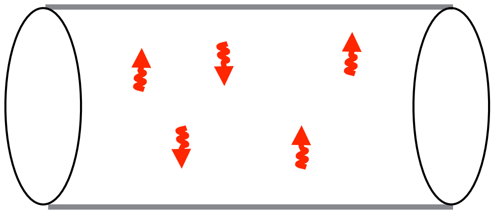
At this energy $$\frac{E_{bh}V}{G} \sim \frac{L^{16}}{G^2}$$

The area entropy expression is $$S_{bh} \sim \frac{A}{G} \sim \frac{L^8}{G}$$

Thus to match onto the area entropy of black holes, we need

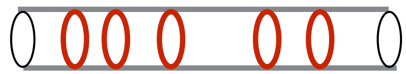
$$f\left(\frac{EV}{G}\right) \sim \sqrt{\frac{EV}{G}}$$

What is the meaning of the requirement: The expression for S in the limit $E \to \infty$ is invariant under S,T dualities ?



Gas in a box: at large box size, the entropy comes from gravitons

$$S \sim E^{\frac{9}{10}} \left(\prod_{i=1}^{8} L_i^{\frac{1}{10}} \right) L^{\frac{1}{10}}$$



After T-duality, the entropy is given by winding strings ….

$$S \sim E^{\frac{9}{10}} \left(\prod_{i=1}^{8} L_i^{\frac{1}{10}} \right) L^{-\frac{1}{10}}$$

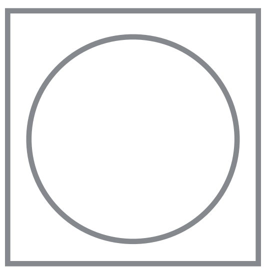
But in black holes, the entropy is given by the area, both before and after the T-duality

$$S = \frac{A}{4G}$$

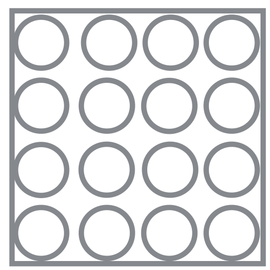
$$S = \frac{A}{4G}$$

Thus it seems plausible that

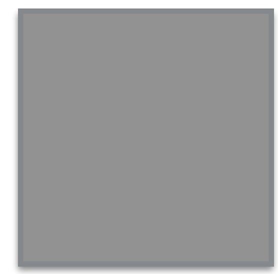
$$S = K\sqrt{\frac{EV}{G}} \qquad \text{(equivalently,} \quad s = K\sqrt{\frac{\rho}{G}} \quad \text{)}$$



Just enough energy to make a black hole of box size

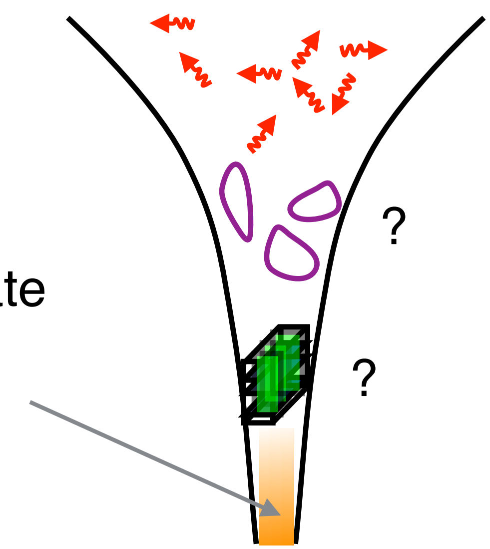


$$\rho_{bh} \lesssim \rho \lesssim \rho_p$$



Black holes of planck size each:
Energy density is planck density
Entropy is one bit per planck volume

Let us see what happens if we use this equation of state near the big bang …

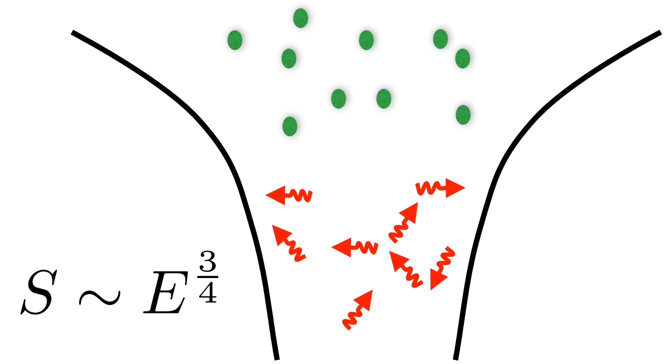


Some comments:

(A) Should we start with ‘maximal entropy’ states or ‘minimum entropy’ states ?

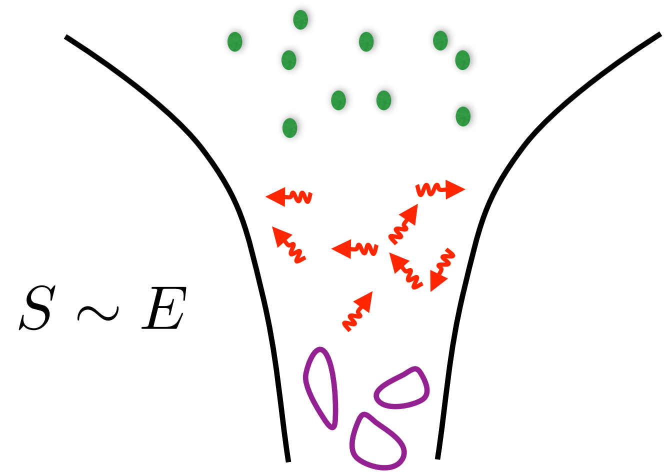
Actually, we have always taken maximal entropy states …

When we thought radiation had the maximum entropy, then we took ‘radiation dominated cosmology’ near the big bang



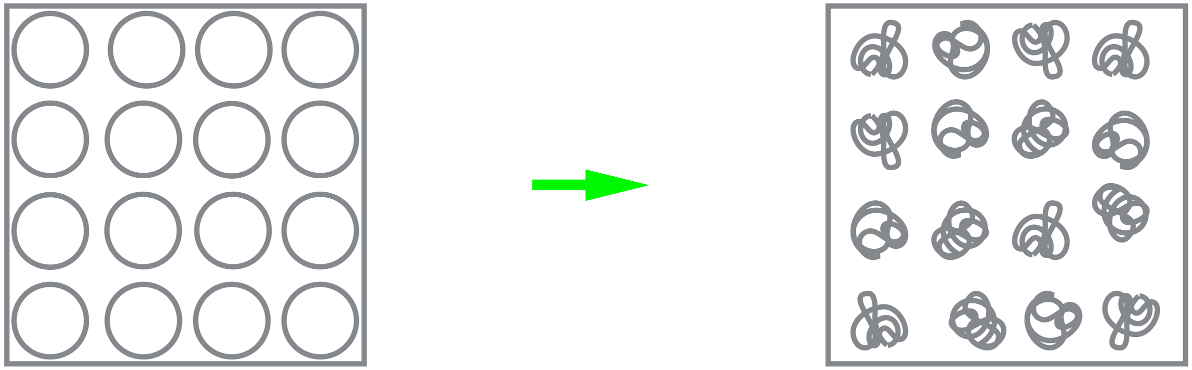
When we thought more entropy can be obtained from a string gas, we assumed a string gas

(Brandenberger - Vafa ‘89)



Thus we can always take the maximal entropy state in a given volume of the universe … when the universe expands, then the phase can change and give yet more entropy

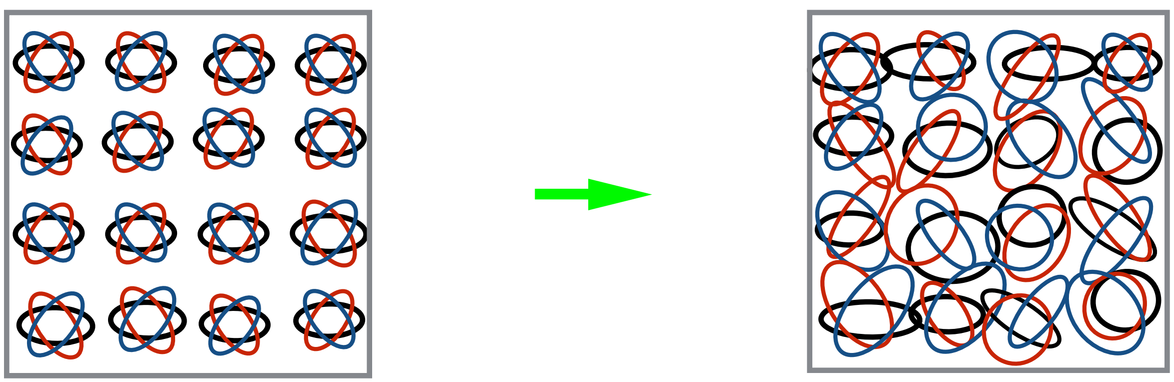
(B) If we start with a ‘black hole gas’, then will we ever get rid of these black holes ?



Veneziano ’99: strings at the Horowitz-Polchinski correspondence point

(on the borderline of being black holes)

In string theory, we have learnt that black holes are not a new kind of esoteric object; they are just messy states of intersecting branes …



The brane sets will of course interact and merge, but the distance over which a given brane intersects other branes would be the size of the ‘holes’

Intersecting brane states $S \sim (\sqrt{n_1}+\sqrt{\bar{n}_1})\ldots(\sqrt{n_k}+\sqrt{\bar{n}_k})$

Thermodynamics:

$$E, \ V$$

The entropy is $$S = K\sqrt{\frac{EV}{G}}$$

The first law of thermodynamics gives $$TdS = dE + pdV$$

Thus we get $$T = \left(\frac{\partial S}{\partial E}\right)_V^{-1} = \frac{2}{K}\sqrt{\frac{EG}{V}}$$

$$p = T\left(\frac{\partial S}{\partial V}\right)_E = \frac{E}{V} = \rho$$

Writing $p = w\,\rho$ gives $w = 1$

which is the stiffest equation of state compatible with reasonable physics:

Speed of sound $$v = \frac{\partial p}{\partial \rho} = 1$$

Cosmological evolution:

We take the metric for a flat cosmology:

$$ds^2 = -dt^2 + \sum_{i=1}^{d} a_i^2(t) dx_i dx_i$$

The Einstein equations are

$$-\frac{1}{2}\left(\sum_i \frac{\dot{a}_i}{a_i}\right)^2 + \frac{1}{2}\sum_i \frac{\dot{a}_i^2}{a_i^2} = -8\pi G\rho$$

$$\frac{\ddot{a}_k}{a_k} + \frac{\dot{a}_k}{a_k}\left(\sum_i \frac{\dot{a}_i}{a_i}\right) - \frac{\dot{a}_k^2}{a_k^2} - \sum_i \frac{\ddot{a}_i}{a_i} 8\pi G(1+w_k)\rho = 16\pi G\rho$$

We assume the ansatz:

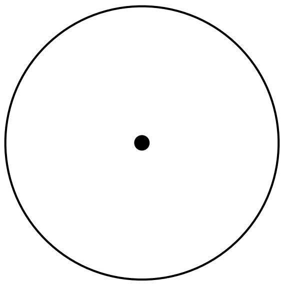
$$a_i = a_{0i} t^{C_i}, \quad i = 1, \ldots d$$

The solution is

$$\sum_i C_i = 1\,, \qquad \rho = \frac{1}{16\pi G t^2}\left(1 - \sum_i C_i^2\right)$$

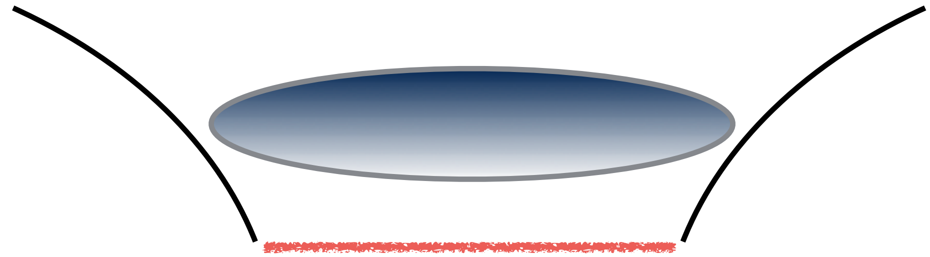
(B) Entropy bounds in cosmology

Black holes



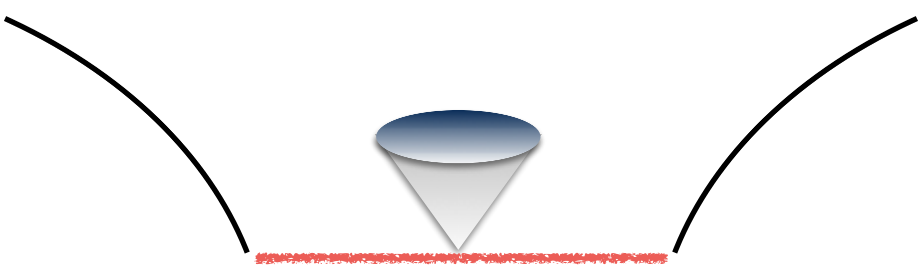
$$S = \frac{A}{4G}$$

Homogenous cosmology



Entropy is proportional to volume …
For a large enough volume, $S > \frac{A}{4G}$

How should we look for entropy bounds in cosmology ?

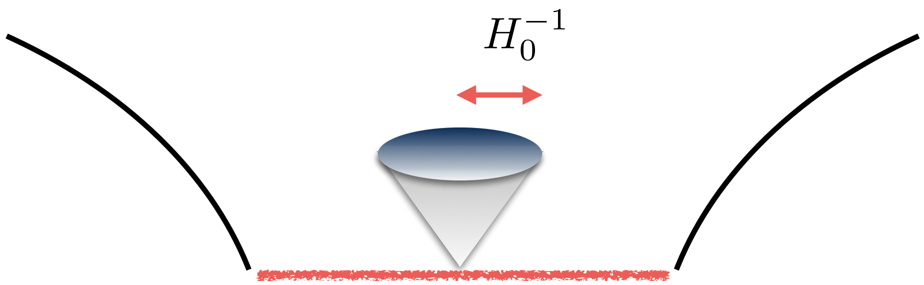


We should look only inside a region of size the cosmological horizon

Then we must have $S \leq \frac{A}{4G}$

(Fischler+Susskind 98)

Another derivation of the expression $s = K\sqrt{\frac{\rho}{G}}$



$$S \leq \frac{A}{4G}$$

The horizon scale is set by the Hubble constant H_0

We have $A \sim H_0^{-2}$, $H_0^2 = \left(\frac{\dot{a}}{a}\right)^2 \sim G\,\rho$

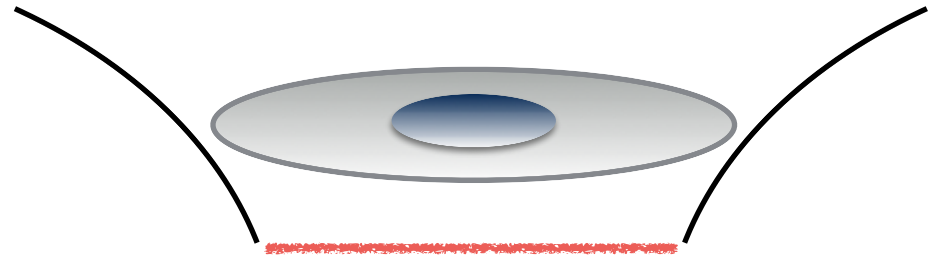
Then we see that $S \leq \frac{A}{4G} \quad \leftrightarrow \quad s \lesssim \frac{H_0^{-2}}{H_0^{-3}G} \sim \frac{H_0}{G} \sim \sqrt{\frac{\rho}{G}}$

(Fischler+Susskind 98)

So we seem to have yet another way of getting the expression $s \sim \sqrt{\frac{\rho}{G}}$

Instead of using the cosmological horizon, one should use the apparent horizon

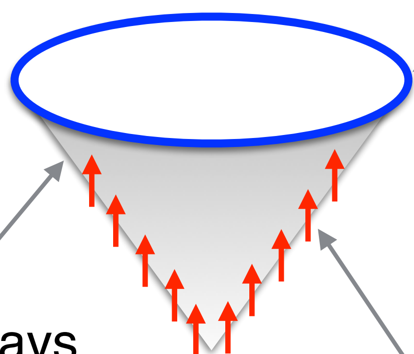
(Bak+Rey 99)



$$S \leq \frac{A}{4G}$$

These ideas were made precise by Bousso, and encoded in a principle called the 'covariant entropy bound'

Area A of surface bounding the light sheet



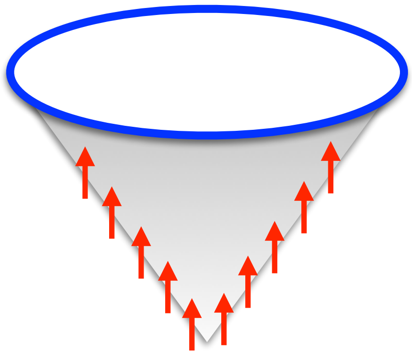
Light sheet, made of light rays orthogonal to the bounding surface

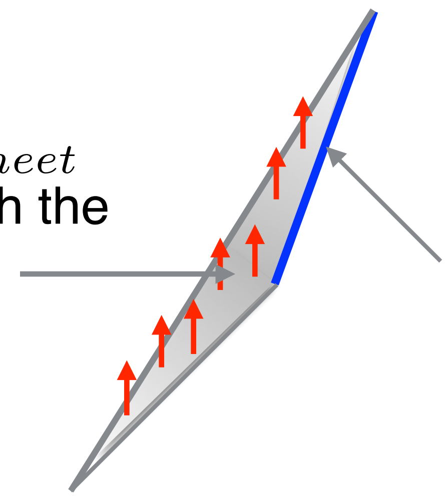
Entropy S_{sheet} flows through the light sheet

Covariant entropy bound
(Bousso 99)

$$S_{sheet} \leq \frac{A}{4G}$$

The bounding surface could be closed or open:





What we will do:

(a) Take the flat cosmology with equation of state $p = \rho$

(b) Consider an open surface at some time t_0

(c) Find the backwards light sheet, which terminates at the initial singularity

(d) Find the entropy S_{sheet} through this sheet, using the entropy density $s = K\sqrt{\frac{\rho}{G}}$

(e) See that the Bousso bound $S_{sheet} \leq \frac{A}{4G}$ is violated for suitable initial conditions

Carrying out these steps:

(a) Take the flat cosmology with equation of state $p = \rho$

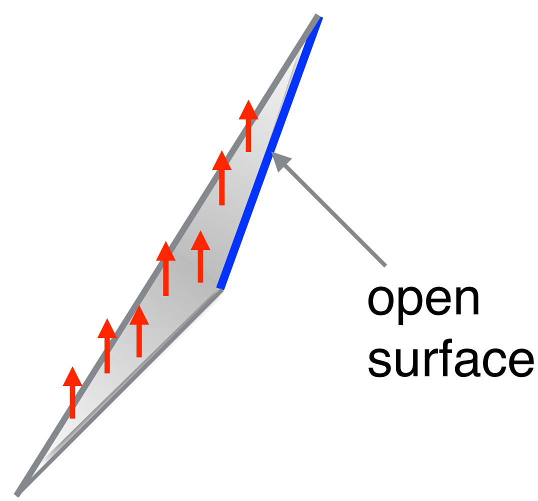
$$ds^2 = -dt^2 + \sum_{i=1}^{d} a_i^2(t) dx_i dx_i$$

$$a_i = a_{0i} t^{C_i}, \quad i = 1, \ldots d \qquad \sum_i C_i = 1$$

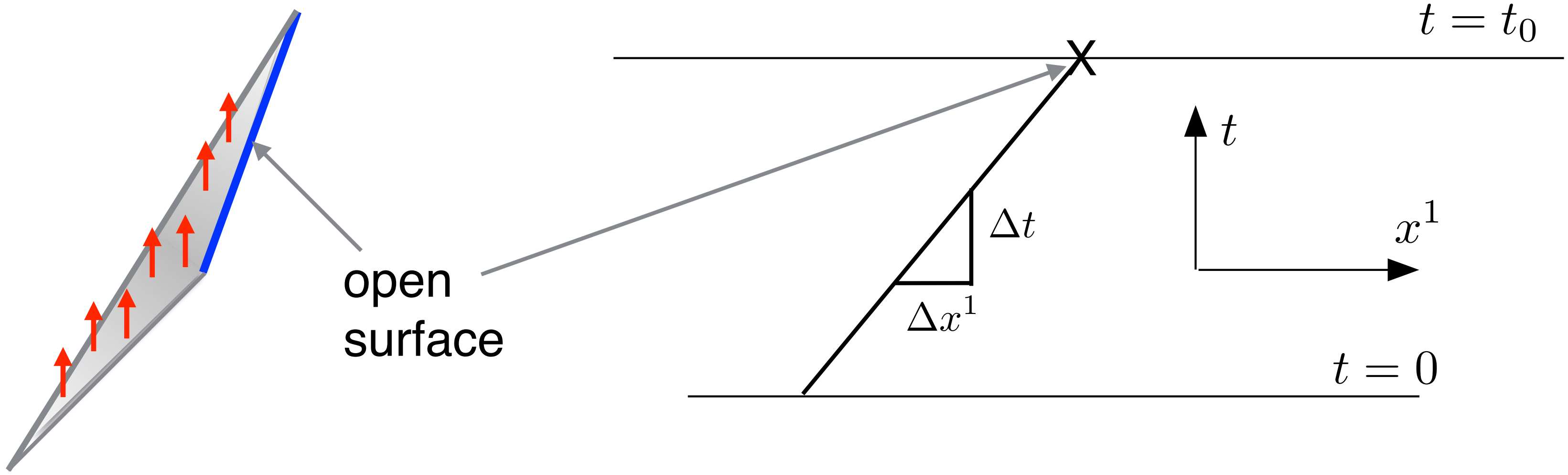
$$\rho = \frac{1}{16\pi G t^2}\left(1 - \sum_i C_i^2\right)$$

(b) Consider an open surface at some time t_0

$$0 \leq x^i \leq L_i, \qquad i = 2, \ldots d$$



(c) Find the backwards light sheet, which terminates at the initial singularity



$$\frac{dx^1}{dt} = \frac{1}{a_1(t)}$$

(d) Find the entropy S_{sheet} through this sheet, using the entropy density $s = K\sqrt{\frac{\rho}{G}}$

$$S_{\text{sheet}} = \frac{K\prod_{i=2}^{d} a_{i0} L_i}{G}\left[\frac{1-\sum_i C_i^2}{16\pi}\right]^{\frac{1}{2}} \frac{t_0^{1-C_1}}{1-C_1}$$

(e) See that the Bousso bound $S_{sheet} \leq \frac{A}{4G}$ is violated for suitable initial conditions

$$S_{\text{bound}} = \frac{A}{4G} = \frac{1}{4G}\left(\prod_{i=2}^{d} L_i a_{i0}\right) t_0^{1-C_1}$$

$$r \equiv \frac{S_{\text{sheet}}}{S_{\text{bound}}} = \frac{K\left(1-\sum_i C_i^2\right)^{\frac{1}{2}}}{\sqrt{\pi}(1-C_1)}$$

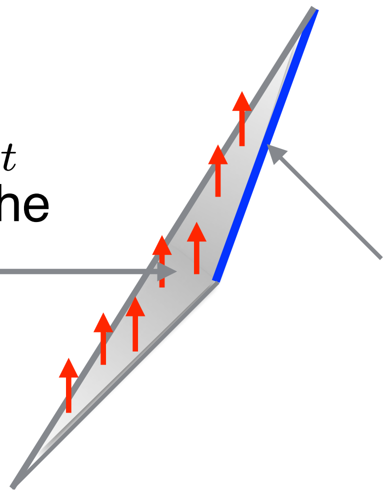
If the bound were true, we would always have $r \leq 1$

But we find that we can violate the bound. For example, set $C_i = \tilde{C}, \quad i = 2, \ldots d$

Then we get $r > 1$ for

$$1 - \frac{\frac{2K^2}{\pi}}{1+\frac{K^2 d}{\pi(d-1)}} < C_1 < 1$$

The Bekenstein bound and the Area formula for black hole entropy had all led to a general notion of the covariant entropy bound …

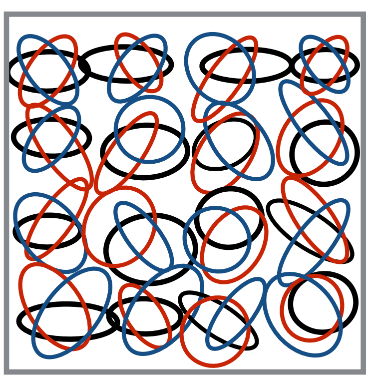


$$S_{sheet} \leq \frac{A}{4G}$$

But we have seen that we can make a set of states in string theory which have

$$s = K\sqrt{\frac{\rho}{G}}$$

The evolution under this equation of state violates the bound if the initial conditions are sufficiently asymmetric



For example we may take

$$d = 10, \quad C_1 = .6, \quad C_i = .044, \quad i = 2, \dots 10$$

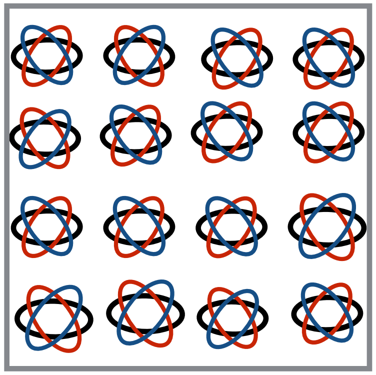
and this would violate the bound

Physical implications ?

We have seen that the covariant entropy bound is violated by a set of states in string theory. What are the implications of this observation?

(A) There could be something wrong in the derivation of this violation:

(i) There may be some physical effect which prevents us from placing black hole like states on a lattice



Then we would not get $s = K\sqrt{\frac{\rho}{G}}$

(ii) There may be some reason why standard thermodynamics

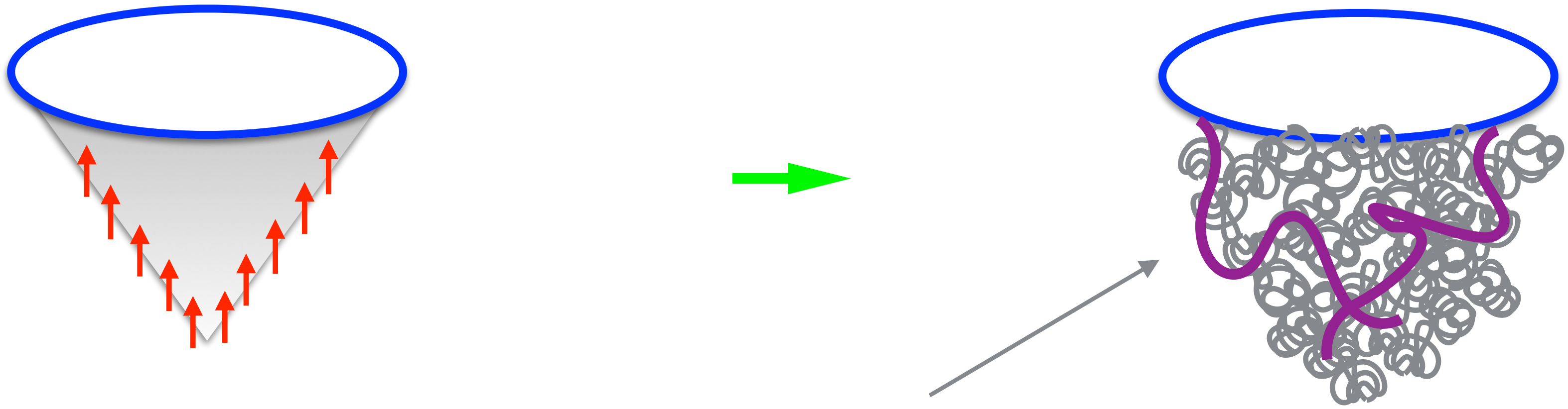
$$TdS = dE + pdV$$

cannot be used to get the pressure. Then we would not get $p = \rho$

(iii) There may be some reason why the classical Einstein equations cannot be used here (maybe quantum measure effects are too large) (Masoumi+SDM 14)

Then we would not get the space-time we used

(B) Perhaps we cannot draw light sheets through matter which is very quantum and stringy, so we cannot check the covariant entropy bound the way we have done …

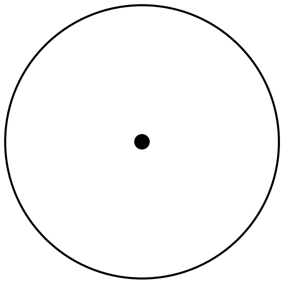


When matter is a stringy mess, then light rays may not have well defined paths …

But in that case it is unclear how we would ever hope to us the covariant entropy bound in the early universe, where matter is like to be stringy anyway …

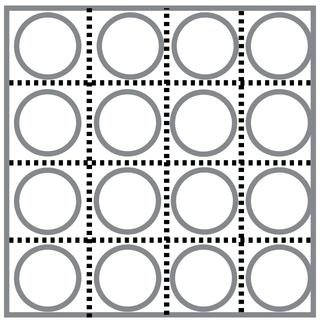
(The main role of the bound is in the early universe, since it is too easily satisfied in other situations..)

(C) Is it really correct to search for entropy bounds in terms of area?



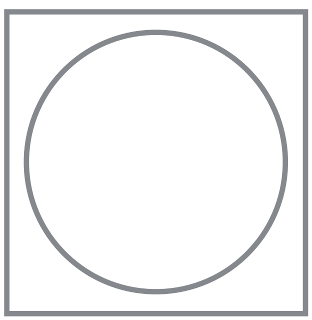
$$S = \frac{A}{4G}$$

Hole allowed to expand freely to its natural size



$$S = K\sqrt{\frac{EV}{G}} = K\sqrt{\frac{\rho}{G}}\;V$$

Entropy proportional to volume



$$E = M \sim \frac{R}{G}, \quad V \sim R^3$$

$$S \sim \sqrt{\frac{EV}{G}} \sim \sqrt{\frac{R^4}{G^2}} \sim \frac{R^2}{G} \sim \frac{A}{G}$$

Area entropy arises as a special case

Conjecture: $S = K\sqrt{\frac{EV}{G}}$ $\left(s = K\sqrt{\frac{\rho}{G}} \right)$ $\rho_{bh} \lesssim \rho \lesssim \rho_p$

Perhaps we have been asking the wrong question …

Bigger stars are hotter …

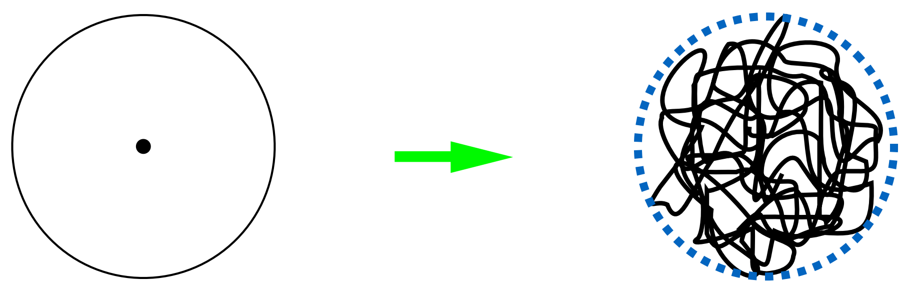
Suppose someone said that gases obey

$$T \propto M$$

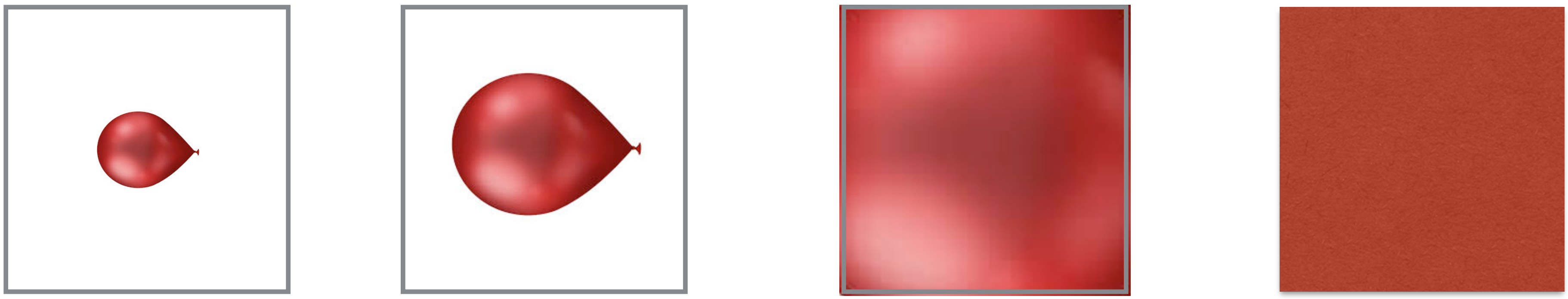
This would be incorrect …. the law actually is $PV = nkT$

Under the complicated interplay of heat generation and gravitational attraction, the law $T \propto M$ emerges as an effective description in the case that we allow the gas ball to expand to its natural size

A black hole is just a state of stringy matter … if it is allowed to expand to its natural size, we get the usual black hole …

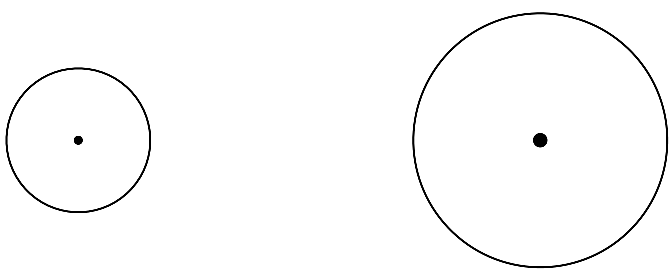


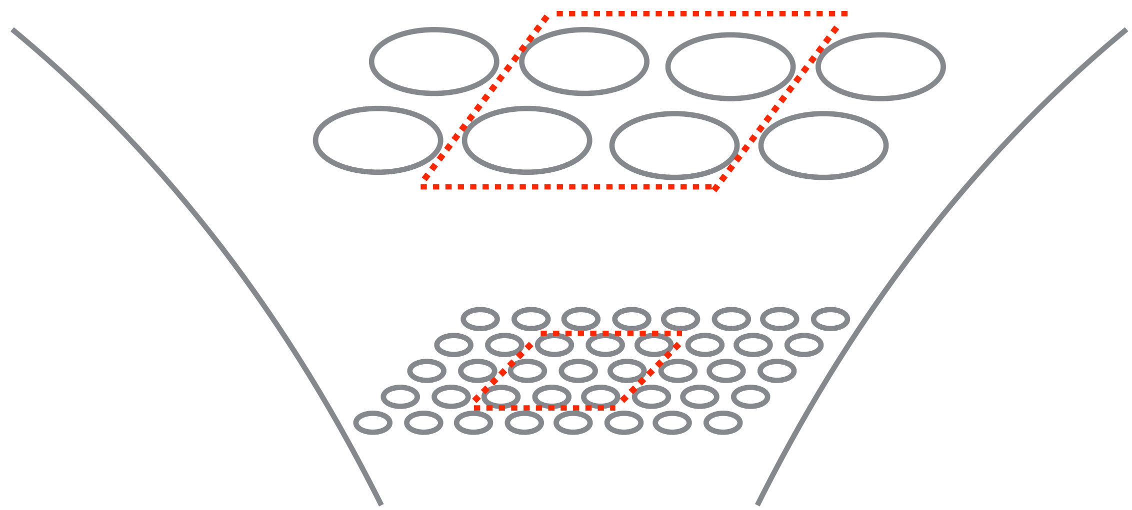
Consider a balloon in a box …



Volume rises, pressure does not change much ….

Different law … volume stops increasing, pressure rises faster than before …





$$S = \frac{A}{4G}$$

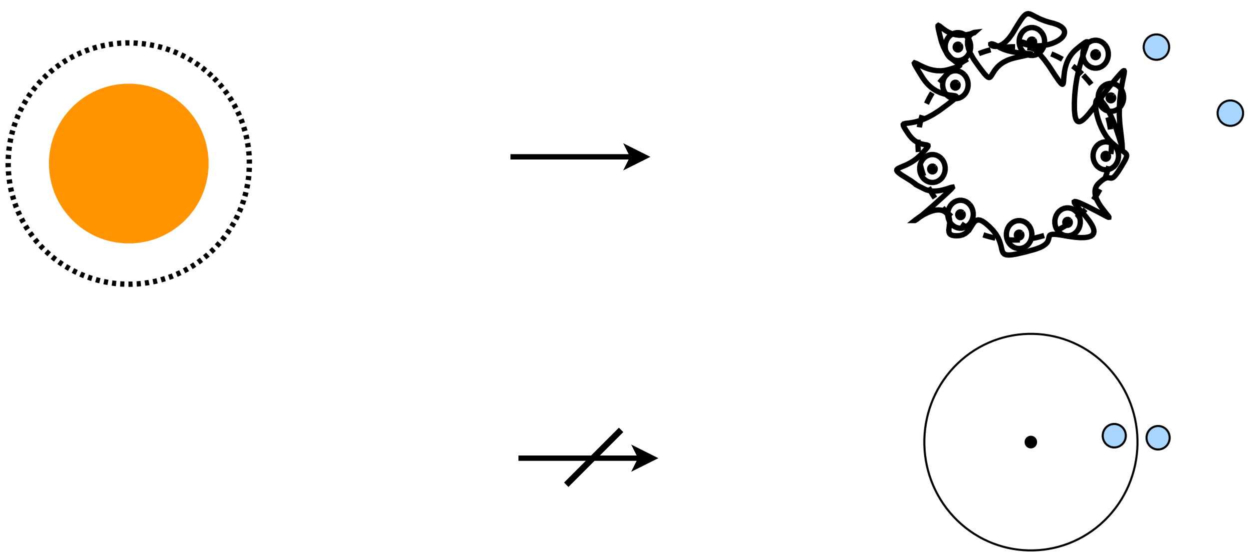
$$S \sim \sqrt{\frac{\rho}{G}}\, V$$

Philosophical overview:

A principal goal of any theory of quantum gravity (like string theory) is to resolve the singularities of classical general relativity

(A) Black hole singularity: This is resolved by the fuzzball conjecture

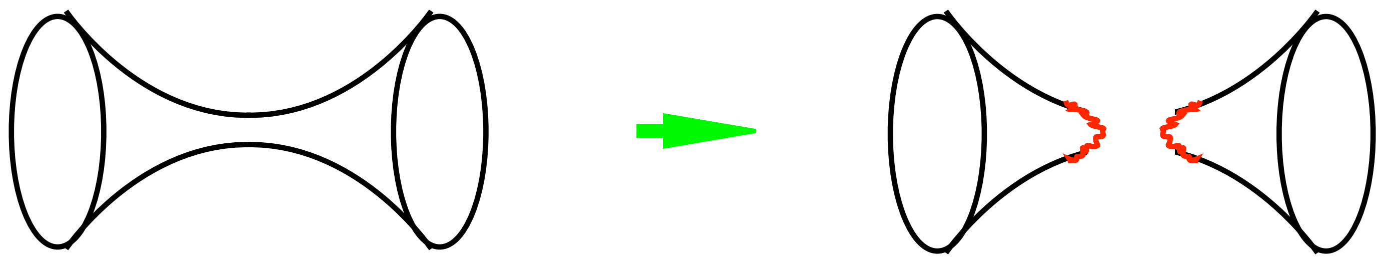
A collapsing star is unstable to tunneling into fuzzballs.



The large degeneracy $e^{S_{bek}}$ of fuzzball states leads to a violation of the semiclassical approximation of general relativity

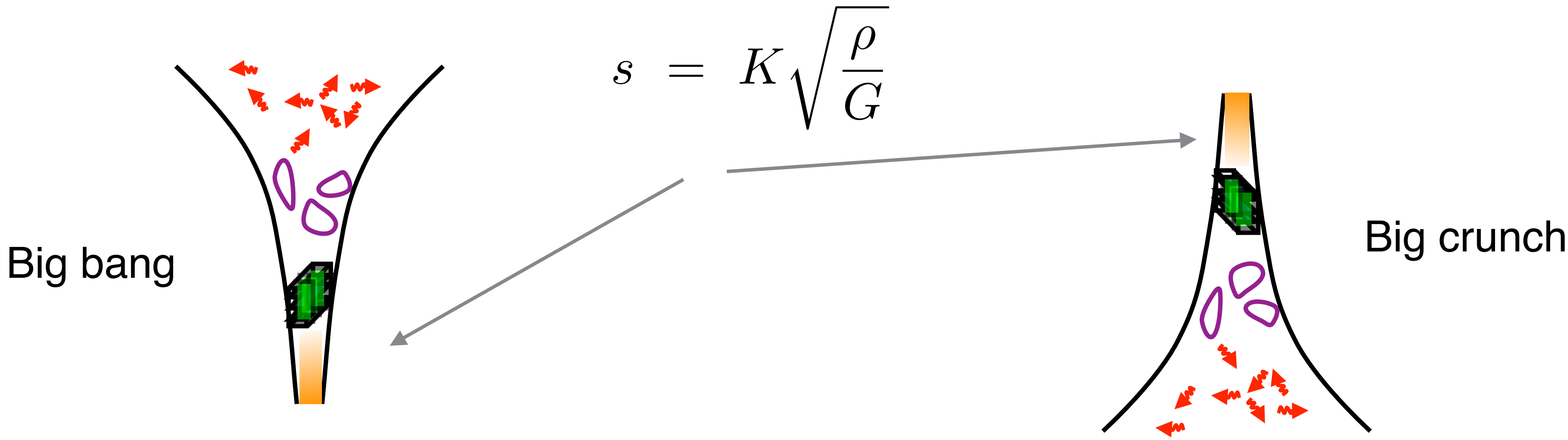
If we try to compress a mass M to a region of radius less than $2GM$ then the tunneling process destabilizes the space-time

In particular, if we have a throat with radius less than $2GM$, then the space-time fractures into two disconnected but entangled pieces



What are the implications of the fuzzball picture for cosmology ?

(B) The initial singularity of the universe:



As the universe compresses in a big crunch, it fractures into parts that are disconnected but entangled …

The reverse should happen at the big bang …