Undoing Reductions: from 11 to 4 and back

Mahdi Godazgar
(with Hadi Godazgar and Hermann Nicolai)

DAMTP, University of Cambridge

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Introduction

- General feature of gravitational theories: appearance of enhanced symmetries under reduction.

Relativity: existence of commuting Killing isometeries [Ehlers 1957; Matzner, Misner 1967; Geroch 1972].

Supergravity: presence of matter fields gives richer structures [Cremmer, Julia 1979; Julia 1981].

Classical example: toroidal reduction of $D = 11$ supergravity to $D = 4$ gives theory controlled by $E_{7(7)}/SU(8)$.

- Dualisation of fields important in realising enhancement of symmetries [Cremmer, Julia, Lu, Pope 1998].

The existence of enhanced symmetries useful:

- solution generating method [Ehlers 1957; Geroch 1971, 1972],
- clearer understanding of reduced theory [Cremmer, Julia 1979],
- efficient way of classifying gaugings in supergravity: duality covariant embedding formalism [Nicolai, Samtleben 2001a, 2001b; de Wit, Samtleben, Trigiante 2003, 2007].

Question we would like to address in this talk:

Does the enhanced symmetry play a role in the higher-dimensional unreduced theory?

Two approaches to answering this question.

- Symmetry-based approach: [Eric's talk] construct field theories that are built from objects and structures derived from the enhanced symmetry of interest.
  - Useful for investigating possible generalisations.
  - Leads to novel mathematical structures.
  - Relation to original theory must be established.

- Theory-based approach: reformulate the higher-dimensional theory in such a way as to make explicit, features of the enhanced symmetry.
  - Relation to original theory (on-shell equivalence) trivial.
  - Much simpler setting in which to study reductions, their consistency and uplift ansätze:
    new perspective on Kaluza-Klein theory.
  - Any relation of the symmetry-based approaches to the original theory would be via the theory-based reformulation.
    [H Godazgar, M.G., Hohm, Nicolai, Samtleben 2014]

$D = 11$ supergravity and $E_{7(7)}/SU(8)$

- $D = 11$ supergravity [Cremmer, Julia, Scherk 1978] thought to be unique.
- Maximum dimension in which one can define a consistent supergravity.
- Field content:

$$E_M{}^A, \; A_{MNP}, \; \Psi_M.$$

- Reduction on T^7 gives maximal ungauged supergravity with $E_{7(7)}/SU(8)$ symmetry [Cremmer, Julia 1979].

Aim: reformulation of $D = 11$ supergravity making manifest as much of the $E_{7(7)}/SU(8)$ structures as is possible.

de Wit-Nicolai formalism [de Wit, Nicolai 1986] ([dWN86])

de Wit-Nicolai formulation goes some way into addressing this aim:

- On-shell equivalent refomulation of $D = 11$ supergravity.
- Gives up manifest Lorentz invariance and spacetime covariance $(11 \to 4 + 7)$.
- There exists manifest local $SU(8)$ symmetry $(SO(7) \to SU(8))$.
- Analysis of the supersymmetry tranformations in 4+7 split used to achieve this.
- Consistency of the S^7 reduction to maximal gauged supergravity proved using this formalism [de Wit, Nicolai 1987; Nicolai, Pilch 2012].

Natural to work in de Wit-Nicolai formalism.

Find [GGN13b]: completion of de Wit-Nicolai reformulation to include the role of the global $E_{7(7)}$ group.

Note: $E_{7(7)}$ is *not* a symmetry of the $D = 11$ theory.

Introduce six-form dual $A_{(6)}$ using equation of motion of 3-form potential $A_{(3)}$:

$$d \star F_{(4)} + \cdots = 0$$
$$d(\star F_{(4)} + \dots) = 0$$
$$\star F_{(4)} + \cdots = dA_{(6)}.$$

Can determine SUSY transformation of $A_{(6)}$.

Decompose fields in terms a 4+7 split: $z^M \to (x^\mu, y^m)$

'vierbein'	'scalars'	'vectors'	'2-forms'	...
${e_\mu}^\alpha$	${e_m}^a$	${B_\mu}^m$		
	A_{mnp}	$\mathcal{A}_{\mu mn}$	$\mathcal{A}_{\mu\nu m}$	...
	$A_{m_1 \dots m_6}$	$\mathcal{A}_{\mu m_1 \dots m_5}$	$\mathcal{A}_{\mu\nu m_1 \dots m_4}$	...

- Equivalent to full $D = 11$ theory: full dependence on (x, y).

Similarly [Cremmer, Julia 1979; dWN86],

$$\Psi_A \equiv E_A{}^M \Psi_M = (\Psi_\alpha, \Psi_a) \to (\varphi_\mu{}^A, \chi_{ABC})$$

Chiral $SU(8)$ indices.

Now consider the SUSY transformations of the redefined fields.

For the vierbein we have

$$\delta e_\mu{}^\alpha = \tfrac{1}{2}\bar{\epsilon}^A \gamma^\alpha \varphi_{\mu A} \; + \; \text{h.c.}.$$

Now consider variation of $B_\mu{}^m$ [dWN86]

$$\delta B_\mu{}^m = \tfrac{\sqrt{2}}{8} e^m_{AB} \left[2\sqrt{2}\bar{\varepsilon}^A \varphi^B_\mu + \bar{\varepsilon}_C \gamma'_\mu \chi^{ABC} \right] + \text{h.c.},$$

where

$$e^m_{AB} \sim e^m{}_a \Gamma^a_{AB}.$$

Can think of e^m_{AB} as facilitating embedding of $SO(7) \subset SU(8)$.

Moreover [dWN86]

$$\delta e^m_{AB} = -\sqrt{2}\Sigma_{ABCD} e^{mCD}$$

Σ_{ABCD} complex self-dual $SU(8)$ tensor.

Roughly,

$$\delta(\text{vector}) = \text{scalar} \times \text{fermions}.$$

Therefore, consider variation of other vector degrees of freedom [de Wit, Nicolai 2013; GGN13b].

Result: we find the same pattern:
$\delta(\text{vector}) = \text{scalar} \times \text{fermions!}$

Furthermore [GGN13b]

All bosonic degrees of freedom assemble into $E_{7(7)}$ objects in accordance with the decomposition
$\mathbf{56} \;\to\; \mathbf{28} \oplus \overline{\mathbf{28}} \;\to\; \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{21}} \oplus \overline{\mathbf{7}}$

- $e_\mu{}^\alpha$: 'vierbein'—singlet under $E_{7(7)}$.
- $\mathcal{V}_{\mathcal{M}\,AB}(e_m{}^a, A_{mnp}, A_{m_1\ldots m_6})$: 'scalars'—$E_{7(7)}/SU(8)$ element.
- $\mathcal{A}_\mu{}^{\mathcal{M}}$: 'vectors'—part of the **56** of $E_{7(7)}$.

$$\delta e_\mu{}^\alpha = \tfrac{1}{2}\bar{\epsilon}^A\gamma^\alpha\varphi_{\mu A} \;+\; \text{h.c.},$$

$$\delta\mathcal{V}_{\mathcal{M}\,AB} = \sqrt{2}\,\Sigma_{ABCD}\,\mathcal{V}_{\mathcal{M}}{}^{CD},$$

$$\delta\mathcal{A}_\mu{}^{\mathcal{M}} = i\,\Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{N}\,AB}\left(2\sqrt{2}\bar{\varepsilon}^A\varphi_\mu^B + \bar{\varepsilon}_C\gamma_\mu\chi^{ABC}\right) \;+\; \text{h.c.}.$$

Identical in form to the supersymmetry transformation of maximal gauged theories in four dimensions [de Wit, Samtleben, Trigiante 2007]!

Uplift formulae

This can be used to derive uplift formulae:

- The linear Kaluza-Klein ansatz for the vector fields is exact

$$\mathcal{A}_\mu{}^{\mathcal{M}}(x,y) = \mathcal{B}_\mu{}^{\mathcal{N}}(x) R_{\mathcal{N}}{}^{\mathcal{M}}(y)$$

- Comparing the $D = 11$ and $D = 4$ supersymmetry variations, the generalised vielbeine (components of an element of $E_{7(7)}/SU(8)$ coset) are related to the $D = 4$ scalars

$$\mathcal{V}^{(11)}{}_{\mathcal{M}}{}^{AB}(x,y) = -\mathcal{V}^{(4)}{}_{\mathcal{N}}{}^{ij}(x)\, R^{\mathcal{N}}{}_{\mathcal{M}}(y)\, \eta_i^A(y)\eta_j^B(y)$$

Note: By rewriting the 'scalars' in an $E_{7(7)}$ matrix, the highly non-linear uplift formulae transform into a linear relation, e.g. $\mathcal{V}$ is cubic in A_{mnp} and highly non-linear in $e_m{}^a$.

Generalised vielbein postulates (GVPs)

[dWN86; GGN13b; (GGN 2014)]

- Components of $E_{7(7)}$ vielbein $\mathcal{V}_{\mathcal{M}\,AB}$ satisfy generalised vielbein postulates.

Internal GVP:

$$\partial_m \mathcal{V}_{\mathcal{M}\,AB} - \mathbf{\Gamma}_{m\mathcal{M}}{}^{\mathcal{N}} \mathcal{V}_{\mathcal{N}\,AB} + \mathcal{Q}^{C}_{m[A} \mathcal{V}_{\mathcal{M}\,B]C} = \mathcal{P}_{m\,ABCD} \mathcal{V}_{\mathcal{M}}{}^{CD}.$$

- $\mathbf{\Gamma}_{m\mathcal{M}}{}^{\mathcal{N}}$: generalised affine connection.
- $\mathcal{Q}^{C}_{mA}$: generalised spin connection.
- $\mathcal{P}_{m\,ABCD}$: generalised non-metricity.

There is a rich and beautiful structure here: matter and gravitational degrees of freedom packaged into $E_{7(7)}$ connections.

External GVP:

$$\partial_\mu \mathcal{V}_{\mathcal{M}\,AB} + 2\hat{\mathcal{L}}_{\mathcal{A}_\mu}\mathcal{V}_{\mathcal{M}\,AB} + \mathcal{Q}^{C}_{\mu\,[A}\mathcal{V}_{\mathcal{M}\,B]C} = \mathcal{P}_{\mu\,ABCD}\mathcal{V}_{\mathcal{M}}{}^{CD},$$

where,

$$\hat{\mathcal{L}}_{\mathcal{A}_\mu}\mathcal{V}_{\mathcal{M}\,AB} \;=\; \tfrac{1}{2}\mathcal{A}_\mu{}^{\mathcal{N}}\partial_{\mathcal{N}}\mathcal{V}_{\mathcal{M}AB} \;+\; 6(t^\alpha)_{\mathcal{M}}{}^{\mathcal{N}}(t_\alpha)_{\mathcal{P}}{}^{\mathcal{Q}}\partial_{\mathcal{Q}}\mathcal{A}_\mu{}^{\mathcal{P}}\mathcal{V}_{\mathcal{N}AB}$$

is the $E_{7(7)}$ generalised Lie derivative [Coimbra, Strickland-Constable, Waldram 2011; Berman, Cederwall, Kleinschmidt, Thompson 2013].

Compare this with the Cartan equation that defines $D = 4$ maximal gauged theories [de Wit, Samtleben, Trigiante 2007]:

$$\partial_\mu \mathcal{V}_{\mathcal{M}\,AB} - g\mathcal{B}_\mu{}^{\mathcal{P}}X_{\mathcal{P}\mathcal{M}}{}^{\mathcal{N}}\mathcal{V}_{\mathcal{N}ij} + \mathcal{Q}^{k}_{\mu[i}\mathcal{V}_{\mathcal{M}\,j]k}\mathcal{V}_{\mathcal{N}\,ij} = \mathcal{P}_{\mu\,ijkl}\mathcal{V}_{\mathcal{M}}{}^{kl},$$

where $X_{\mathcal{M}}$ generate the gauge algebra and are constructed from the embedding tensor

$$X_{\mathcal{M}} = \Theta_{\mathcal{M}}{}^{\alpha}(t_\alpha).$$

The external GVP reduces directly to the Cartan equation in four dimensions:

Higher-dimensional understanding of the embedding tensor [GGN13b; GGN13c; (GGN14)]

The last component of the vectors $\mathcal{A}_{\mu m}$ drops out of the external GVP (consequence of generalised Lie derivative). Understanding of $\mathcal{A}_{\mu m}$ irrelevant for any discussion related to reductions and their consistency. Furthermore, this proves that [GGN13d]

New ω-deformed $SO(8)$ gauged maximal theories [Dall'Agata, Inverso, Trigiante 2012] can not be realised as a reduction from $D = 11$ supergravity.

Could they come from a reduction of a deformed $D = 11$ theory? [GGN13c; GGN13d] This seems unlikely...

The formalism completed in [GGN13b] can be applied to concrete examples of reductions:

- It can be used to derive in a much simpler manner the $SO(8)$ gauging in the S^7 reduction [GGN13c].
- Full uplift formulae for all fields of maximal gauged theory [GGN13c].
- Applied to give new highly non-trivial non-supersymmetric, stable (?) solution of $D = 11$ supergravity [H Godazgar, MG, Krüger, Nicolai, Pilch 2014].
- Provides a derivation of the embedding tensor of Scherk-Schwarz compactifications with flux from $D = 11$ [GGN13d].

More generally, these ideas apply to any reduction

- Type IIB on S^5 [de Wit *et. al.*, forthcoming], ...

Take home message

A new way of studying and understanding reductions: formulate the higher-dimensional theory fully in terms of the duality symmetry obtained under reduction.