Euclidean Quantum Supergravity

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Work with Carl Turner
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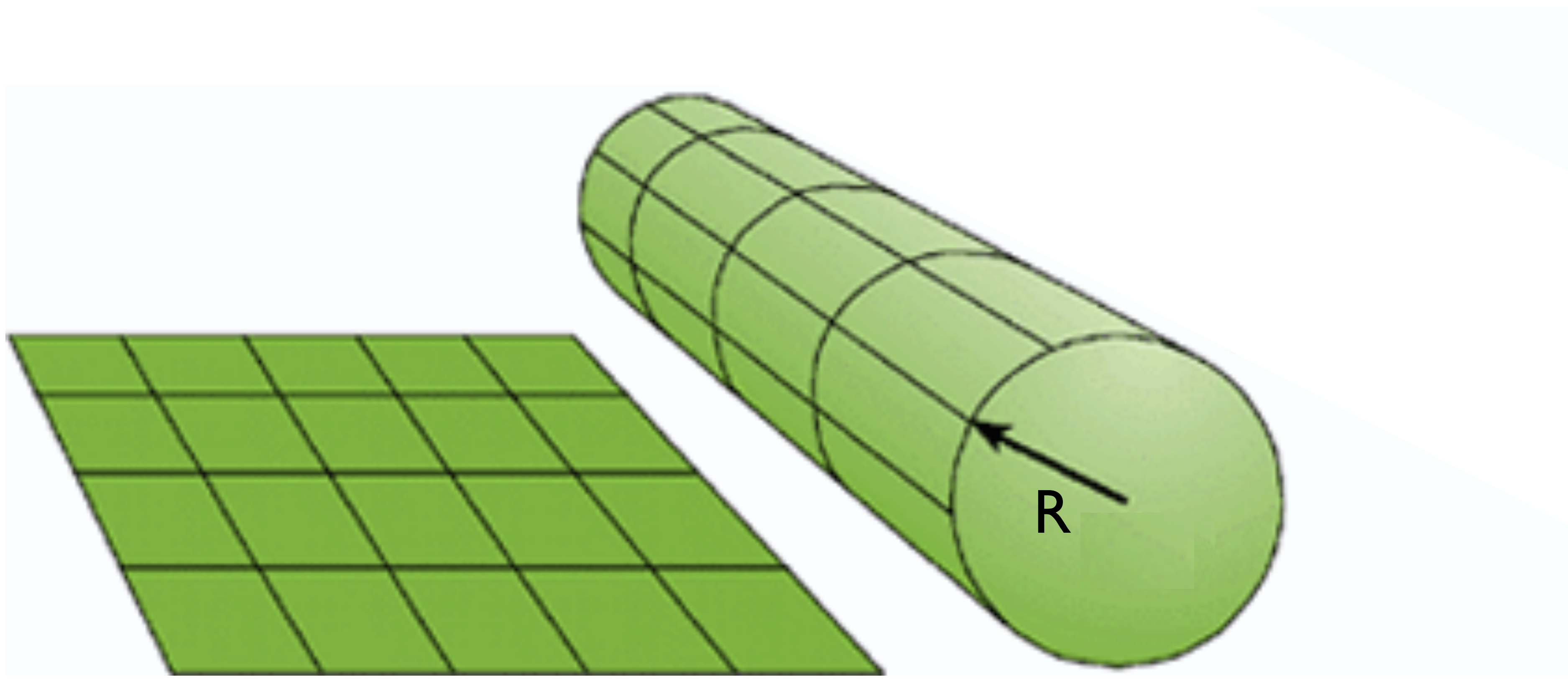
A Preview of the Main Results

Kaluza-Klein Theory: $\mathcal{M} = \mathbb{R}^{1,d-1} \times \mathbf{S}^1$



There is a long history of quantum instabilities of these backgrounds

- Casimir Forces
- Tunneling to “Nothing”

Appelquist and Chodos ‘83
Witten ‘82

The Main Result

Kaluza-Klein compactification of *N=1* Supergravity is unstable.

$$\boxed{\mathcal{W} \sim \exp\left(-\frac{\pi R^2}{4G_N} - i\sigma\right)}$$

Kaluza-Klein dual photon: $\partial_\mu \sigma \sim \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$

Grimm and Savelli 2011

And Something Interesting Along the Way…

Quantum Gravity has a hidden infra-red scale!

$$\Lambda_{\mathrm{grav}} \ll M_{\mathrm{pl}}$$

This is the scale at which gravitational instantons contribute

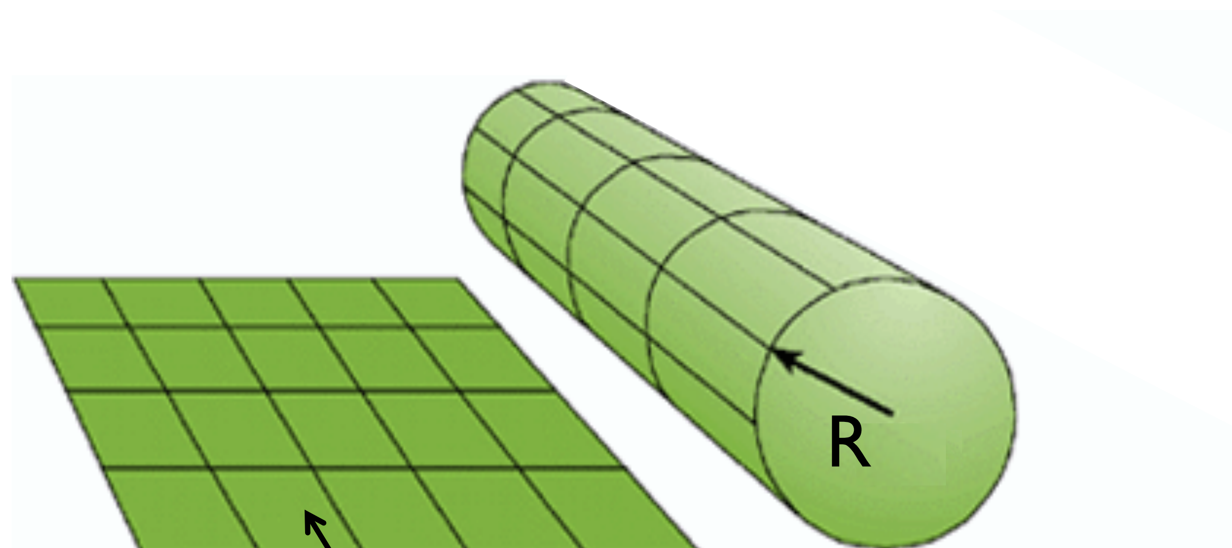
The Theory: *N=1* Supergravity

$$S = \frac{M_{\rm pl}^2}{2} \int d^4x \, \sqrt{-g} \, \left(\mathcal{R}_{(4)} + \bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho \right)$$

Compactify on a Circle

$$ds^2_{(4)} = \frac{L^2}{R^2} ds^2_{(3)} + \frac{R^2}{L^2}\left(dz^2 + A_i dx^i\right)^2 \qquad z \in [0, 2\pi L)$$

$$\mathcal{M} = \mathbb{R}^{1,2} \times \mathbf{S}^1$$



Fields $R(x^i)$ and $A_i(x^i)$ live here

L is fiducial scale

Classical Low-Energy Physics

$$S_{\rm eff} = \frac{M_{\rm pl}^2}{2}\int d^4x\,\sqrt{-g}\,\mathcal{R}_{(4)}$$
$$= \frac{M_3}{2}\int d^3x\,\sqrt{-g_{(3)}}\left[\mathcal{R}_{(3)} - 2\left(\frac{\partial R}{R}\right)^2 - \frac{1}{4}\frac{R^4}{L^4}F_{ij}F^{ij}\right]$$

$$M_3 = 2\pi L M_{\rm pl}^2$$

Or, if we work with the dual photon $\partial_\mu\sigma \sim \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$

$$S_{\rm eff} = \int d^3x\,\sqrt{-g_{(3)}}\,\left[\frac{M_3}{2}\,\mathcal{R}_{(3)} - M_3\left(\frac{\partial R}{R}\right)^2 - \frac{1}{M_3}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2\right]$$

Goal: Understand quantum corrections to this action.

Perturbative Quantum Corrections

One-Loop Divergences

't Hooft and Veltman '74

At one-loop in pure gravity, there are three logarithmic divergences

$$\mathcal{R}^2 \quad , \quad \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \quad , \quad \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

These two can be absorbed by a field redefinition of the metric

The Riemann^2 term can be massaged into Gauss-Bonnet.

$$\chi = \frac{1}{8\pi^2}\int d^4x \; \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2$$

This is purely topological. It doesn't affect perturbative physics around flat space.

The Gauss-Bonnet Term

$$S_\alpha = \frac{\alpha}{8\pi^2}\int d^4x\; \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2 = \alpha\chi$$

The coupling runs logarithmically

$$\alpha(\mu) = \alpha_0 - \alpha_1 \log\left(\frac{M_{UV}^2}{\mu^2}\right)$$

where the *beta function* is given by

Christensen and Duff '78
Perry '78; Yoneya '78

$$\alpha_1 = \frac{1}{48\cdot 15}\left(848N_2 - 233N_{3/2} - 52N_1 + 7N_{1/2} + 4N_0\right)$$

For us…

$$\boxed{\alpha_1 = 41/48}$$

A New RG-Invariant Scale

$$\alpha(\mu) = \alpha_0 - \alpha_1 \log\left(\frac{M_{UV}^2}{\mu^2}\right)$$

As in Yang-Mills, we can replace the log running with an RG invariant scale

$$\boxed{\Lambda_{\mathrm{grav}} = \mu \exp\left(-\frac{\alpha(\mu)}{2\alpha_1}\right)}$$

This scale will be associated with physics arising from non-trivial topologies.

(We will see an example)

Another Divergence: The Anomaly

The classical action is invariant under rotations of the phase of the fermion.

This $U(1)_R$ symmetry does not survive in the quantum theory.

$$\nabla_\mu J_5^\mu = \frac{1}{24 \cdot 16\pi^2} \left(21 N_{3/2} - N_{1/2}\right) {}^\star\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma}$$

The phase of the fermion can be absorbed by shifting the theta term

$$S_\theta = \frac{\theta}{16\pi^2} \int d^4x \sqrt{-g}\; {}^\star\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma}$$

Topological Terms

One-loop effects tell us that we should consider two topological terms

$$S_\alpha = \frac{\alpha}{32\pi^2}\int d^4x\sqrt{g}\ {}^\star\mathcal{R}^\star_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma}$$

$$S_\theta = \frac{\theta}{16\pi^2}\int d^4x\sqrt{-g}\ {}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

In supergravity, these two coupling constants sit in a chiral multiplet

$$\tau_{\rm grav} = \alpha + 2i\theta$$

Townsend and van Nieuwenhuizen '79
Ferrara, Sabharwal, Villasante '88

Finite Quantum Corrections

Casimir Energy: $$V_{\text{eff}} = -\frac{N_B - N_F}{720\pi}\frac{L^3}{R^6}$$

Appelquist and Chodos '83
Gross, Perry, Yaffe '82

Supersymmetry means that $N_B = N_F$ and this Casimir energy vanishes.

But there are other effects….

Finite Quantum Corrections

One loop corrections to the kinetic terms give

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(M_3 + \frac{5}{16\pi}\frac{L}{R^2}\right)\mathcal{R}_{(3)} - \left(M_3 - \frac{1}{6\pi}\frac{L}{R^2}\right)\left(\frac{\partial R}{R}\right)^2 - \left(M_3 + \frac{11}{24\pi}\frac{L}{R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2$$

Finite Quantum Corrections

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$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(M_3 + \frac{5}{16\pi}\frac{L}{R^2}\right)\mathcal{R}_{(3)} - \left(M_3 + \frac{1}{6\pi}\frac{L}{R^2}\right)\left(\frac{\partial R}{R}\right)^2 - \left(M_3 + \frac{11}{24\pi}\frac{L}{R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2$$

Something important: these two numbers are different!

The Complex Structure

The two fields *R* and σ must combine in a complex number

Classically:

$$\mathcal{L}_{\rm eff} = \left(\frac{\partial R}{R}\right)^2 + \frac{1}{M_3^2}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2$$
$$= \frac{1}{(\mathcal{S}+\mathcal{S}^\dagger)^2}\partial\mathcal{S}\partial\mathcal{S}^\dagger$$

$$\boxed{\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + i\sigma}$$

The Complex Structure

The two fields *R* and σ must combine in a complex number

At one-loop

$$\mathcal{L}_{\rm eff} = \left(1 - \frac{1}{6\pi}\frac{L}{M_3 R^2}\right)\left(\frac{\partial R}{R}\right)^2 + \left(1 + \frac{11}{24\pi}\frac{L}{M_3 R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2$$

We want to write this in the form

$$\mathcal{L}_{\rm eff} = K(\mathcal{S},\mathcal{S}^\dagger)\,\partial\mathcal{S}\partial\mathcal{S}^\dagger$$

$$\Longrightarrow \quad \boxed{\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + \frac{7}{48}\log(M_{\rm pl}^2 R^2) + i\sigma}$$

Non-Perturbative Quantum Corrections

Gravitational Instantons

Look for other saddle points of the action

We want these to contribute to the (super)potential. They must obey

$$\mathcal{R}_{\mu\nu\rho\sigma} = \pm {}^\star\mathcal{R}_{\mu\nu\rho\sigma}$$

Taub-NUT Instantons

The appropriate metrics are given by the multi-Taub-NUT solutions

Gibbons and Hawking '78

$$ds^2 = U(\mathbf{x}) d\mathbf{x} \cdot d\mathbf{x} + U(\mathbf{x})^{-1} \left(dz + \mathbf{A} \cdot d\mathbf{x}\right)^2$$

with $$U(\mathbf{x}) = 1 + \frac{L}{2} \sum_{a=1}^{k} \frac{1}{|\mathbf{x} - \mathbf{X}_a|} \quad \text{and} \quad \nabla \times \mathbf{A} = \pm \nabla U$$

From the low-energy 3d perspective, these look like Dirac monopoles.

This is the gravitational verson of Polyakov's famous calculation. Polyakov '77

Gross '84
Hartnoll and Ramirez '13

Zero Modes of Taub-NUT

$$ds^2 = U(\mathbf{x}) d\mathbf{x} \cdot d\mathbf{x} + U(\mathbf{x})^{-1} \left(dz + \mathbf{A} \cdot d\mathbf{x}\right)^2$$

$$U(\mathbf{x}) = 1 + \frac{L}{2} \sum_{a=1}^{k} \frac{1}{|\mathbf{x} - \mathbf{X}_a|} \quad \text{and} \quad \nabla \times \mathbf{A} = \pm \nabla U$$

3k bosonic zero modes

2k fermionic zero modes

$\Longrightarrow$ Only *k=1* solution contributes to the superpotential

Doing the Computation

Action, Zero Modes, Jacobians, Determinants, Propagators….

The Determinants

$$\mathrm{dets} = \frac{\det(\mathrm{Fermions})}{\det(\mathrm{Bosons})}$$

Supersymmetry $\Longrightarrow$ $\mathrm{dets} = 1\ ?$

Hawking and Pope '78

The Determinants

In a self-dual background, you can write the determinants as

$$\text{dets} = \left.\frac{\det' \not{D}^\dagger \not{D}}{\det \not{D} \not{D}^\dagger}\right|_{\text{spin}-3/2}^{1/4} \left.\frac{\det' \not{D}^\dagger \not{D}}{\det \not{D} \not{D}^\dagger}\right|_{\text{spin}-1/2}^{-1/2}$$

A somewhat detailed calculation gives

$$\text{dets} = A \left(\mu^2\right)^{41/48} \left(\frac{1}{R^2}\right)^{7/48}$$

An ugly number

UV cut-off scale

We've seen these fractions before!

The Superpotential

The calculation gives

$$\mathcal{W} = C \left(\frac{\mu^2}{M_{\rm pl}^2}\right)^{41/48} \left(\frac{1}{M_{\rm pl}^2 R^2}\right)^{7/48} e^{-S_{\rm TN} - i\sigma} e^{-\tau_{\rm grav}}$$

$$C = \frac{\left(4e^{24\zeta'(-1)-1}\right)^{7/48}}{2(4\pi)^{3/2}}$$

$$S_{\rm TN} = 2\pi^2 M_{\rm pl}^2 R^2$$

Topological terms

All the pieces now fit together

$$\boxed{\mathcal{W} = C \left(\frac{\Lambda_{\rm grav}^2}{M_{\rm pl}^2}\right)^{41/48} e^{-\mathcal{S}}}$$

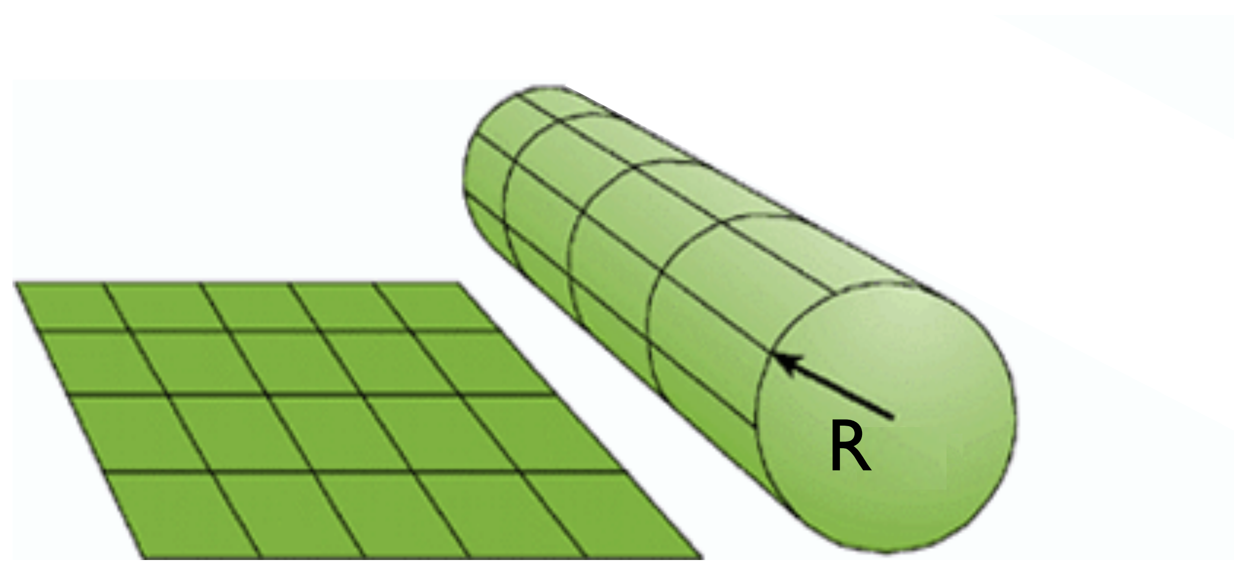
with $\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + \frac{7}{48}\log(M_{\rm pl}^2 R^2) + i\sigma$

The Potential

Kaluza-Klein compactification of *N=1* supergravity is unstable

$$\boxed{V \sim M_3^3 (R\Lambda_{\text{grav}})^{41/24} \exp\left(-4\pi^2 M_{\text{pl}}^2 R^2\right)}$$

The ground state has $R \longrightarrow \infty$



Open Questions

$$\Lambda_{\rm grav}$$

What is this good for?

Thank you for your attention