LOCALIZATION on HOPF SURFACES

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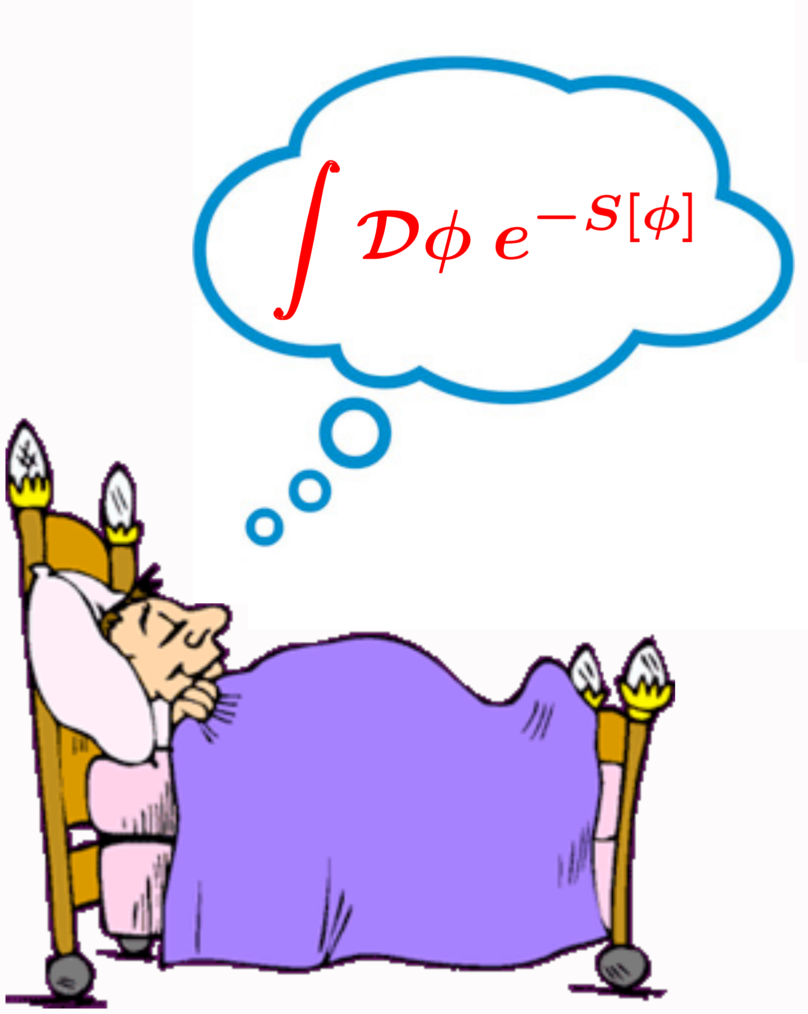
LPTHE - Université Paris 6

Indian Strings Meeting 2014

Puri, 19/12/2014

As all precious things,

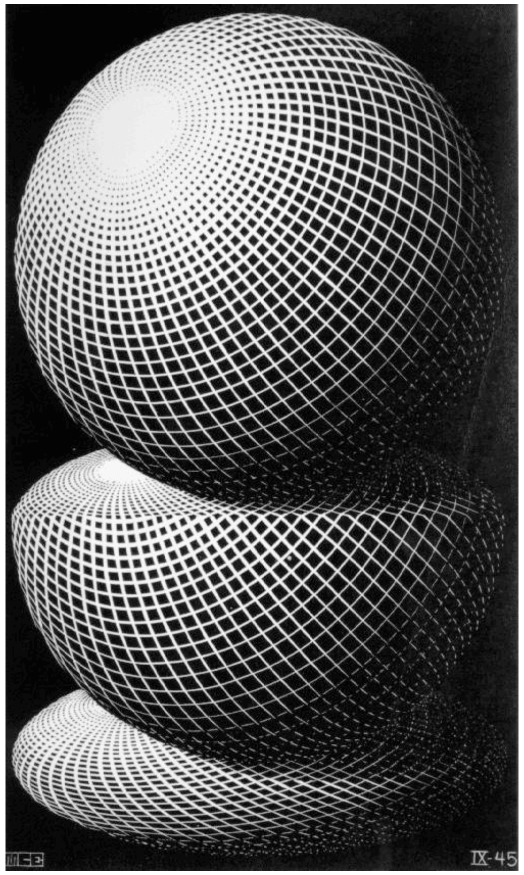
exact results in QFT are highly desirable but rare



- **Supersymmetry** has proven a very successful theoretical tool

- In the last few years: **many new exact results** about **supersymmetric QFT in curved space** based on the technique of **supersymmetric localization**

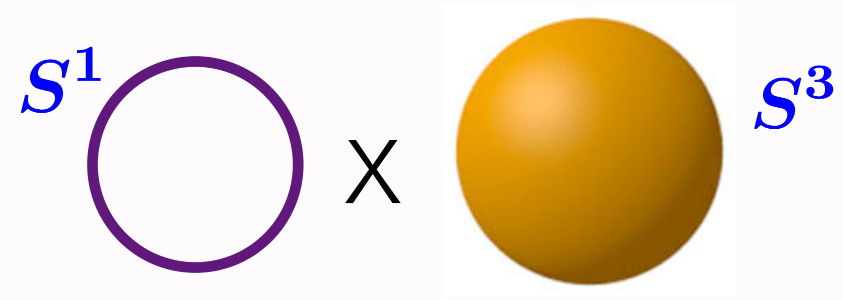
- In this talk :
  - focus on partition function Z
  - study dependence of Z on geometry
  - probe the theory varying shape & size parameters



Outline

✤ Will consider $S^1 \times S^3$ topology

as a complex manifold: **Hopf surface**



Compute the **exact partition function of N=1 QFTs**

(with an R-symmetry)

- reproduce **old** results (the **index**) with a different approach
- define a **new** interesting quantity (**susy Casimir energy**)
- put this in the context of **gauge-gravity duality**

Based on 1402.2278 with D. Martelli

1405.5144, 1410.6487 with B. Assel and D. Martelli

Localization

- Under some assumptions, the Euclidean supersymmetric path integral can be *deformed* by a susy-exact term, so that
  - it is dominated by supersymmetric configurations $\boldsymbol{\Phi_0}$
  - **saddle point approximation becomes exact**

➜ **huge simplification !**

$$Z = \int \mathcal{D}\Phi_0 \, e^{-S[\Phi_0]} \frac{1}{\mathrm{Sdet}[\text{kinetic operator for } \delta\Phi]}$$

often $\Phi_0 = \mathrm{const}$, so $\mathcal{D}\Phi_0 \to d\Phi_0$

➜ infinite-dimensional integral reduces to a finite one

- In the last years: exact partition function computed for many theories on various geometries, in different dimensions.

Partition function with sources

Need to place our field theory on a Riemannian manifold, preserving susy

- Couple it to background fields :

$$S[\Phi; A_\mu, g_{\mu\nu}] \;=\; S_0[\Phi] + \int (A_\mu j^\mu + g^{\mu\nu} T_{\mu\nu} + \ldots)$$

background gauge field

background curved metric

conserved current

energy-momentum tensor

supergravity multiplet

super-current multiplet

- Partition function :

$$Z[A_\mu, g_{\mu\nu}] \;=\; \int \mathcal{D}\Phi \, e^{-S[\Phi; A_\mu, g_{\mu\nu}]}$$

Supersymmetric backgrounds

- Which curved backgrounds preserve supersymmetry?

method : start from off-shell supergravity and take a rigid limit $\psi_\mu = 0$

$\delta\psi_\mu = 0$ constrains background metric and auxiliary fields

Festuccia, Seiberg '11

answer for a **four-dimensional N=1 theory with an R-symmetry** :

- **one** supercharge $\Longleftrightarrow$ **complex manifold with Hermitian metric**
- **two** supercharges (of opposite R-charge) ➜ **complex isometry** K

other background fields, including A_μ coupling to R-current, determined

Klare, Tomasiello, Zaffaroni; Dumitrescu, Festuccia, Seiberg

Focus on second case : localization more powerful

Hopf surfaces

- Choose $S^1 \times S^3$ topology.

Assel, DC, Martelli,
also Closset, Shamir

Hopf surfaces

Assel, DC, Martelli,
also Closset, Shamir

- Choose $S^1 \times S^3$ topology.

- Complex manifolds with $S^1 \times S^3$ topology are **Hopf surfaces** $\mathcal{H}_{p,q}$

  defined as a quotient of $\mathbb{C}^2 - (0,0)$ $\qquad (z_1, z_2) \sim (p z_1, q z_2)$

  $p = \mathrm{e}^{-2\pi b_1}, \quad q = \mathrm{e}^{-2\pi b_2}$: complex structure moduli

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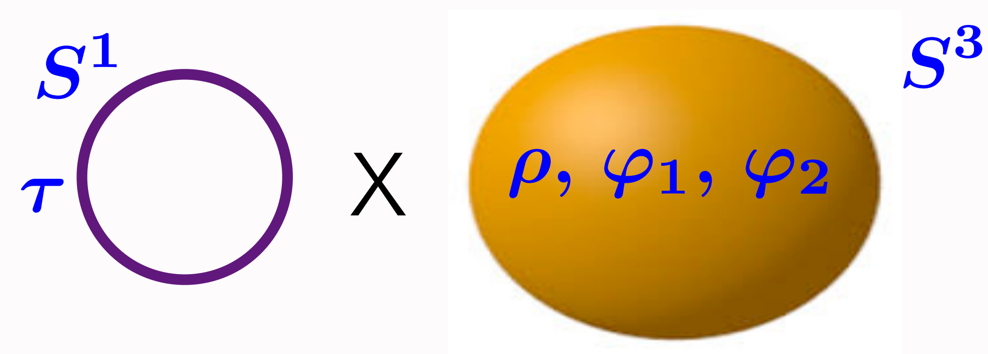
- two supercharges: complex Killing vector

$$K = b_1 \frac{\partial}{\partial\varphi_1} + b_2 \frac{\partial}{\partial\varphi_2} - i\frac{\partial}{\partial\tau}$$



S^3 as torus fibration over an interval; $b_1\,,\ b_2 \in \mathbb{R}$ for simplicity

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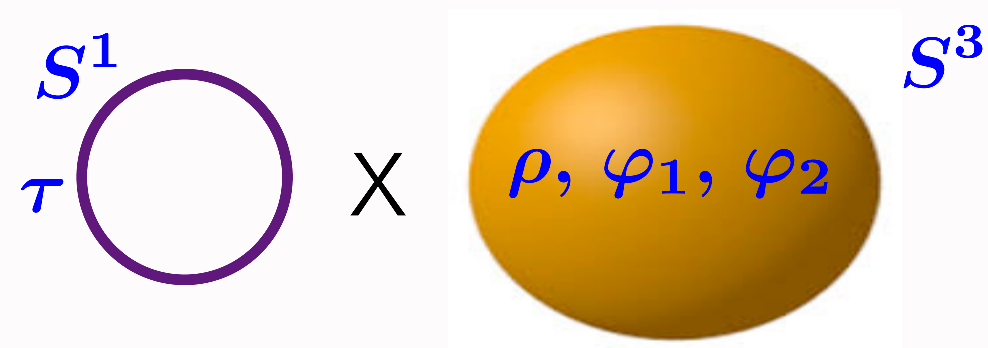

S^3 as torus fibration over an interval; $b_1\,,\, b_2 \in \mathbb{R}$ for simplicity

- compatible metric has U(1)xU(1)xU(1) symmetry, still very general :

$$\mathrm{d}s^2 = \Omega^2(\rho)\mathrm{d}\tau^2 + f^2(\rho)\mathrm{d}\rho^2 + m_{IJ}(\rho)\mathrm{d}\varphi_I \mathrm{d}\varphi_J \qquad I, J = 1, 2$$

Localization on Hopf surfaces

- Consider partition function of an Euclidean theory on $\mathcal{H}_{p,q}$, with
  - vector multiplet for general gauge group
  - (charged) chiral multiplets, with superpotential

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- Adding a suitable susy-exact deformation term,

localization locus : $\Phi_0 \left\{ \begin{array}{l} \text{flat connections } \mathcal{F} = 0 \rightarrow \mathcal{A}_\tau = \mathbf{const} \\ \text{all other fields vanish} \end{array} \right.$

Localization on Hopf surfaces

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- Adding a suitable susy-exact deformation term,

$$\text{localization locus :} \quad \Phi_0 \begin{cases} \text{flat connections } \mathcal{F} = \mathbf{0} \rightarrow \mathcal{A}_\tau = \mathbf{const} \\ \text{all other fields vanish} \end{cases}$$

- Integral over field fluctuations around localization locus :

$$\mathbf{Sdet}[\mathbf{kinetic\ operator\ for\ }\delta\Phi]$$

  - cancellations between bosonic and fermionic eigenvalues
  - left with ∞ product over 3 integers (from Fourier modes on $U(1)^3$)
  - regularized using generalised version of Γ and ζ functions

Localization on Hopf surfaces

Result :

$$Z[\mathcal{H}_{p,q}] = \underbrace{\mathrm{e}^{-\mathcal{F}(p,q)}}_{\text{prefactor}} \underbrace{\mathcal{I}(p\,,q)}_{\text{supersymmetric index}}$$

$$\mathcal{I}(p,q) = \frac{(p;p)^{r_G}(q;q)^{r_G}}{|\mathcal{W}|}\int_{T^{r_G}}\frac{\mathrm{d}z}{2\pi i z}\prod_{\alpha\in\Delta_+}\theta\left(z^{\alpha},p\right)\theta\left(z^{-\alpha},q\right)\prod_J\prod_{\rho\in\Delta_J}\Gamma_e\left(z^{\rho}(pq)^{\frac{r_J}{2}},p,q\right)$$

Localization on Hopf surfaces

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defined as

$$\mathcal{I}(p,q) = \mathrm{tr}\left[(-1)^F p^{J+J'-\frac{R}{2}}q^{J-J'-\frac{R}{2}}\right]$$

fugacities

refined Witten index $\mathbf{tr}(-1)^F$ counting certain BPS states

Romelsberger;
Kinney et al.

Localization on Hopf surfaces

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Romelsberger; Kinney et al.

- General arguments show that Z does not depend on Hermitian metric and is a holomorphic function of the complex structure parameters

  Closset, Dumitrescu, Festuccia, Komargodski

- $Z[\mathcal{H}_{p,q}]$ conjectured to compute $\mathcal{I}(p\,,q)$

➜ we have explicitly checked this. Found an **extra contribution** $\mathcal{F}(p,q)$

Localization on Hopf surfaces

$$Z[\mathcal{H}_{p,q}] \;=\; e^{-\mathcal{F}(p,q)}\mathcal{I}(p\,,\,q)$$

index

$$\boxed{\mathcal{F}(p,q) = \frac{4\pi}{3}\,(b_1+b_2)\,(a-c) + \frac{4\pi}{27}\,\frac{(b_1+b_2)^3}{b_1 b_2}(3\,c-2\,a)}$$

$$p = e^{-2\pi b_1}\,, \quad q = e^{-2\pi b_2}$$

$$a \;=\; \frac{3}{32}\left(3\,\mathrm{tr}R^3 - \mathrm{tr}R\right), \qquad c \;=\; \frac{1}{32}\left(9\,\mathrm{tr}R^3 - 5\,\mathrm{tr}R\right)$$

R : fermionic R-charge

SCFT central charges

Localization on Hopf surfaces

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is this physical or ambiguous?

Ambiguities

- two choices of renormalisation scheme may differ by *finite* counterterms
- allowed counterterms are restricted by symmetries
- a physical observable should not depend on the chosen scheme

➜ allowed finite counterterms parametrise ambiguities in the computation of partition function

We have classified the supersymmetric, local counterterms.

All finite ones vanish whenever two supercharges are preserved.

➜ **there is no ambiguity in the partition function on Hopf surfaces**

Localization on Hopf surfaces

$$Z[\mathcal{H}_{p,q}] = e^{-\mathcal{F}(p,q)}\mathcal{I}(p\,,q)$$

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- physical (non-removable by supersymmetric local counterterm)
- defines a **supersymmetric Casimir energy**
- related to anomalies? which regularisation is *correct*?

  in progress - Assel, DC, Di Pietro, Komargodski, Lorenzen, Martelli

- dominates Z at large N ➜ **prediction for dual supergravity solutions**

Gravity duals

AdS/CFT master equation (at large N)

$$\mathrm{e}^{-S_{\text{gravity}}[M_5]} = Z[M_4]$$

$M_4 = \partial M_5$ QFT background fields $\Longleftrightarrow$ gravity boundary conditions

➜ can explore new corners of AdS/CFT

- When $\partial M_5 = \mathcal{H}_{p,q}$, our prediction from localization (at large N) :

$$S_{\text{5d sugra}}[M_5] = \frac{\pi^2}{54 G_5} \frac{(b_1 + b_2)^3}{b_1 b_2}$$

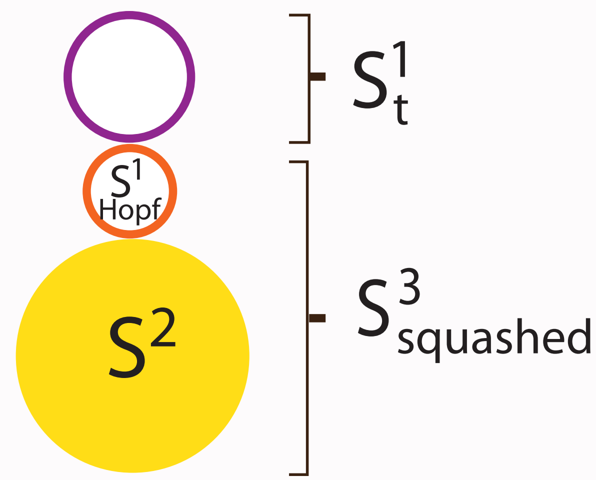
(entirely from prefactor)

New supergravity solution

We took a first step :

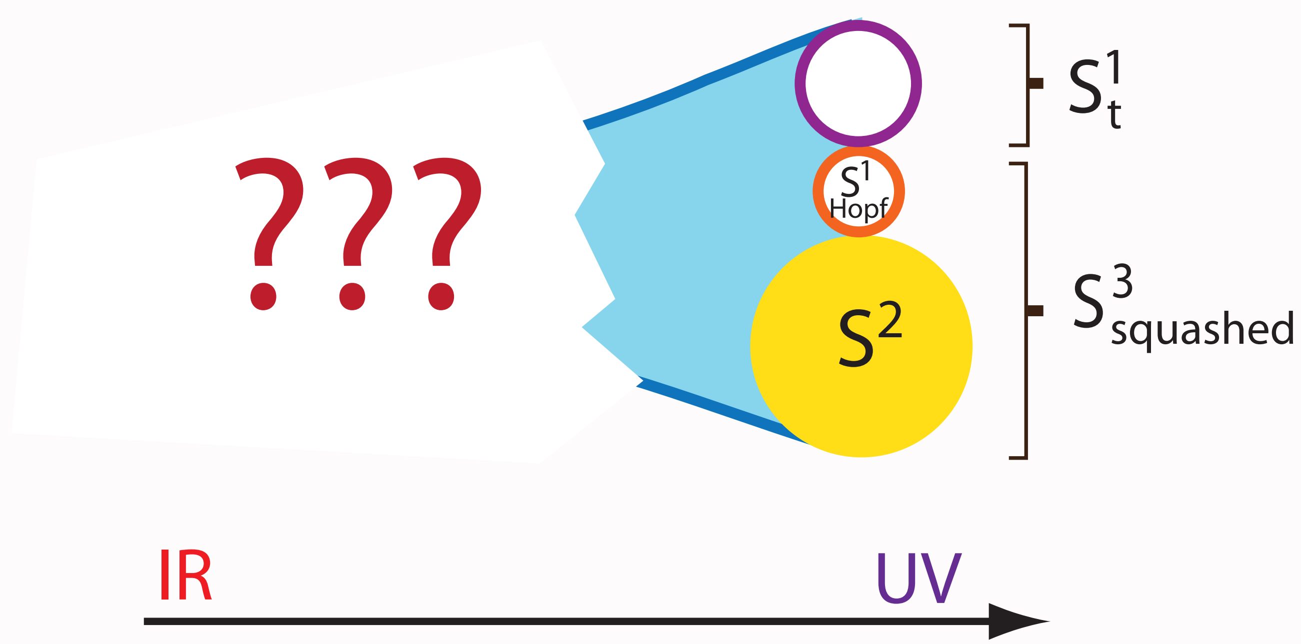
D.C., Martelli '14

- considered $S^1 \times S^3_{\text{squashed}}$



impose enhanced symmetry U(1)xU(1)xU(1) ➜ SU(2)xU(1)xU(1)

- studied 5d supergravity susy equations with these boundary conditions

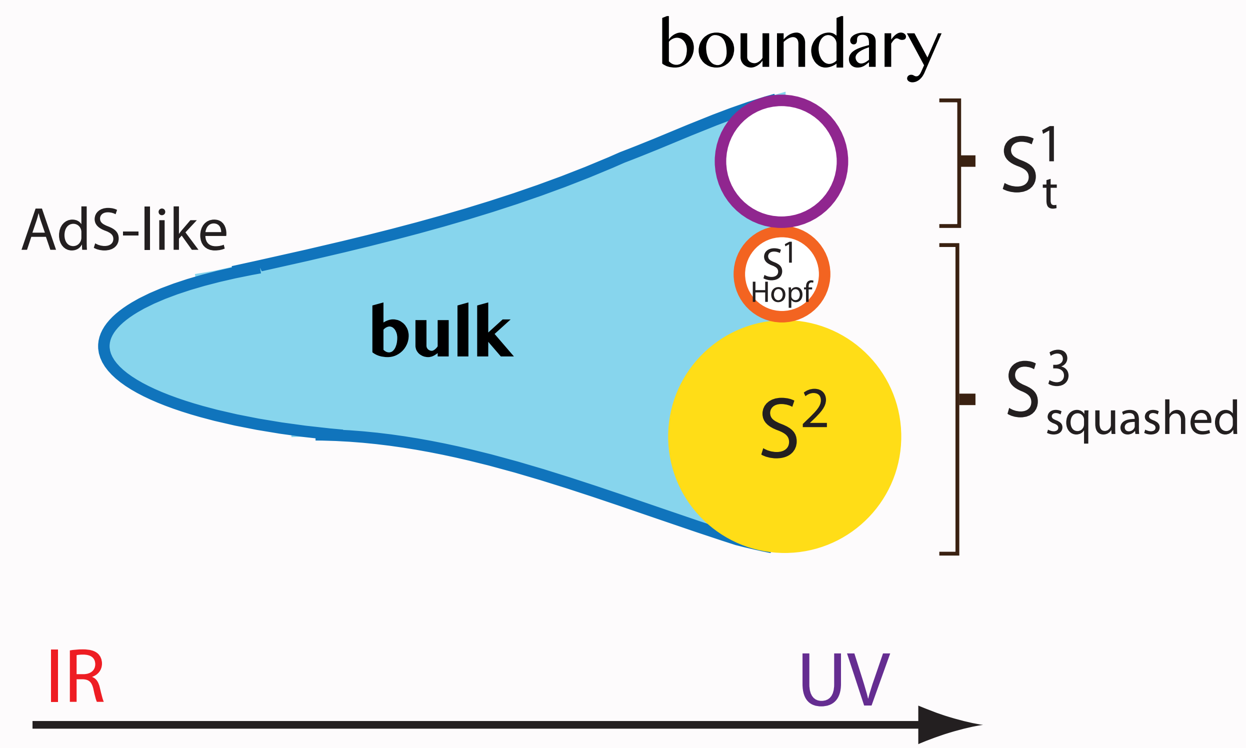


New supergravity solution

D.C., Martelli '14

found a new one-parameter family of solutions

- regular
- no horizon
- family parameterized by squashing of S^3



- solution obtained in Lorentzian signature

  analytic continuation $t \longrightarrow i\,t$ yields a **complex bulk metric**

- However, boundary metric and on-shell action remain **real**

New supergravity solution

- on-shell action agrees with field theory formula, with

$$b_1 = b_2 = \beta\,, \qquad \beta = \frac{r_{S^1}}{r_{S^1_{\rm Hopf}}}$$

field theory prediction : $S_{\rm 5d\,sugra}[M_5] = \dfrac{\pi^2}{54G_5}\dfrac{(b_1+b_2)^3}{b_1 b_2} = \dfrac{4\pi^2}{27G_5}\beta$

gravity : $S_{\rm 5d\,sugra}[M_5] = \dfrac{4\pi^2}{27G_5}\beta + \ldots$ ← terms independent of β

⚠ need better understanding of supersymmetric holographic renormalization

→ something to learn about (supersymmetric) AdS/CFT !

Conclusions

- We saw an explicit computation of the partition function of N=1 gauge theories on a Hopf surface $\mathcal{H}_{p,q}$, allowing for a general metric

$$Z[\mathcal{H}_{p,q}] = \mathrm{e}^{-\mathcal{F}(p,q)}\mathcal{I}(p\,,q)$$

- First holographic check by constructing a new supergravity solution

- $\mathcal{F}(p,q)$ defines a supersymmetric Casimir energy

Future work :

- to explore more its (universal?) meaning in field theory
- to retrieve it in full generality in a holographic setup

➜ refine our understanding of gauge-gravity correspondence

- implications for matching black hole entropy?

... thank you !