GEROCH GROUP DESCRIPTION OF BLACK HOLES

Bidisha Chakrabarty, Amitabh Virmani

Institute of Physics , Bhubaneswar

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References:

- JHEP11 (2014) 068 by B. Chakrabarty, A. Virmani
- JHEP02(2013)011 by D. Katsimpouri, A. Kleinschmidt and A. Virmani
- Annales Henri Poincare A 46 (1987) 215 by P. Breitenlohner and D. Maison
- Unpublished notes of Breitenlohner and Maison from June 1986.

Outline

Motivation

- Gravity in ($d > 4$) is rich.

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- So, **efforts to study exact solution generating techniques.**
- In the present talk we consider cases with $d \leq 5$

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- These symmetries have been used to study various features of BHs.
- Higher dimensional gravity theories when reduced to 2D has infinite no of symmetries $\rightarrow$ **Integrability.**
  The symmetry group is called the **Geroch group**.
- these symmetries can be useful in constructing various exact sol^ns

Outline

Dimensional Reduction from 5D to 2D

Perform dimensional reduction of a five-dimensional gravity theory to 2Dim^{n}s in two steps.

1. Reduce the theory to 3D
2. Reduce it from 3 to 2 dim^{n}s.

Outline

Step 1: 5D to 3D

- **vacuum gravity in 5D**:

$$\boxed{\mathcal{L}_5 = R_5 \star 1} \quad (1)$$

- Assume two commuting Killing vectors exist : $\frac{\partial}{\partial x^5}$ (space-like) and $\frac{\partial}{\partial x^4}$(time-like) .

- Dimensionally reduce theory from 5D to 3D, first reduction over x^5, then over x^4.

Dimensional Reduction of 5D vacuum Gravity to 3D

- **Kaluza-Klein metric ansatz**:

$$ds_5^2 = e^{\frac{1}{\sqrt{3}}\phi_1 + \phi_2} ds_3^2 + \epsilon_2 e^{\frac{\phi_1}{\sqrt{3}} - \phi_2} \left(dz_4 + \mathcal{A}^2\right)^2$$
$$+\epsilon_1 e^{-\frac{2\phi_1}{\sqrt{3}}} \left(dz_5 + \chi_1 dz_4 + \mathcal{A}^1\right)^2 \qquad (2)$$

$$g^{(5)}_{\mu\nu} = \left(\begin{array}{c|c|c} g_{ab} & \mathcal{A}^2 & \mathcal{A}^1 \\ \hline \star & \phi_2 & \chi_1 \\ \hline \star & \star & \phi_1 \end{array}\right)$$

3D fields are:
—metric g,
—Dilatons ϕ_1, ϕ_2 ,
—one axion χ_1,
—two Maxwell-like one form potentials $\mathcal{A}^1$ & $\mathcal{A}^2$

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- **3D fields are independent of x^4 and x^5**

Dualising 1-form potentials into axions

- Dualise the Maxwell-like one form potentials $\mathcal{A}^1$ & $\mathcal{A}^2$ in 3D into scalar axions χ_2 and χ_3.

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- **3D Lagrangian in dualised variables**:

$$\begin{aligned}\mathcal{L}_3 \;=\; & R_3 \star 1 - \frac{1}{2} \star d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2}\epsilon_1 \epsilon_2 e^{-\sqrt{3}\phi_1 + \phi_2} \star d\chi_1 \wedge d\chi_1 \\ & -\frac{1}{2}\epsilon_2 e^{\sqrt{3}\phi_1 + \phi_2} \star d\chi_2 \wedge d\chi_2 \\ & -\frac{1}{2}\epsilon_1 e^{2\phi_2} \star (d\chi_3 - \chi_1 d\chi_2) \wedge (d\chi_3 - \chi_1 d\chi_2). \end{aligned} \tag{3}$$

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where

$\epsilon_1, \epsilon_2 = \pm 1$

Coset Model Construction

- 3D scalar Lagrangian can be parametrised by the $\frac{SL(3,R)}{SO(2,1)}$ coset representative

$$\mathbf{V} = \mathbf{e}^{\frac{1}{2}\phi_1 \mathbf{h}_1} \mathbf{e}^{\frac{1}{2}\phi_2 \mathbf{h}_2} \mathbf{e}^{\chi_1 \mathbf{e}_1} \mathbf{e}^{\chi_2 \mathbf{e}_2} \mathbf{e}^{\chi_3 \mathbf{e}_3}. \tag{4}$$

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where $h_1, h_2 \rightarrow$ Cartan Generators of sl(3)
$e_1, e_2, e_3 \rightarrow$ positive root generators of sl(3)

V → Upper triangular matrix.

Coset Model Construction

- Construct

$$M = V^T V, \tag{5}$$

- The 3D Lagrangian

$$\boxed{\mathcal{L}_3' = \mathbf{R} \star \mathbf{1} - \frac{\mathbf{1}}{\mathbf{4}} \mathrm{tr}(\star(\mathbf{M}^{-1}\mathbf{dM}) \wedge (\mathbf{M}^{-1}\mathbf{dM}))}. \tag{6}$$

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is manifestly **SL(3,R) invariant**

$$M \to g^T M g$$
$$\therefore M^{-1} dM \to g^{-1}(M^{-1} dM) g$$
$$\therefore \mathrm{tr}(\star(M^{-1} dM) \wedge (M^{-1} dM)) \to \text{invariant.}$$

Outline

Step 2: 3D to 2D

- In Step2 of Dimensional Reduction, reduce over a space-like Killing vector to 2D.

- 3D metric ansatz: $\boxed{ds_3^2 = f^2(d\rho^2 + dz^2) + \rho^2 d\varphi^2}$;

  $\rho, z \rightarrow$ Weyl Canonical Coordinates, $f \rightarrow$Conformal factor, $\partial_\varphi \rightarrow$ Spacelike Killing Vector.

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Dimensional Reduction to 2 *dim*^{n} s

- The 2D system is Integrable$\Rightarrow$ **Lax pair** exists and its compatibility condition is the *eq*^{n}*s* of the 2D gravity system.
- Lax equations require the generalization $V(x) \to \mathcal{V}(t, x)$ with $\mathcal{V}(0, x) = V(x)$
- t satisfies certain space-time dependent Differential eq^n
- $t \to$ Space-time dependent Spectral Parameter

Dimensional Reduction to 2 dim^ns

- Solves to
  $t_{\pm}(w,x) = \frac{1}{\rho}\left[(z-w) \pm \sqrt{(z-w)^2+\rho^2}\right] = -\frac{1}{t_{\mp}}(w,x)$,
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  $w \rightarrow$ Integration const(Space-time Independent Spectral Parameter)

- $(\mathcal{V}(t,x))^T = \mathcal{V}^T\left(-\frac{1}{t},x\right)$, like $M(x)$ before here **Monodromy matrix**

$$\mathcal{M}(t,x) = \mathcal{V}^T\left(-\frac{1}{t},x\right)\mathcal{V}(t,x). \tag{7}$$

- $\partial_\pm \mathcal{M}(t,x) = 0 \rightarrow \mathcal{M}(t,x)$ is space-time independent (using Lax Eqns).

$$\boxed{\mathcal{M}(t,x) = \mathcal{M}(w)}. \tag{8}$$

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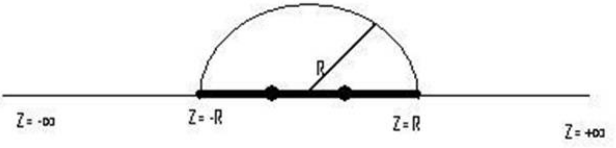
- **The Geroch group allows one to associate a space-time independent matrix to a space-time configuration that effectively depends on only two coordinates.**

Relation between $M(x)$ and $\mathcal{M}(w)$

- 2D space spanned by (ρ, z) coordinates$\rightarrow$ **Factor Space**
- **Boundary** $\rho = 0$ consists of a union of Intervals [**Hollands** & **Yazadjiev gr-qc 0707.2775**].
- Two adjacent intervals meet at the **corners.**

Relation between $M(x)$ and $\mathcal{M}(w)$

- We concentrate in the $\rho = 0$, $z < -R$ region of the factor space.



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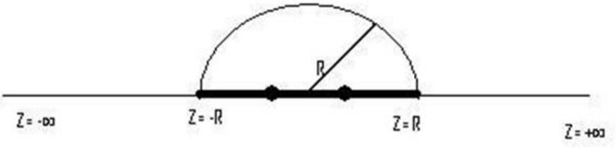


The important relation is

$$M(\rho = 0, z = w \;\text{ with }\; z < -R) = \mathcal{M}(w). \quad (9)$$

This relation is obtained via Lax equations.

Geroch Group Matrices

Consider SL(3) matrices with simple poles in w with constant residue matrices of rank one:

$$\mathcal{M}(w) = Y + \sum_{k=1}^{N} \frac{A_k}{w - w_k} \tag{10}$$

with residue matrix $A_k = \alpha_k a_k a_k^T$ where α's are constants chosen to satisfy coset conditions.

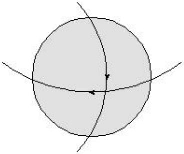
Solitonic matrices SL(3)

- Consider the case when $\mathcal{M}(w)$ has two poles at $w_1 = +c$ and $w_2 = -c$.
- This works well for the two examples we consider:

1. 5D Myers-Perry
2. 5D Dyonic Kaluza Klein

Geroch Group Matrices for Black Holes

Example 1 : 5D Myers-Perry



- Consider a **doubly spinning Myers-Perry BH in 5D** with three independent parameters (mass m, angular momenta l_1 and l_2). In 5D two independent rotation planes.
- Perform KK reduction over appropriately chosen space-like and time-like Killing directions.

Geroch Group Matrices for Black Holes

- Resulting matrix $M(r, x)$ has the asymptotic behaviour

$$\boxed{M(r, x) = Y + \mathcal{O}\left(\frac{1}{r^2}\right)}, \tag{11}$$

- To construct the monodromy matrix $\mathcal{M}(w)$ from $M(r, x)$ change to canonical coordinates (ρ, z) and take the limit $\rho \to 0$, z near $-\infty$.

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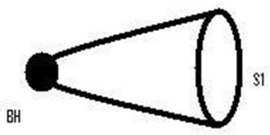
- Final form of $\mathcal{M}(w)$:

$$\boxed{\mathcal{M}(w) = Y + \frac{A_1}{w-\alpha} + \frac{A_2}{w+\alpha}}$$

where $A_1 = \alpha_1 a_1 a_1^T$, $A_2 = \alpha_2 a_2 a_2^T$

Geroch Group Matrices for Black Holes

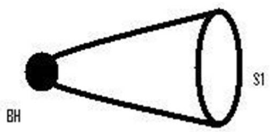
Example 2 : Dyonic Kaluza Klein



- Kaluza Klein Black Hole is written in terms of four parameters p, q, m, a corresponding to electric and magnetic KK charges , mass and angular momentum.

Geroch Group Matrices for Black Holes

Example 2 : Dyonic Kaluza Klein



- Kaluza Klein Black Hole is written in terms of four parameters p, q, m, a corresponding to electric and magnetic KK charges , mass and angular momentum.

- In this case $\boxed{M(x) = g^T M_{Kerr}(x) g}$; $g \in \mathrm{SO}(2,1)$

Geroch Group Matrices for Black Holes

- $$M_{Kerr}(x) = \begin{pmatrix} 1 + \frac{2mr}{r^2-2mr+a^2x^2} & 0 & -\frac{2amx}{r^2-2mr+a^2x^2} \\ 0 & 1 & 0 \\ -\frac{2amx}{r^2-2mr+a^2x^2} & 0 & 1 + \frac{2m(2m-r)}{r^2-2mr+a^2x^2} \end{pmatrix}$$

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$$g = \exp(-\gamma k_3) \cdot \exp(-\beta k_1) \cdot \exp(\alpha k_2). \tag{12}$$

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- Finally $\boxed{\mathcal{M}(w) = I + \frac{A_1}{w-c} + \frac{A_2}{w+c}}$,

  where $A_1 = \alpha_1 a_1 a_1^T$, $A_2 = \alpha_2 a_2 a_2^T$.

Outline

Charge matrix

- The charge matrix $\mathcal{Q}$ for a 4 D asymptotically flat configuration is defined as
$$M(x) = I - \frac{\mathcal{Q}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{13}$$
(Bossard, Nicolai, Stelle JHEP $0907, 003(2009)$)

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- $\mathcal{Q}$ satisfies characteristic eq.

$$\mathcal{Q}^3 - \frac{1}{2}\mathrm{Tr}(\mathcal{Q}^2)\mathcal{Q} = 0, \tag{14}$$

Charge matrix

- Asymptotic form of $\mathcal{M}$ (w) in terms of $\mathcal{Q}$ is

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Charge matrix

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  $$\mathcal{M}(w) = I + \frac{\mathcal{Q}}{w} + \mathcal{O}\left(\frac{1}{w^2}\right).$$

- $\mathcal{Q} = \sum_{i=1}^{N} \alpha_i a_i (a_i)^T$.

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- We presented some of the **relations between the Geroch group matrices and the charge matrices.**
- **Future interest can be in studying cases where monodromy matrix is not a constant at infinity i.e *w* has a pole at infinity.**

THANK YOU!

3D Lagrangian

- The **reduced 3D Lagrangian** :

$$\mathcal{L}_3 = R_3 \star 1 - \frac{1}{2} \star d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2}\epsilon_1\epsilon_2 e^{-\sqrt{3}\phi_1+\phi_2} \star \mathcal{F}_{(1)} \wedge \mathcal{F}_{(1)}$$
$$-\frac{1}{2}\epsilon_1 e^{-\sqrt{3}\phi_1-\phi_2} \star \mathcal{F}^1_{(2)} \wedge \mathcal{F}^1_{(2)} - \frac{1}{2}\epsilon_2 e^{-2\phi_2} \star \mathcal{F}^2_{(2)} \wedge \mathcal{F}^2_{(2)} \tag{15}$$

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where

$$\epsilon_1, \epsilon_2 = \pm 1$$

$$\& \quad \mathcal{F}_{(1)} = d\chi_1, \quad \mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} + \mathcal{A}^2_{(1)} \wedge d\chi_1, \quad \mathcal{F}^2_{(2)} = d\mathcal{A}^2_{(1)} \quad (16)$$

are the field strengths for χ_1, $\mathcal{A}^1_{(1)}$, and $\mathcal{A}^2_{(1)}$ respectively.

Dualising 1-form potentials into axions

- **The Hodge Dual of a 1-form potential (2-form Field strength) in 3D is a scalar**.
- The full hidden symmetry of a theory can be manifested after the gauge potentials are dualised into scalar axions.

$$\mathcal{A}^1_{(1)} \to \chi_2 \,, \quad \mathcal{A}^2_{(1)} \to \chi_3$$

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- **For Dualisation add Lagrange Multiplier terms**

$$-\chi_2 d(\mathcal{F}^1_{(2)} - \mathcal{A}^2_{(1)} \wedge d\chi_1) - \chi_3 d\mathcal{F}^2_{(2)} \tag{17}$$

to the 3D Lagrangian.

Dualising 1-form potentials into axions

- Eliminating $\mathcal{F}^1_{(2)}$ and $\mathcal{F}^2_{(2)}$ we obtain **Duality Relations**

$$\epsilon_1 e^{-\sqrt{3}\phi_1-\phi_2} \star \mathcal{F}^1_{(2)} = d\chi_2, \quad \epsilon_2 e^{-2\phi_2} \star \mathcal{F}^2_{(2)} = d\chi_3 - \chi_1 d\chi_2 \quad (18)$$

(Eqns of motion and Bianchi identities get interchanged).

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Relation between $M(x)$ and $\mathcal{M}(w)$

- $t_\pm$ have two branch points at $\rho = \pm\text{Im}(w)$, z=Re(w)
- $\mathcal{V}_\pm(w,\rho,z) = \mathcal{V}(t_\pm(w,\rho,z),\rho,z)$
- $\mathcal{V}_+(w,0,z) = V(0,z)$,
  $\mathcal{V}_-(w,0,z) = (V^T(0,z))^{-1}C(w)$, solving Lax Eqs

Relation between $M(x)$ and $\mathcal{M}(w)$

- At each branch point $t_\pm$ have same values, therefore

$$\mathcal{V}_+(w,\rho,z)\Big|_{\rho=\mathbf{Im}(w),\, z=\mathbf{Re}(w),} = \mathcal{V}_-(w,\rho,z)\Big|_{\rho=\mathbf{Im}(w),\, z=\mathbf{Re}(w)} \quad (19)$$

- Using eq. (14) and $\mathbf{Im}(w) \to 0$,

$$C(w) \;=\; M(0,w). \quad (20)$$

- Therefore

$$\begin{aligned} \mathcal{M}(w) &= \mathcal{V}_-^T(w,0,z)\mathcal{V}_+(w,0,z), \\ &= ((V^T(0,z))^{-1}M(0,w))^T V(0,z), \\ &= M(0,w). \end{aligned} \quad (21)$$

Geroch Group Matrices for Black Holes

- For MP BH: For $l_1 \to 0$ and $l_2 \to 0$ (**5D Schwarzschild case**)

$$\alpha_1 = m, \qquad a_1 = \{0, 0, 1\}, \tag{22}$$

$$\alpha_2 = -1, \qquad a_2 = \left\{\sqrt{2}\ell, -\frac{m}{2\sqrt{2}\ell}, 0\right\}. \tag{23}$$

Geroch Group Matrices for Black Holes

- For MP BH: For $l_1 \to 0$ and $l_2 \to 0$ (**5D Schwarzschild case**)

$$\alpha_1 = m, \qquad a_1 = \{0, 0, 1\}, \tag{22}$$

$$\alpha_2 = -1, \qquad a_2 = \left\{\sqrt{2}\ell, -\frac{m}{2\sqrt{2}\ell}, 0\right\}. \tag{23}$$

- Further for $m \to 0$, residue at $w = +\alpha$ vanishes (**5D Minkowski case** monodromy matrix simplifies to

$$\mathcal{M}(w) = Y + \frac{\alpha_2 a_2 a_2^T}{w}, \tag{24}$$

with $\alpha_2 = -1$ and $a_2 = \left\{\sqrt{2}\ell, 0, 0\right\}$.).

Geroch Group Matrices for Black Holes

- For KK BH : Vectors a_1 and a_2 are

$$a_1 = g^T a_1^{Kerr}, \qquad a_1^{Kerr} = \{\zeta, 0, 1\}, \tag{25}$$

$$a_2 = g^T a_2^{Kerr}, \qquad a_2^{Kerr} = \{1, 0, \zeta\}, \tag{26}$$

where $\zeta = \frac{m-\sqrt{m^2-a^2}}{a}$.

Geroch Group Matrices for Black Holes

- For KK BH : Vectors a_1 and a_2 are

$$a_1 = g^T a_1^{Kerr}, \qquad a_1^{Kerr} = \{\zeta, 0, 1\}, \tag{25}$$
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where $\zeta = \frac{m-\sqrt{m^2-a^2}}{a}$.

- In the limit $a \to 0$, the residue vectors are smooth.

$$a_1 = g^T a_1^{Kerr}, \qquad a_1^{Kerr} = \{0, 0, 1\}, \tag{27}$$
$$a_2 = g^T a_2^{Kerr}, \qquad a_2^{Kerr} = \{1, 0, 0\}, \tag{28}$$