ISM-2014

Constraining gravity using entanglement

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Based on JHEP 1405 (2014) 029 (Arxiv:- 1401.5089)
with Shamik Banerjee, Apratim Kaviraj, Kallol Sen and Aninda Sinha

Also one can see arXiv:1405.3743 by Banerjee, Kaviraj and Sinha

Outline

– Introduction

– Holographic realization of Relative Entropy

– Constraints for two derivative gravity

– Towards Einstein point

– Constraining higher curvature duals

– Extremal surface constraints

– Conclusions

Introduction-Relative Entropy

– Relative Entropy is a fundamental quantum statistical measure of how "distinguishable" two states are.

$$\boxed{S(\rho_1|\rho_0) = Tr(\rho_1 \ln \rho_1) - Tr(\rho_1 \ln \rho_0)}$$

– For unitary theories, $\boxed{S(\rho_1|\rho_0) \geq 0}$

– If ρ_0 is thermal with a temperature T, $\boxed{\rho_0 = \frac{e^{-H/T}}{Tre^{H/T}}}$

$$\boxed{S(\rho_1|\rho_0) = \Delta H - T\Delta S \geq 0}$$

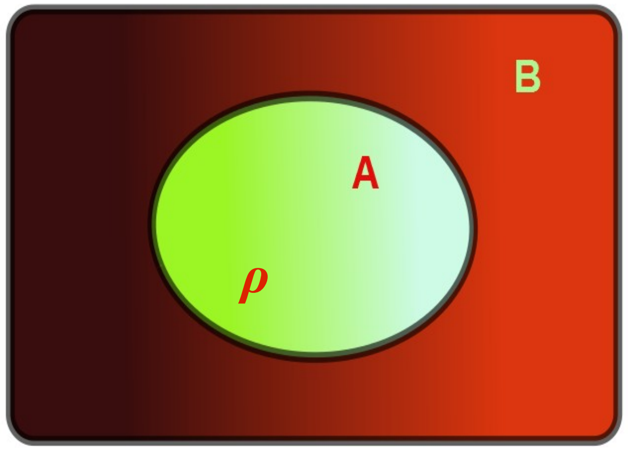
– Equality corresponds to the usual first law of thermodynamics.

(Also see Shouvik Datta and Aninda Sinha's talk)

In context of entanglement

– ρ_0 and ρ_1 are two density matrices of 2 states of an entangled subsystem.

$$\boxed{\rho_0 = \frac{e^{-H}}{Tr\, e^{-H}}}$$



–"H" is called a "Modular Hamiltonian".

–So, $$\boxed{S(\rho_1|\rho_0) \geq 0 \Rightarrow \Delta H \geq \Delta S}$$

$\Delta H = \Delta S$ gives a First Law for Entanglement entropy

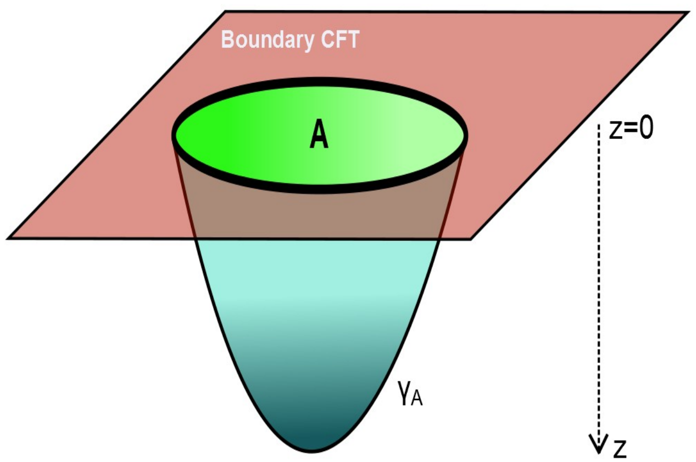
(JHEP 1308 (2013) 060 by Blanco, Casini, Hung,Myers, JHEP 1403 (2014) 051 by Faulkner, Guica,Hartman, Myers,Raamsdonk)

(Various applications of this "First Law" are discussed by Aninda Sinha , Parijat Dey and Shouvik Datta in their talks)

Holographic Realization

- We will restrict ourselves first to those theories where the holographic entanglement entropy can be calculated using Ryu-Takayanagi ('06) prescription.

- We will only consider a "Spherical" entangling surface.



- **Entanglement Entropy:**

Find a minimal surface γ_A: $z = f(r)$

Eg: $z = \sqrt{R^2 - r^2}$ (spherical surfaces)

and EE is $$\boxed{S(A) = \frac{2\pi}{l_p^d}\,\text{Area}(\gamma_A)}$$

- And for the Modular Hamiltonian part:

Calculate holographic stress tensor at boundary

$$T_{\mu\nu} = \frac{1}{l_p^3}\left(K_{\mu\nu} - g_{\mu\nu} K\right)$$

$g_{\mu\nu}$: Boundary metric
$K_{\mu\nu}$: Extrinsic curvature

and use T_{00} to get H

$$\boxed{H = 2\pi \int_{|x|<R} d^{d-1}x \frac{R^2 - r^2}{2R} T_{00}}$$ (for spherical surfaces)

Continue….

ρ_0 corresponds : Spherical surface at the boundary of empty AdS. (dual to a CFT vacua)

ρ_1 corresponds : A small perturbation by a constant stress tensor

$$\boxed{\begin{aligned} ds^2 &= \frac{L^2}{z^2}(dz^2 + g_{\mu\nu}dx^\mu dx^\nu) \\ g_{\mu\nu} &= \eta_{\mu\nu} + a\, z^4\, T_{\mu\nu} + a^2 z^8 (n_1 T_{\mu\alpha}T^\alpha_\nu + n_2 \eta_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta}) \\ a &= \frac{2}{d}\frac{\ell_p^{d-1}}{L^{d-1}} \end{aligned}}$$

- At the linear order $\Delta H = \Delta S \Rightarrow$ Einstein equation

(JHEP 1308 (2013) 060 by Blanco, Casini, Hung,Myers, JHEP 1403 (2014) 051 by Faulkner, Guica,Hartman, Myers,Raamsdonk,
JHEP 1404 (2014) 195 by Lashkari, McDermott, Raamsdonk, JHEP 1310 (2013) 219 Joytirmoy Bhattacharya, Takayanagi)

- From the positivity of relative entropy it follows,

$$\boxed{\Delta^2 S \leq 0}$$

Constraints for two derivative gravity

Now we calculate the second order change in the entanglement entropy due this perturbation.

$$\boxed{\Delta^2 S = \Delta^2\Big(\frac{2\pi}{\ell_p^{d-1}}\int d^{d-1}x\sqrt{h}\Big)}$$

Not only the metric but at this order the entangling surface also gets perturbed.

$$\boxed{z = \sqrt{R^2 - r^2} - \frac{\ell_p^{d-1}R^2(R^2 - r^2)^{d-1}}{d(d+1)L^{d-1}}(T^i_i + \frac{x_i x_j T^{ij}}{R^2})}$$

Finally the second order change can be written as,

$$\boxed{\Delta^2 S = V^T M V}$$

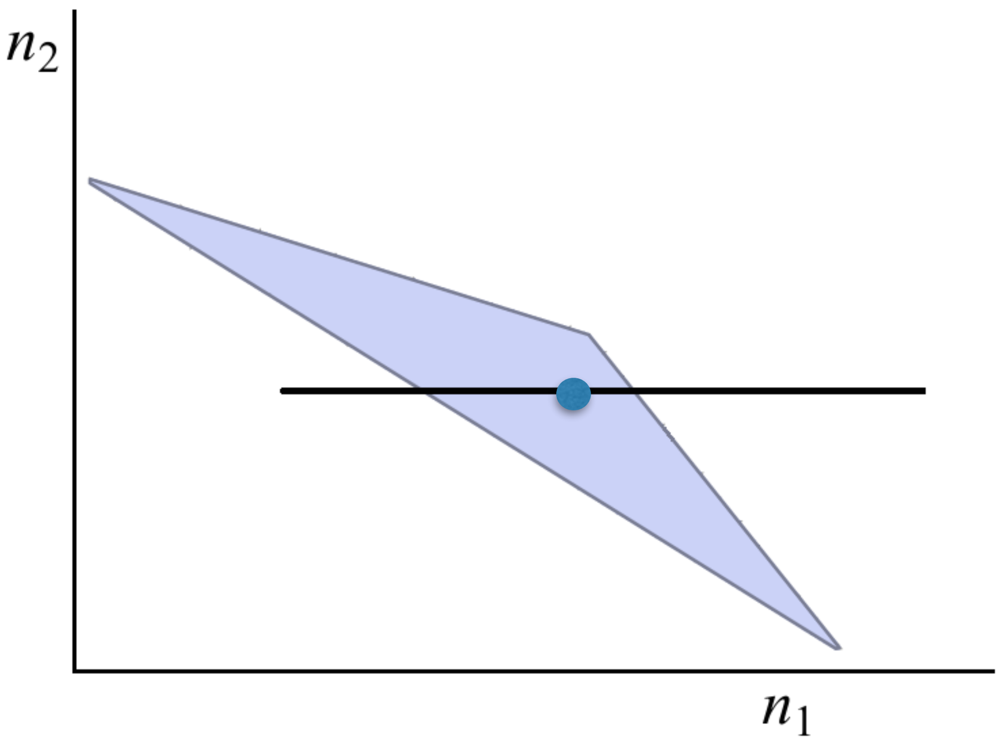
The second order change to be negative, all the eigenvalues of “M” has to be negative.

The "Constraints"-Towards Einstein Point

So we get the following constraints,

$$\boxed{\begin{aligned} n_1 + 2(d-1)n_2 &\geq 0 \\ 2d + 1 - 4(d+1)n_1 - 4(d^2-2)n_2 &\geq 0 \\ d + 2 - 4(d+1)n_1 - 4d(d^2-1)n_2 &\geq 0 \end{aligned}}$$

$$\boxed{\mathrm{Area_d} = \frac{\mathrm{d}^2}{8(\mathrm{d}+1)^2(\mathrm{d}-2)}}$$



In $d \to \infty$ limit the triangle collapses to the "Einstein" point.

$$\boxed{(n_1, n_2) = \left(\frac{1}{2}, -\frac{1}{8(d-1)}\right)}$$

(JHEP 1405 (2014) 029 Banerjee, AB, Kaviraj, Sen and Sinha)

Further constraints……

Now we write the metric as

$$\boxed{\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + az^4 T_{\mu\nu} + a^2 z^8 \Big((\frac{1}{2} + \delta n_1) T_{\mu\alpha} T^{\alpha}_{\nu} + (-\frac{1}{8(d-1)} + \delta n_2) \eta_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} \Big) \\ \delta n_1 &= n_1 - \frac{1}{2} \\ \delta n_2 &= n_2 + \frac{1}{8(d-1)} \end{aligned}}$$

It satisfies, $\boxed{R_{AB} - \frac{1}{2}(R + \frac{12}{L^2}) = \mathcal{T}_{AB}}$

As in case of relative entropy we can write, $\boxed{\mathcal{T} = V^T.M.V}$

V is a (d-1)(d+1)/2 component vector consists of independent components of $T_{\alpha\beta}$

We impose "Null Energy" condition $\boxed{T_{AB}\zeta^A\zeta^B \geq 0}$

Continue……

We get,

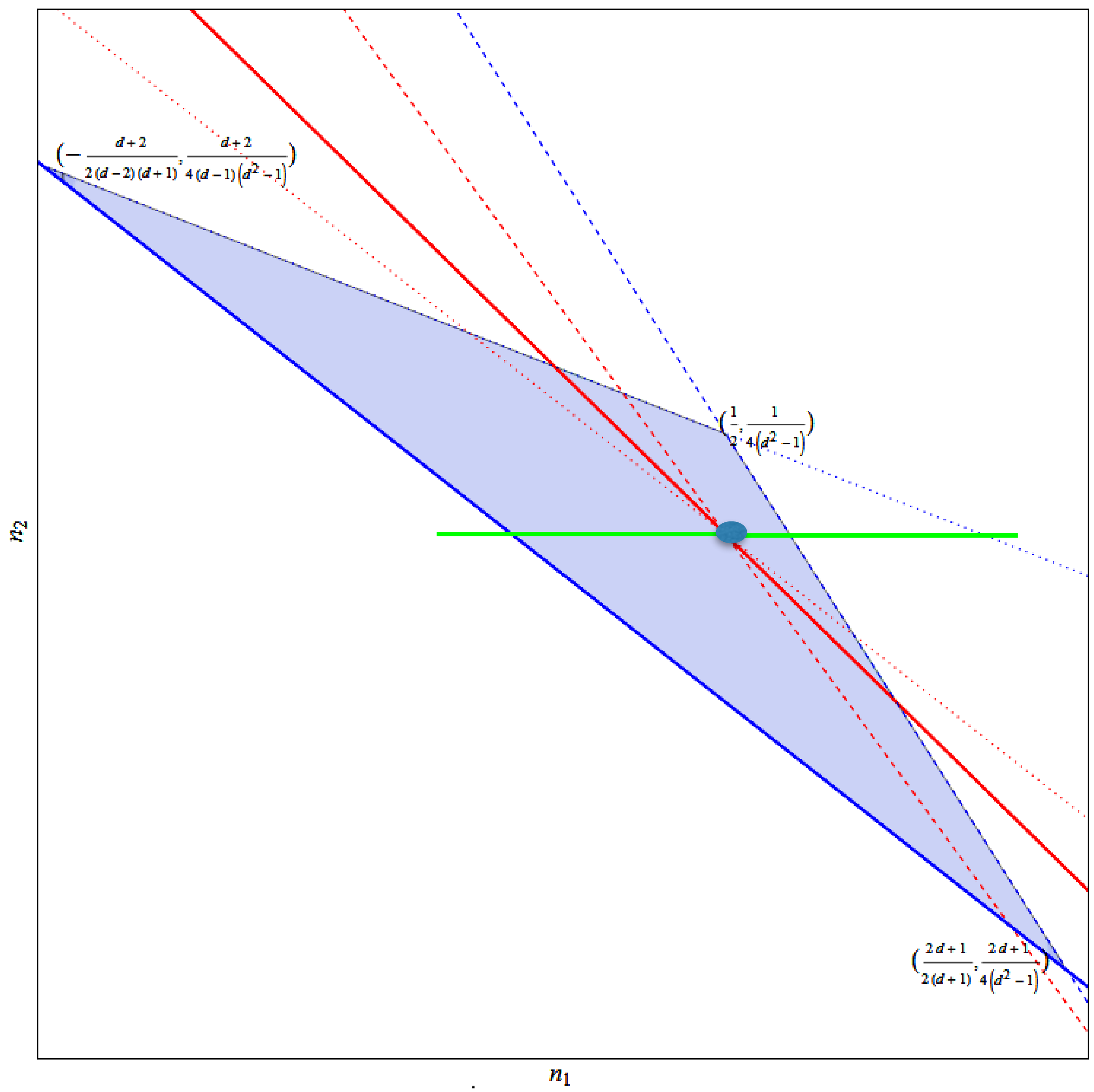
$$\boxed{\begin{aligned}(3d-2)\delta n_1 + 4d(d-1)\delta n_2 &\geq 0\\ (2d-1)\delta n_1 + 2d(d-1)\delta n_2 &\leq 0\\ \delta n_1 + 2(d-1)\delta n_2 &\leq 0\end{aligned}}$$

The solution is obvious,

$$\boxed{\delta n_1 = \delta n_2 = 0}$$

So the null energy condition by itself picks out only the “Einstein Point”

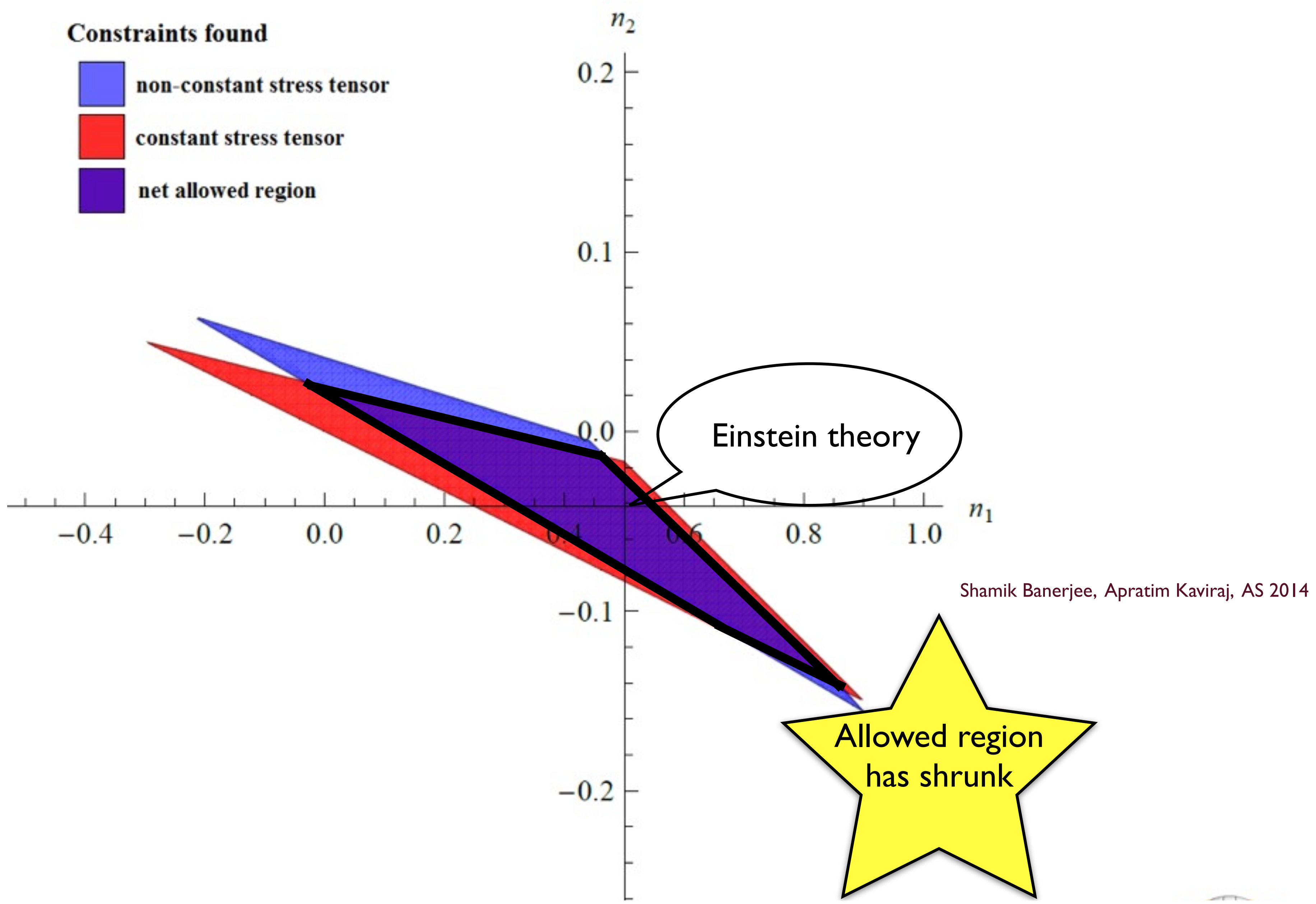
Combining Null energy and relative entropy constraints we get,



(JHEP 1405 (2014) 029 Banerjee, AB, Kaviraj, Sen and Sinha)

Perturbation by non-constant stress tensor

– Perturbing by non-constant stress tensor- but restricted to only two derivative acting on the stress tensors.



Higher derivative Gravity

- We will consider Gauss-Bonnet gravity in 5 dimensions.

$$\boxed{S = -\frac{1}{2\ell_p^3}\int d^5x\Big[R + \frac{12}{L^2} + \lambda L^2 (R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2)\Big]}$$

- Entanglement area functional for this case is the Jacobson-Myers entropy functional.

$$\boxed{S_{EE} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\left(1 + \lambda L^2 \mathcal{R}\right)}$$

(Jacobson-Myers'95, Hung,Myers, Smolkin '10)

- We then calculate the second variation for this case.

$$\boxed{\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + z^4 T_{\mu\nu} + z^8 (n_1 T_{\mu\alpha}T^{\alpha}_{\nu} + n_2 \eta_{\mu\nu} T_{\alpha\beta}T^{\alpha\beta}) \\ n_1 &= \frac{1}{2}\frac{1+2f_\infty\lambda}{1-2f_\infty\lambda}, \quad n_2 = -\frac{1}{24}\frac{1+6f_\infty\lambda}{1-2f_\infty\lambda} \end{aligned}}$$

$$\boxed{1 - f_\infty + f_\infty^2 \lambda = 0}$$

Gauss-Bonnet Gravity

– Finally we get the result for the second order change,

$$\boxed{\Delta^2 S = -\frac{8\pi^3 L_{AdS}^3 (1 - 2f_\infty \lambda)}{\ell_p^3} (C_1 T^2 + C_2 T_{ij}^2 + C_3 T_{i0}^2)}$$

$$C_1 \,, C_2 \,, C_3 > 0$$

– From this we get, $\boxed{\Delta^2 S \leq 0 \Rightarrow 1 - 2f_\infty \lambda \geq 0 \Rightarrow \lambda > \frac{1}{4}}$

–This is equivalent of positivity of two point function of stress tensor.

Extremal Surface Constraints

-Demanding the smoothness of the extremal surface inside the bulk space time we can get some bound on the Gauss-Bonnet coupling.

We start off with a ansatz for the extremal surface:

$$\boxed{f(z) = \sum_{i=0}^{\infty} c_i (z_h - z)^{\alpha+i}}$$

z_h is a point inside the bulk where the extremal surface closes off.

$$f'(z = z_h) \to \infty \Rightarrow 0 < \alpha < 1$$

and $\boxed{c_0 \in Real}$

-We then solve the extremal surface equation coming from minimizing Jacobson -Myers functional and find out c_0

-We do this for different types of entangling surfaces.

Continue…

-And we get ,

Type of Entangling Region	c_0	$c_0 \in Real$
Sphere	Independent of Gauss-Bonnet coupling	No constraints
Cylinder	$\boxed{\sqrt{\frac{2}{3}\sqrt{z_h(1+4f_\infty\lambda \pm \sqrt{1-10f_\infty\lambda+16f_\infty^2\lambda^2})}}}$	$\boxed{\lambda \leq \frac{7}{64}}$
Slab(Strip)	$\boxed{\sqrt{\frac{2}{3}\sqrt{z_h(1+4f_\infty\lambda)}}}$	$\boxed{-\frac{5}{16} \leq \lambda \leq \frac{1}{4}}$

Combining and in terms of central charge we get,

$$\boxed{\frac{1}{3} \leq \frac{a}{c} \leq \frac{5}{3}}$$

(JHEP 1405 (2014) 029 Banerjee, AB, Kaviraj, Sen and Sinha)

The lower bound matches with the bound for non-supersymmetric theories with free bosons coming from the positive energy constraints.

(Maldacena, Hoffman '08)

Conclusions

- We have shown that using the positivity of the relative entropy one can constrain gravity theories

- For higher curvature gravity one gets a bound on the coupling

- A non perturbative statement?

- Using smoothness of the entangling surface one can obtain non trivial bounds on the higher curvature couplings and hence on the central charges.

- Also one might get more non trivial bounds from smoothness analysis if one consider other entangling surfaces.

Lot more to explore !!!

Thank you