COUNTING WOBBLING DUAL-GIANTS

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What this talk is about ..

- Study classically the most general D3 branes in $AdS_5 \times S^5$ that preserve 2 out of 32 supercharges.

  Solve κ-symmetry equations. D-branes are classified by constraints on the pullback of space-time forms.

- Focus on a particular sub-class of solutions (wobbling dual-giants). Quantize the configuration space of these branes / count D3 branes. Compare with gauge theory.

Outline

- Background and motivation
- Classical analysis
  - Geometry of $AdS_5 \times S^5$
  - Kappa symmetry constraints and some solutions
- Quantization
  - Classical phase space and symplectic 2-form (examples)
  - Geometric quantization
- Results and open questions

Background and Motivation

Most discussions of branes in $AdS_5 \times S^5$ use either the embedding of the background in the ambient space $C^{1,2} \times C^3$ or use explicit embedding ansatz such as the round S^3.

Important to improve existing technology so as to generalize to other branes/backgrounds.

Duality between giant gravitons and dual-giant gravitons. This has been confirmed in the (J_1, J_2, J_3) sector by [BGLM, MS]. States in a 3d harmonic oscillator. Is this also true for (S_1, S_2, J_1) sector ?

Tests of AdS/CFT in the BPS sector.

Match counting of operators in gauge theory to counting D-brane states in the gravity theory.

Counting states that preserve 2 supercharges is potentially useful for accounting for the entropy of the Gutowski-Reall black hole.

Original motivation for project. A simple and complete characterization of 1/16-BPS objects conducive to quantization still missing.

Geometry

$$\Phi_0 = \cosh\rho e^{i\phi_0} \qquad Z_1 = \sin\alpha e^{\xi_1}$$

$$\Phi_1 = \sinh\rho\cos\theta e^{i\phi_1} \qquad Z_2 = \cos\alpha\sin\beta e^{i\xi_2}$$

$$\Phi_2 = \sinh\rho\sin\theta e^{i\phi_2} \qquad Z_3 = \cos\alpha\cos\beta e^{i\xi_3}$$

such that

$$|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 = 1 \qquad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$$

$$ds^2_{AdS_5} = -\cosh^2\rho d\phi_0^2 + d\rho^2 + \sinh^2\rho(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2)$$

$$ds^2_{S^5} = \sin^2\alpha\, d\xi_1^2 + d\alpha^2 + \cos^2\alpha(d\beta^2 + \sin^2\beta\, d\xi_2^2 + \cos^2\beta\, d\xi_3^2)$$

It is possible to choose a frame such that both spaces are written as Hopf fibrations. S^5 is given by a $U(1)$ fibration over CP^2. AdS_5 is given by a time-like Hopf fibration over $\widetilde{CP}^2$.

$$(\rho, \theta, \phi_{01}, \phi_{02}) \in \widetilde{CP}^2 \qquad (\alpha, \beta, \xi_{12}, \xi_{13}) \in CP^2$$

$$e^0 \qquad e^{1,2,3,4} \qquad\qquad e^{5,6,7,8} \qquad e^9$$

The kappa symmetry equations for the embedding of a D3-brane in this background is given by

$$\gamma_{\tau\sigma_1\sigma_2\sigma_3}\,\epsilon = i\sqrt{det\,h}\;\epsilon$$

where $\qquad \gamma_i = \partial_i X^\mu \Gamma_\mu$

Impose projections that preserve 2 supercharges. These were already discussed in the context of dual-giants by [MS].

The Killing spinor of the background is of the form

$$\epsilon = e^{-\frac{1}{2}(\Gamma_{79} - i\Gamma_5\tilde{\gamma})\alpha)} \dots e^{\frac{1}{2}\Gamma_{57}\xi_1} \dots e^{-\frac{1}{2}\phi_2\Gamma_{24}}\epsilon_0$$

ϵ_0 is a constant 32-component Weyl spinor that satisfies

$$\Gamma_{11}\epsilon_0 = -\epsilon_0$$

The projection conditions take the form

$$\Gamma_{09}\epsilon_0 = \Gamma_{13}\epsilon_0 = \Gamma_{24}\epsilon_0 = -\Gamma_{57}\epsilon_0 = \Gamma_{68}\epsilon_0 = -i\,\epsilon_0$$

The projection conditions leads to a particularly simple form of the Killing spinor

$$\epsilon = e^{i(\phi_0 + \phi_1 + \phi_2 + \xi_1 + \xi_2 + \xi_3)}\,\epsilon_0$$

If we define new combinations of vielbein

$$E^0 = e^0 + e^9 \quad E^1 = e^1 - i\, e^3 \quad E^2 = e^2 - i\, e^4 \quad E^5 = e^5 + i\, e^7 \quad E^6 = e^6 + i\, e^8$$

the (15) BPS equations can be packaged into the simple form

$$P[E^{ABCD}] = 0$$

$$P[E^{AB} \wedge (e^{09} + i(\tilde{\omega} - \omega)] = 0$$

One can check that all known supersymmetric embeddings of D3 branes satisfy these equations.

Volume form on the D3 brane: $P[e^{09} \wedge (\tilde{\omega} - \omega)]$

What about solutions ?

A D3 brane in ten dimensions is specified by 6 real equations.

There are two types of solutions studied in the literature.

Giants are point-like in AdS_5 and wrap 3-surfaces in the S^5. The volume form takes the form

$$dvol_3 = P[e^9 \wedge \omega]$$

Dual-giants are point-like in the S^5 and wrap 3-surfaces in AdS_5. The volume form is

$$dvol_3 = P[e^0 \wedge \tilde{\omega}]$$

There are four classes of $\frac{1}{8}$-BPS D3-branes:

Giants	Mikhailov giants	Wobbling giants	DUALITY
Dual-giants	Mandal-Suryanarayana dual- giants	Wobbling dual-giants (new)	
quantum no.	(J_1, J_2, J_3)	(S_1, S_2, J_1)	

In what follows, we will restrict ourselves to those branes that carry angular momenta (S_1, S_2, J_1).

These preserve 4 supercharges. Without loss of generality, we choose the brane to have angular momentum along the $\alpha = \dfrac{\pi}{2}$ circle. This automatically implies that

$$Z_2 = Z_3 = 0$$

These are already 2 complex constraints, so we require one more equation to completely specify the D3-brane:

$$F(\rho, \theta, \phi_i, \alpha, \beta, \xi_i) = 0$$

This leads to the differential constraint $dF = 0$. This can be written in terms of the vielbein introduced earlier.

$$\sum_i \left(a_i E^i + \bar{a}_i E^{\bar{i}} \right) = 0$$

$$Z_2 = Z_3 = 0 \qquad \Rightarrow \qquad P[E^5] = P[E^6] = 0$$

This, combined with the non-trivial differential constraint can be used to solve for three of the ten one-forms. Substituting this into the BPS equations lead to differential equations whose solution is given by

$$F(\Phi_0 Z_1, \Phi_1 Z_1, \Phi_2 Z_1) = 0$$

Here, Z_1 is a pure phase.

$$Z_1 = e^{-i\frac{t}{l}}$$

These are the "wobbling" dual-giants [AS]. Exchanging the roles played by $E^{\{1,2\}}$ and $E^{\{3,4\}}$ lead to Mikhailov's giant gravitons.

$$F(\Phi_0 Z_1, \Phi_0 Z_2, \Phi_0 Z_3) = 0 \qquad\qquad \Phi_0 = e^{i\frac{t}{l}}$$

Both of them are solutions to the same set of BPS equations.

Mini Summary

- Choice of frame (Hopf fibration)
- kappa-symmetry constraints imposed
- Found 2 simples classes of solutions
  - Wobbling dual-giants
  - Mikhailov giants
- There is a dual description for each of these D-branes found by [MS] that is much easier to quantize and count.

Classical Phase Space

Consider world-volume theory of a probe D3-brane that solves the BPS equations. These are given by

$$F(Y_i) = \sum_{i,j,k} c_{ijk} Y_0^i \, Y_1^j \, Y_2^k \qquad\qquad Y_a = Z_1 \Phi_a$$

The c's span the space of classical solutions to the BPS equations. This is *also* the classical phase space of the probe theory. The symplectic form on this phase space is given by [C-W, Z]

$$\theta = \int_\Sigma p_\mu \, \delta x^\mu \qquad\qquad \omega = \delta\theta \, .$$

Here, p's are the momenta (or charges) carried by the D-brane. The variations δx^μ are those that do *not* take one away from the space of classical solutions. They are 1-forms on phase space.

The idea is to write $\omega = f_{ijklmn}(c) \, \delta c_{ijk} \wedge \delta c_{lmn}$ after integration.

General expressions for the momenta can be easily obtained for a probe D-brane using

$$p_a = \frac{\partial L}{\partial e^a} \quad \Rightarrow \quad p_\mu = e^a_\mu \, p_a$$

Variations can be computed for specific examples. For instance, for the spherical dual-giant

$$f(Y_i) = Y_0 - c = 0$$

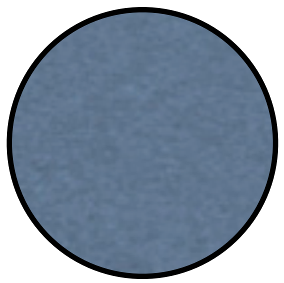
Embedding $\rho \;\; \theta \;\; \phi_i \;\; \alpha \;\; \beta \;\; \xi_i$

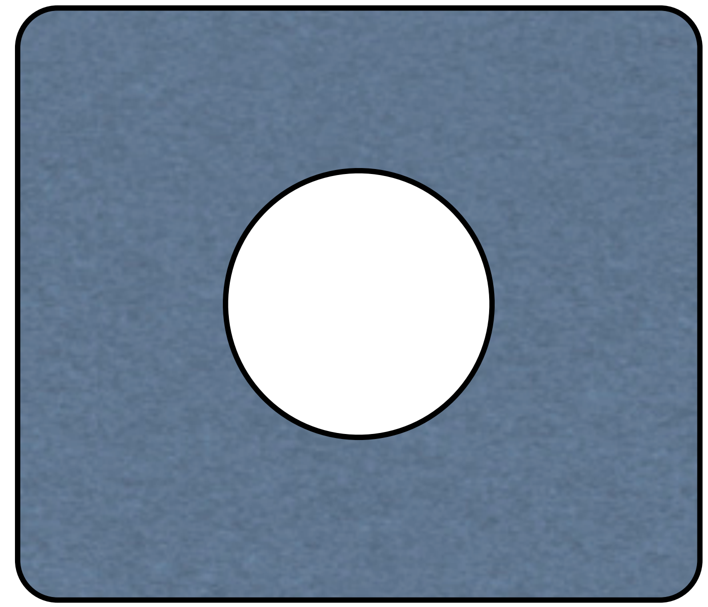
$\sigma_1 \;\; \sigma_i$

The one-form on phase space is given by $\theta = \int_\Sigma [p_\rho \delta\rho + p_{\xi_1} \delta\xi_1]$.

Result: The symplectic 2-form $\omega = -i\, N\, \delta\bar{c} \wedge \delta c \qquad |c| > 1$.

For a spherical giant, we get an identical result, except $|c| < 1$.

 $\longrightarrow CP^1$

 $\longrightarrow C^1 \longrightarrow \widetilde{CP}^1$

$$\zeta = \frac{r}{l}\,e^{i\phi} \qquad c = \sqrt{1 + \frac{r^2}{l^2}}e^{i\phi} \qquad \Rightarrow \omega = \delta\bar{c} \wedge \delta c = \delta\bar{\zeta} \wedge \delta\zeta$$

This maps the exterior of the unit disk to the complex plane. Now, let us start with the complex plane with the symplectic form equal to the Kahler form. Consider the further change of variables

$$\zeta = b\,\sqrt{\frac{1}{1 \pm |b|^2}} \qquad \Longrightarrow \qquad \begin{tikzcd} \omega_C \arrow[r] \arrow[d] & \omega_{CP^1} \\ \omega_{\widetilde{CP}^1} & \end{tikzcd}$$

The phase spaces are Kahler manifolds and their quantization is well known using the methods of geometric (holomorphic) quantization.

So far, we have discussed the phase space of the single giant and single dual-giant with quantum numbers $(0,0,J_1)$.

We will discuss two generalizations

$$Y_0 = c$$

$SU(1,2)$ rotation

$$\prod_{i=1}^{k}(Y_0 - c_i) = 0 \qquad\qquad c_0Y_0 + c_1Y_1 + c_2Y_2 = c$$

$\frac{1}{2}$-BPS

quantum numbers (S_1, S_2, J_1).

What is the classical phase space for these generalizations ?

For the 6 parameter single dual-giant with (S_1, S_2, J_1), we find that the classical phase space is just $\widetilde{CP}^3$.

For the multiple dual-giants which are 1/2-BPS, the configuration space is just the symmetric product space. However, we have yet to take into account the stringy exclusion principle.

For giants, there is an upper limit on the angular momenta

$$P_{\xi_1} \leq N$$

For dual-giants, there is an upper bound on their number $\leq N$

$$deg\left(f\left(Y_0\right)\right) \leq N$$

Configuration space is the phase space $\widetilde{CP}^N$.

Can we generalize this to $f(Y_i) = \sum_{i,j,k} c_{ijk} Y_0^i Y_1^j Y_2^k$?

Conjecture for implementing stringy exclusion principle

$$f(Y_i) = \sum_{i=0}^{N} \sum_{j,k} c_{ijk} Y_0^i Y_1^j Y_2^k$$

Repeating our analysis for this more general polynomial seems difficult for technical reasons. However there seems a natural guess for the classical phase space of the general polynomial.

$$\widetilde{CP}^{m-1}$$

where m is a regulator and is the number of terms in the polynomial. Of course, this agrees with the earlier results which have been derived carefully.

We cannot prove this conjecture, however, we can do some checks.

For Mikhailov giants, CP^{m-1} was proposed and proved by [BGLM] based on a discussion of homogeneous polynomials that define the giant gravitons.

Geometric (Kahler) Quantization

Symplectic 2-form is the Kahler form.

Choose (holomorphic) polarization $\bar{D}\phi = (\bar{\partial} + \theta_{\bar{z}})\phi = 0$.

Choose adapted Kahler potential for the Kahler-covariant derivative.

$$\theta = -i\,\partial K\,dz$$

Wave-functions are automatically holomorphic functions.

Functions on phase space get mapped to operators in the quantum theory: $f \longrightarrow i\,\partial_i f\,\omega^{ij} D_j + f$ and these act on the wave-functions in the Hilbert space.

Partition functions can be defined to be

$$Z = Tr_H\, e^{-\beta_i J_i}$$

where the J_i are the operator representations of the angular momenta.

Partition Function

The angular momenta of a given dual-giant configuration is given by

$$\frac{J_1}{N} = f(b) \sum_{n_0,n_1,n_2} n_0 |b_{n_0,n_1,n_2}|^2 \qquad \frac{S_i}{N} = f(b) \sum_{n_0,n_1,n_2} n_i |b_{n_0,n_1,n_2}|^2$$

where the b's are related to the c's in the defining equation. These are now functions on phase space and can be written as operators in the quantum theory.

The wave-functions are holomorphic functions of the b's. Choosing a basis of monomials

$$\phi(b) = \prod_{n_0,n_1,n_2} (b_{n_0,n_1,n_2})^{p_{n_0,n_1,n_2}}$$

$$J_1 = \sum_{n_0,n_1,n_2} n_0\, p_{n_0,n_1,n_2} \qquad S_i = \sum_{n_0,n_1,n_2} n_i\, p_{n_0,n_1,n_2}$$

$$0 < n_0 \leq N$$

When the regulator is taken to infinity, the Hilbert space coincides with that of an arbitrary number of bosons in a 3-dim harmonic oscillator such that level # of one of them is $n_0 \leq N$.

In this Hilbert space, the partition function defined earlier takes the simple form

$$Z = \prod_{n_0=1}^{N} \prod_{n_1,n_2=0}^{\infty} \frac{1}{1 - q_0^{n_0} q_1^{n_1} q_2^{n_2}}$$

It matches the one obtained using giants with the same quantum numbers [MS].

Immediate open questions

Include EM fields on the brane and repeat quantization methods. Does not seem very straightforward.

Find simple way to characterize the 1/16-BPS D-brane configurations.

Count 1/16-BPS D-branes from the gravity side to account for entropy of Gutowski-Reall black hole.

Generalize to Sasaki-Einstein spaces.