Topics in Fluid-Gravity Correspondence

Spenta R. Wadia

Tata Institute of Fundamental Research

A : Navier-Stokes from Gravity

1. Horizons and Thermodynamics

N D3 branes

Supergravity :

$$ds^2 = \frac{r^2}{R^2}\left(-f(r)\,dt^2 + d\vec{x}^{\,2}\right) + \frac{R^2}{r^2 f(r)}\,dr^2$$

$$+ R^2 d\Omega_5^2$$

(Dilaton = constant)

$$f(r) = 1 - r_0^4/r^4$$

horizon at $r = r_0$

Hawking temp : $T = \dfrac{r_0}{\pi R^2}$

Bekenstein-Hawking entropy :

$$s = \frac{S}{V_3} = \frac{\pi^3 r_0^3 R^2}{4 G_{10}} = \frac{\pi^6 R^8 T^3}{4 G_{10}}$$

N D3 branes $\Longleftrightarrow$ $SU(N)$, $\mathcal{N}=4$ SYM.

AdS/CFT $\Longrightarrow$

$$R^4 = \frac{\sqrt{8\pi G_{10}}}{2\pi^{5/2}} N$$

$$\Longrightarrow \quad s = \frac{\pi^2}{2} N^2 T^3$$

Conformal invariance $\Rightarrow$

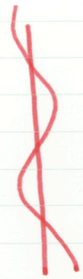
$$\varepsilon = 3P$$

$$\varepsilon + P = Ts$$

$$P = \frac{1}{4} Ts = \frac{\pi^2}{8} N^2 T^4$$

$$\frac{s}{\varepsilon} \sim \frac{1}{T}$$

2. Hydrodynamics + Displaced Horizon



$r = r_0$

$$r_0 \Rightarrow \delta r_0(\vec{x}, t) + r_0$$

$$T_0 \Rightarrow T_0 + \delta T_0(\vec{x}, t)$$

$$\frac{\partial}{\partial x^\mu} \frac{\delta T_0}{T_0} \sim \frac{1}{L T_0} \ll 1$$

Hydrodynamic description valid for largest time and length scales.

$$u^\mu(\vec{x}, t) \quad , \quad u^\mu u_\mu = -1$$

$$T(\vec{x}, t)$$

4 independent variables

Bhattacharya et. al (BHMR):

$G_{10} \sim N^{-2} \to 0$

$\therefore$ use "infalling" Edington-Finklestein coordinates

$$ds^2 = 2\,dt\,dv - r^2 f(r)\,dt^2 + r^2 d\vec{x}^{\,2}$$

$\Big\downarrow$ Boosted metric well behaved solution

$$ds^2 = -2u_\mu dx^\mu dv - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu$$

$$P_{\mu\nu} = (\eta_{\mu\nu} + u_\mu u_\nu).$$

(u_μ, b) 4 parameters of $\dfrac{SO(3,1)}{SO(3)} \times D$.

Slowly varying $u^\mu(\bar{x},t)$, $b(\bar{x},t)$

(Nambu- Goldstone modes)

Solve Einstein eqn. in AdS_5

provided : (Relativistic Hydro-dynamics).

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\varepsilon + P)\, u^\mu u^\nu + P\eta^{\mu\nu}$$

$$-2\eta \left(P^{\mu\alpha} P^{\nu\beta} \left[\partial_\alpha u_\beta + \partial_\beta u_\alpha \right] \frac{1}{2} \right.$$

$$\left. - \frac{1}{3} P^{\mu\nu} \partial_\alpha u^\alpha \right)$$

$$\varepsilon = 3P\ , \quad P = \frac{\pi^2}{8} N^2 T^4\ , \quad \eta = \frac{\pi}{8} N^2 T^3$$

Since $s = \frac{\pi^2}{2} N^2 T^3$

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

(Policastro-Son-Starinets)

Non-Relativistic Limit

(Bhattacharya, Minwalla, SRW)
0810.1545

Speed of sound in fluid:

$$v_s^2 = \frac{\partial P}{\partial \mathcal{E}} = \frac{1}{3}$$

$$\frac{|\vec{V}|}{v_s} \ll 1 \quad , \quad \vec{V}(\vec{x},t)$$

$$u^0 = \frac{1}{\sqrt{1-\vec{V}^2}} \quad , \quad u^i = \frac{V^i}{\sqrt{1-\vec{V}^2}}$$

Scaling limit to project out sound mode:

$$\vec{V}(\vec{x},t) = \epsilon\, \vec{v}(\epsilon \vec{x}, \epsilon^2 t)$$

$$P(\vec{x},t) = P_0 + \epsilon^2 p(\epsilon\vec{x}, \epsilon^2 t)\rho_e$$

$$\rho_e = 4P_0$$

$\epsilon \to 0$ in relativistic eqn: $\Rightarrow$

$$\partial_\mu T^{\mu 0} = \epsilon^2 \left[4P_0 \, \partial_i v^i \right] + O(\epsilon^4)$$

$$\Rightarrow \quad \partial_i v^i = 0$$

$$\partial_\mu T^{\mu i} = \epsilon^3 4P_0 \left[\partial_t v^i + v^j \partial_j v^i - \nu \partial^2 v^i + \partial^i p \right] + O(\epsilon^5)$$

$$\nu = \frac{\eta}{\varepsilon_0 + P_0} = \frac{\eta}{4P_0}$$

$$\Rightarrow \quad \partial_t v^i + v^j \partial_j v^i = \nu \partial^2 v^i - \partial^i p$$

Navier-Stokes eqns. for incompressible fluid.

Basic idea :

(i) $E = \sqrt{m^2 + \vec{p}^{\,2}}$

$$E = m + \epsilon^2 e \quad , \quad \vec{P} = \epsilon \vec{p}$$

$$\epsilon^2 e = \epsilon^2 \frac{\vec{p}^{\,2}}{2m} + \# \epsilon^4 \frac{\vec{p}^{\,4}}{m^4} + \cdots$$

$$\therefore \quad \epsilon \to 0 \quad \Rightarrow \quad e = \frac{\vec{p}^{\,2}}{2m}$$

(ii) Perfect fluids $(\eta = 0)$

$$T^{\mu\nu} = 4P\, u^\mu u^\nu + P\, \eta^{\mu\nu}$$

$$T^{0i} = \frac{4P\, V^i}{1 - \vec{V}^2} \ , \quad T^{00} = \frac{4P}{1 - \vec{V}^2} - P$$

$$V^i(\vec{x},t) = \epsilon\, v^i(\epsilon\vec{x}, \epsilon^2 t)$$

$$P(\vec{x},t) = \epsilon^2 4P_0\, p(\epsilon\vec{x}, \epsilon^2 t) + P_0$$

$$\partial_0 T^{00} \sim \partial_0\left[\frac{4P}{1-\vec{V}^2} - P\right] \sim o(\epsilon^4)$$

$$\partial_i T^{0i} \sim \partial_i\left(\frac{4PV^i}{1-\vec{V}^2}\right)$$

$$\sim \partial_i\left\{\frac{4\,[P_0 + \epsilon^2 p(\epsilon\vec{x}, \epsilon^2 t)P_0]\,[\epsilon\, v^i(\epsilon\vec{x}, \epsilon^2 t)]}{1 - \epsilon^2 \vec{v}^{\,2}(\epsilon\vec{x}, \epsilon^2 t)}\right\}$$

$$\sim \quad 4P_0\epsilon^2\partial_i v^i + o(\epsilon^4)$$

$$\Longrightarrow \quad \partial_i v^i = 0$$

Similarly :

$$\partial_0 T^{0i} = \partial_0 \left(\frac{4P V^i}{1-\vec{V}^2} \right)$$

$$\approx \epsilon^3 \, 4P_0 \, \partial_0 v^i \qquad + O(\epsilon^5)$$

$$\partial_j T^{ji} = \partial_j \left(\frac{4P V^j V^i}{1-\vec{V}^2} \right) + \partial^i P$$

$$\approx \epsilon^3 \left[4P_0 \ \partial_j (v^j v^i) + 4P_0 \, \partial^i p \right] + O(\epsilon^5)$$

$$\Rightarrow \quad \partial_0 v^i + v^j \partial_j v^i = - \partial^i p$$

Driven fluid in curved space

$$G_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu} \qquad |H_{\mu\nu}| << |g_{\mu\nu}|$$

$$g_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + g_{ij}\, dx^i dx^j$$

$$\tilde{u}^\mu = u^\mu + \delta u^\mu$$

$$u^\mu = \frac{1}{\sqrt{1 - g_{ij} V^i V^j}} (1, \vec{V})$$

$$\delta u^\mu = -\frac{u^\mu}{2} u^\alpha u^\beta H_{\alpha\beta}$$

$$H_{00} = \epsilon^2 h_{00}(\epsilon \vec{x}, \epsilon^2 t)$$

$$H_{0i} = \epsilon A_i(\epsilon \vec{x}, \epsilon^2 t)$$

$$H_{ij} = \epsilon^2 h_{ij}(\epsilon \vec{x}, \epsilon^2 t)$$

$$V^i = \epsilon v^i(\epsilon \vec{x}, \epsilon^2 t)$$

$$P = P_0 + \epsilon^2 p(\epsilon \vec{x}, \epsilon^2 t) \rho_e$$

small amplitude

asymetric scaling limit

$\epsilon \to 0$

$$\nabla_\mu T^{\mu 0} = \epsilon^2 \rho_e \nabla_i v^i + o(\epsilon^4) \Rightarrow \nabla_i v^i = 0$$

$$\nabla_\mu T^{\mu i} = \epsilon^3 \Big[\rho_e \nabla^i p + \rho_e \nabla_\mu (v^i v^\mu)$$

$$-2\eta \nabla_j \left(\frac{\nabla^j v^i + \nabla^i v^j}{2} - \frac{g^{ij}}{3} \nabla\cdot\vec{v} \right)$$

$$-\zeta \nabla_i \vec{\nabla}\cdot\vec{v} - f^i \Big] + o(\epsilon^5)$$

$$v^\mu = (1, \vec{v}), \quad \nabla_i v^i = 0, \quad \Rightarrow$$

$$\nabla^i p + \partial_0 v^i + \vec{v}\cdot\vec{\nabla} v^i - \nu (\nabla^2 v^i + R^i_j v^j)$$

$$= \frac{\partial^i h_{00}}{2} - \partial_0 A^i + F^i_j v^j$$

$$F_{ij} = \partial_i A_j - \partial_j A_i, \quad \nu = \eta/\rho_e .$$

Use coulomb gauge : $A_i = a_i + \nabla_i \chi$

$\nabla_i a^i = 0$

Finally:

$$\nabla_i v^i = 0$$

$$\partial_t v_i + \bar{v}\cdot\bar{\nabla}\, v_i = \nu\left(\nabla^2 v_i + R_{ij} v^j\right) - \nabla_i p_e + \mathcal{F}_i^{ext}$$

$$p_e = p - \frac{h_{00}}{2} + \dot{\chi}$$

$$\mathcal{F}_i^{ext} = -\partial_t a^i - v^j f_{ji}$$

(a general driving force).

Eqn. for the pressure:

$$\nabla^2 p_e = -\nabla_i v^j \nabla_j v^i - v^i v^j R_{ij} + \nabla_i\left[\left(-\nu R^i_j + f^i_j\right) v^j\right]$$

These equations are a univeral limit of fluid dynamics. They are invariant under a 14 parameter symmetry algebra.

Symmetries

Symmetries of relativistic fluid dynamics:

Fluid dynamics gives rise to a representation of the conformal group $SO(4,2)$, the global symmetry of the underlying $\mathcal{N}=4$ gauge theory:

6	Lorentz generators	$M_{\mu\nu}$
4	momenta (space-time)	P_μ
4	special conformal gen.	K_μ
1	dilatation	
$\overline{15}$	generators.	

(we restrict to flat space-time on the boundary and set $\mathcal{F}_i = 0$)

In fluid dynamics it is the velocity $u^\mu(x)$ and the temperature $T(x)$ which are the carriers of the representation:

$$[M_{\mu\nu}, \phi_\alpha(0)] = (-M^R_{\mu\nu})_\alpha^{\ \beta}\, \phi_\beta(0)$$

$$[D, \phi_\alpha(0)] = \epsilon_0\, \phi_\alpha(0)$$

(e.g. see Minwalla hep-th/9712074)

$$\phi_\alpha \equiv \{u^\mu, T\}$$

$M^R_{\mu\nu}$ is a Lorentz trans. in the R rep.

ϵ_0 is the scaling dimension.

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) - M^R_{\mu\nu}$$

$$P_\mu = i\partial_\mu$$

$$K_\mu = i[-2x_\mu (x\cdot\partial) + x^2 \partial_\mu] - 2x^\alpha M^R_{\alpha\mu} + 2x_\mu \epsilon_0$$

$$D = i(-x\cdot\partial) + \epsilon_0$$

Field :	ϵ_0	
$T(x)$	1	$\left(x_0^E \simeq x_0^E + \frac{2\pi}{T} \right)$
$u^\mu(x)$	0	$\left(u^\mu u_\mu = -1 \right)$
$T^{\mu\nu}(x)$	4	

Special conformal trans: $[c^\mu k_\mu, T]$

$$\delta T = -\delta x^\nu \partial_\nu T + 2c\cdot x T$$

$$\delta u^\mu = -\delta x^\nu \partial_\nu u^\mu - 2[x^\mu c^\nu - x^\nu c^\mu] u_\nu$$

$$\delta x^\mu = -2c\cdot x\, x^\mu + x^2 c^\mu$$

$$\delta T_0 = 2(c^0 x_0 - c^i x_i) T_0$$

Scaling of c^0, c^i for descent to NR limit:

$$\boxed{c^0 = \epsilon^4 \tilde{c}^0} \quad , \quad \boxed{c^i = \epsilon^3 \tilde{c}^i}$$

$$\delta T_0 = \epsilon^2 (\tilde{c}_0\, \epsilon^2 x_0 - \tilde{c}^i \epsilon x_i) T_0$$

$$\delta T_0 = \epsilon^2\, \delta t\, (\epsilon^2 x_0, \epsilon \vec{x})$$

Correct scaling of temperature.

Special conformal transformation as $\epsilon \to 0$ descends to

$$\delta t = 0 \qquad \text{(speed of sound} = \infty)$$
$$\delta x^i = -t^2 c^i$$
$$\delta v^i = -2c^i t + t^2 c_j \partial_j v^i$$
$$\delta T = 2(-c^0 t + c^i x^i)T + t^2 \partial_j T c_j$$

(c_0 + c_i are scaled parameters).

Only c_i acts non-trivialy on v^i + x^i

The above transformations are a symmetry of the N-S equations.

Full conformal symmetry group of incompressible fluid dynamics:

$$D\, v^j = (-2t\,\partial_t - x^m \partial_m - 1)\, v^j$$

$$K_i\, v^j = -2t\,\delta_{ij} + t^2 \partial_i v^j$$

$$B_i\, v^j = \delta_{ij} - t\,\partial_i v^j$$

$$H\, v^j = -\partial_t v^j$$

$$P_i\, v^j = -\partial_i v^j$$

$$M_{ik}\, v^j = \delta_{ij} v^k - \delta_{kj} v^i - (x^k \partial_i - x^i \partial_k) v^j$$

$D \equiv$ dilatation

$K_i \equiv$ special conformal trans.

$B_i \equiv$ Gallilian boosts

$H \equiv$ time trans.

$P_i \equiv$ space trans., $M_{ij} \equiv$ space rot.

~~D~~ ~~≠~~

Full Lie algebra:

$[D, K_i] = -3K_i$, $[D, B_i] = -B_i$

$[D, H] = 2H$, $[D, P_i] = P_i$

$[D, M_{ij}] = 0$

$[M_{ij}, P_k] = -\delta_{ik} P_j + \delta_{jk} P_i$

$[M_{ij}, K_k] = -\delta_{ik} K_j + \delta_{jk} K_i$

$[M_{ij}, B_k] = -\delta_{ik} B_j + \delta_{jk} B_i$

$[M_{ij}, H] = 0$

$[K_i, P_j] = 0$, $[K_i, B_j] = 0$, $[K_i, H] = -2B_i$

$[B_i, H] = P_i$

$[M_{ij}, M_{kl}]$ ~ standard rot. algebra

(not Schrodinger algebra)

Use of symmetry algebra ?

Kolmogorov scaling ....

Non-conformal fluid dynamics in 1+1 dim. Justin David, M. Mahato, SRW

B: Non-conformal fluid dynamics of D1 brane :

Non-trivial dilaton (unlike the case of D3 branes).

N D1 branes $\Longrightarrow$ $(T \neq 0)$.

SU(N) gauge theory in 1+1 dim. with 16 supersymmetries, at finite temp. T.

dual gravity description in 2-diff. regimes :

$\sqrt{\lambda}\, N^{-2/3} \ll T \ll \sqrt{\lambda}$ non-extremal D1

$\sqrt{\lambda}\, N^{-1} \ll T \ll \sqrt{\lambda}\, N^{-2/3}$ F1

$\lambda = g_{YM}^2 N$

Fluid dynamics:

$$T^{\mu\nu} = (\epsilon + P)\, u^\mu u^\nu + P\, \eta^{\mu\nu} - \xi\, P^{\mu\nu}\, \partial_\lambda u^\lambda$$

(no shear term in 1+1 dim.).

$\xi > 0$ $\qquad \eta_{\mu\nu} T^{\mu\nu} \neq 0$.

If $\eta_{\mu\nu} T^{\mu\nu} = 0$, no dissipative dynamics.

Equation of motion for gauge invariant quasi-normal mode $\Rightarrow$

$$\boxed{\omega = \frac{q}{\sqrt{2}} - \frac{i}{8\pi T} q^2}$$

$$\omega = v_s q - i \frac{\xi}{2(\epsilon + P)} q^2$$

$$\Rightarrow \quad v_s = \frac{1}{\sqrt{2}} \quad + \quad \epsilon = 2P$$

$$\frac{S}{L} \equiv s = N^2 T^2 \sqrt{\frac{2\pi l_s^2}{g_s}}$$

$$\epsilon + P = Ts = N^2 T^3 \sqrt{\frac{2\pi l_s^2}{g_s}}$$

$$\Rightarrow \boxed{\xi = \frac{N^2}{4\pi} T^2 \sqrt{\frac{2\pi l_s^2}{g_s}}}$$

$$\Rightarrow \boxed{\frac{\xi}{s} = \frac{1}{4\pi}}$$

Non-relativistic limit

The standard scaling

$$V' = \epsilon v'(\epsilon \vec{x}, \epsilon^2 t)$$

$$P = \epsilon^2 p(\epsilon \vec{x}, \epsilon^2 t)(P_0 + \mathcal{E}_0) + P_0$$

leads to a trivial equation $\partial_x v' = 0$

Since the sound mode is projected out the dynamics is trivial.

Question:

Is there a model for the Burgers' equation:

$$\partial_t v' + v' \partial_x v' = -\partial_x p + \nu \partial_x^2 v'$$