The Non-commutative S matrix

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9 Dec 2008
(work in progress)

CONTEMPORARY HISTORY

- In the past few years, S-matrix techniques have seen a revival. (Bern et al., Britto et al., Arkani-Hamed et al.)
- In gauge theories, one can use the analytic structure of the S-matrix, to bootstrap from the three pt amplitude to the complete tree level and one loop S-matrix
- Even in intermediate steps, only on shell quantities are used. (goodbye ghosts!)
- This works for $N = 4$ SYM, $N = 8$ SUGRA but also for pure gauge theories including QCD.

MOTIVATION

- WHY USE THESE METHODS?
- Primary driving force is concrete QCD calculations for the LHC! These techniques are much more efficient than Feynman diagrams.
- This is because gauge theory amplitudes have more structure than we expect from the Lagrangian. eg. The Parke-Taylor Formula:

$$|M^{--++\cdots}|^2 \propto \sum_{\text{permutations}} \frac{(p_1 \cdot p_2)^4}{(p_1 \cdot p_2)(p_2 \cdot p_3)\ldots(p_n \cdot p_1)}$$

- From Feynman diagrams, this is a miraculous simplification! Natural in the S-matrix approach.

MOTIVATION II

- Can we reformulate gauge theory using S-matrix techniques?
- One clear advantage is that it reveals new structure in gauge theory amplitudes.
- Second, notice that this formalism:
  1. Keeps unitarity manifest at every level.
  2. Locality is not manifest (contrast with Lagrangian!)
- Perhaps this also teaches us to move away from locality? (Arkani-Hamed, Cachazo, Kaplan '08).
- In the Lagrangian, locality is manifest. Non-local terms create problems with unitarity. Here, the situation is reversed: unitarity is manifest; locality is not

NON-LOCAL S-MATRIX

- Caution required because locality is closely related to the analytic properties of the S-matrix.
- A non-local theory could interact over a distance l through a $\exp\{\vec{l} \cdot \vec{\partial}\}$ interaction.
- In the S-matrix this becomes $e^{i\vec{l}\cdot\vec{k}}$ – ESSENTIAL SINGULARITY in the complex k plane.(In string theory, for example, the hard scattering amplitude goes as $\sim e^{-\alpha' s}$).
- The S-matrix approach relies strongly on the detailed analytic properties of amplitudes (at tree level, they are holomorphic, at one loop you get specific branch cuts etc.). Are these essential singularities tractable?

OBJECTIVE OF THIS TALK

- Study a simple example of non-local theories – non-commutative theories. Object is to reconstruct one loop S-matrix using the analytic properties of the S-matrix.
- This works!
- However, some delicate conspiracies (involving the structure of the non-commutative S-matrix) are required to make this work.

TECHNICALITIES: IR DIVERGENCES

- Even in ordinary gauge theories, gluon S-matrix elements have IR divergences.
- We will work within dimensional regularization.
- Non-commutative theories have worse IR properties as a result of UV-IR mixing; we will keep kinematic invariants large enough that the one loop approximation is valid.

Outline

NON-COMMUTATIVE THEORIES

- Non-commutative theories can be obtained from ordinary theories by replacing the pointwise product of fields with the * product

$$(f * g)(x) = \lim_{y \to x} e^{i\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} f(x) g(y)$$

- Quadratic part of the action is unchanged.
- Interaction vertices develop non-local terms. (will review perturbation theory as we go along)

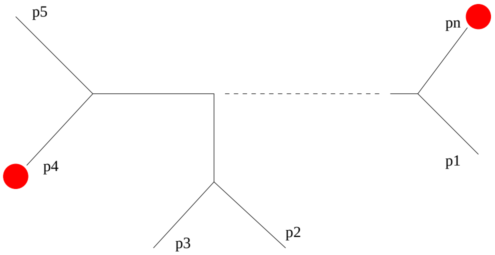
WHY STUDY NC THEORIES?

- Non-commutative theories give us an example of a quantum field theory that is non-local, unitary, relativistic and perturbative
- This is perfect for our purpose of studying how S-matrix techniques could apply to non-local quantum field theories.
- It is not entirely clear if the generic non-commutative gauge theory is well defined (UV-IR mixing may spoil renormalizability and stability of the vacuum). If this is bothersome, think exclusively of $N = 4$ non-commutative SYM theory.

BCFW RELATIONS

- Consider a n-point gluon amplitude.

Figure: BCFW EXTENSION



- Extend *any* two momenta on shell

$$p_4 \to p_4 + qz; \quad p_n \to p_n - qz$$
$$q^2 = q \cdot p_4 = q \cdot p_n = 0$$

- For each p, one of two gauge boson polarization vectors also grows as $O(z)$.

LARGE Z BEHAVIOUR

- How do these amplitudes behave at large z?
- Naive guess:
  - Independent of z for scalars
  - grow fast for gauge theories $O(z^3)$.
  - grow even faster for gravity $O(z^6)$
- Correct Answer: For 3 out of 4 possible polarizations:
  - $M \sim O(1/z)$ for gauge theories
  - $M \sim O(1/z^2)$ for gravity
- For 1 polarization, naive expectation is justified.
- I strongly recommend doing this calculation for $2 \rightarrow 2$ scattering in the $SU(2)$ theory. Its fun ☺

RECURSION RELATIONS

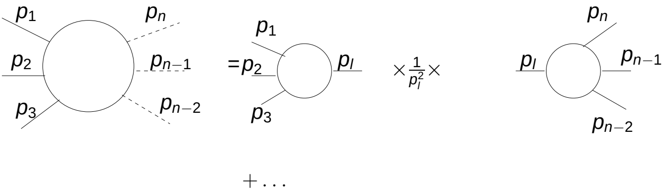
- If a holomorphic function dies off at infty, we can reconstruct it from its poles.
- Poles in the amplitude occur when an internal line goes on shell
- So,

$$M(z) \sim \sum_{\text{partitions}} M_L(p_1, \ldots p_j, P_L = \sum_{i=1}^{j} p_i) \frac{1}{P_L^2(z)} M_R(-P_L(z), p_{j+1} \ldots p_n)$$

- These are the BCFW recursion relations. (Britto et al. '05) They allow us to reconstruct all tree amplitudes from a knowledge of the 3 pt amplitude!

SCHEMATIC BCFW

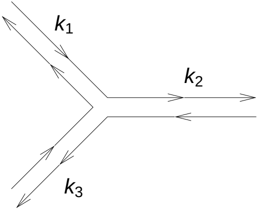
Figure: Recursion Relations



TREE-LEVEL NC PERTURBATION THEORY

- NC perturbation theory is conveniently formulated in double-line notation. Each vertex gets a phase factor (Minwalla et al. '99)

Figure: Non-commutative Vertex



$\sim \exp(i \sum_{i<j} k_i \times k_j)$

$$a \times b = a_\mu \theta^{\mu\nu} b_\nu$$

- In a planar graph the phase-factors from internal lines cancel. Get an overall phase factor from the external legs.
- How can the recursion relations work with these phase factors all over the place?

BCFW FOR NC THEORIES I

- Sum graphs with the same cyclic ordering of external momenta. Call this subamplitude the color-ordered amplitude

$$M^{\mathrm{nc}}_{\mathrm{col-ord}}(k_1 \ldots k_n) = M^{\mathrm{ordinary}}_{\mathrm{col-ord}}(k_1 \ldots k_n) \exp(\sum_{i<j} k_i \times k_j),$$

- Color-ordered amplitudes also satisfy recursion relations in ordinary gauge theories! In fact, this is what the BCFW recursion relations are most frequently used for.

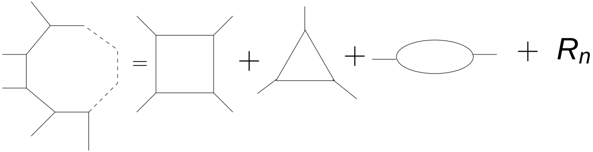
BCFW FOR NC THEORIES II

- So, the generalization of the BCFW relations is simple.
  - Calculate each color-ordered subamplitude in the ordinary theory using the recursion relations.
  - Multiply the result with a phase factor to get the non-commutative answer
- This is very simple but it is a remarkable coincidence that the same object – the color ordered subamplitude – plays an important role both in the recursion relations and non-commutative theories!

ONE LOOP IN ORDINARY THEORIES

- ANY one loop amplitude in an ordinary quantum field theory can be written as a sum of scalar boxes triangles and bubbles with rational coefficients and a possible rational remainder.

Figure: ONE LOOP DECOMPOSITION



- This is surprising. A 1-loop diagram may have 1000 propagators in the denominator. How can we reduce it to something that has at most 4 factors in the denominator

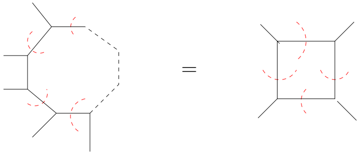
ANALYTIC PROPERTIES OF ONE LOOP AMPLITUDES

- The reason this works is that the analytic structure of the one loop amplitude is tightly constrained.
  - It could have branch cut discontinuities: the 2-cut gives us the discontinuity across the branch cut
  - The discontinuity may itself have a discontinuity: cutting three lines gives us the discontinuity of the discontinuity.
  - In 4 dimensions, not more than 4 lines can go on shell.
- So a box plus triangle plus bubble can reproduce the most general branch cut singularities that can appear at one loop.
- A possible rational remainder is accounted for explicitly

BOX COEFFICIENTS

- Figure out the coefficients of the decomposition using cuts!
- eg. make 4-cut. Only one term from the RHS contributes (Britto et al. '04)

Figure: FOUR CUTS



- a 4-cut localizes the loop momentum up to an ambiguity of two

$$p^2 = 0 = (p + q_1)^2 = (p + q_2)^2 = (p + q_3)^2$$

- Add contribution from both solutions and set the sum to the coefficient of box.

TRIANGLES AND BUBBLES

- Make 3 cuts – solutions are parameterized by one complex number z. Roughly speaking this z can be identified with the BCFW z.
- The 3-cut is given by the product of three tree amplitudes.
- Coefficient of triangle is given by the pole at infinite z of this product. (Forde '08, Arkani-Hamed et al. '08)
- Thus, the coefficients of the triangle is related to the behavior of the product of three tree amplitudes at large "BCFW" deformations.
- A very similar method works for bubbles.
- Used to prove that $N = 4$ SYM has no triangles or bubbles ("no triangle hypothesis").

RATIONAL TERMS

- Rational terms come from UV divergences of the theory.
- At one-loop, they are absent in both $N = 4$ SYM and $N = 8$ SUGRA.
- However, they do appear in QCD where they can be obtained with significant effort. (Bern et al. '06, Berger et al. '06, Ossola et al. '06)
- We will not discuss these further.

ONE LOOP PERTURBATION THEORY IN NC THEORIES

- At one-loop you can have planar and non-planar graphs.
- As usual, planar graphs are given by ordinary graphs multiplied by a phase factor.
- Non-planar graphs behave very differently in the non-commutative theory. They give rise to integrals like

$$I_n = \int \frac{e^{ip\cdot k} d^{4+\epsilon} p}{(p+q_1)^2 (p+q_2)^2 (p+q_3)^2 \dots (p+q_n)^2}$$

k depends on the exact topology of the graph and is linear in θ and the q's.

ONE LOOP NONPLANAR GRAPHS

- At one-loop, non-planar graphs are UV-convergent.

$$I_n = \int dx^i \delta(1 - \sum x_i) \left(\frac{k}{\Delta}\right)^{n-2-\epsilon} K_{n-2-\epsilon}\left(2k\sqrt{\Delta}\right)$$
$$\Delta = (q_i x_i)^2 - q_i^2 x_i$$

- Recall
  1. $K_m(x) \sim \frac{1}{x^{|m|}} + \ldots x^{|m|} \log(x) + \ldots, \quad x << 1$
  2. $K_m(x) \sim \frac{e^{-x}}{\sqrt{x}} + \ldots, \quad x >> 1$
- Note the essential singularity in momentum space.
- k depends linearly on θ. For bubbles and tadpoles, the $\theta \to 0$ limit is not smooth (related to UV-IR mixing)

DECOMPOSING 1 LOOP NC AMPLITUDES

- Non-Commutative nonplanar graphs can be decomposed into a basis of simple graphs as in ordinary theories. However, we require a larger basis.
- Define

$$I_n[\alpha] = \int \frac{e^{i\alpha(p\cdot k)}}{p^2(p+q_1)^2\dots(p+q_{n-1})^2}$$

- Recall for ordinary theories, we need $I_4[0], I_3[0], I_2[0]$.

A ONE-LOOP NON-COMMUTATIVE-NONPLANAR BASIS

- For NC theories, we require a 11 dimensional basis for nonplanar graphs (proof is somewhat involved).
  1. Box — $I_4[1], \left.\frac{\partial I_4}{\partial \alpha}\right|_{\alpha=1}$
  2. Triangle — $I_3[1], \left.\frac{\partial^n I_3}{\partial \alpha^n}\right|_{\alpha=1}, n = 1, 2, 3$
  3. Bubble — $I_2[1], \left.\frac{\partial^n I_2}{\partial \alpha^n}\right|_{\alpha=1}, n = 1, 2$
  4. Tadpole — $I_1[1], \left.\frac{\partial I_1}{\partial \alpha}\right|_{\alpha=1}$
- Rational terms seem to be absent in nonplanar graphs.

OBTAINING THE COEFFICIENTS

- Can we still obtain the coefficients of this basis purely from cuts?
- YES!. The cuts give us exactly enough information to determine the coefficients.
- Box Coefficients:
  1. The new basis has 2 diagrams with 4 propagators in the denominator.
  2. Make 4 cuts.
  3. Putting 4 lines on shell gives us two solutions for the loop momentum.
  4. For the ordinary theory, we add contributions from both solutions and set it to the coefficient of the box.
  5. For the non-commutative theory, we match the cut on both solutions individually.

TRIANGLES

1. We have 4 terms with 3 propagators in the denominator.
2. Make 3 cuts. $p^2 = (p + q_1)^2 = (p + q_2)^2 = 0$ leaves us with a free parameter in p.
3. Set $p \cdot k = z$.
4. The 3-cut is given by a product of three amplitudes.
5. In any gauge theory, this can grow at most as $z^3 \leftarrow$ Important
6. The four coefficients we need are given by the coefficient of the $1, z, z^2, z^3$ terms for large z. (In the ordinary theory, we just use a specific linear combination of these coefficients)

BUBBLES AND RATIONAL TERMS

- Exactly the same technique works for bubbles.
  1. We have 3 undetermined coefficients
  2. Make a "double-cut". Laves us with two free parameters. Set $p \cdot k = z$.
  3. The double-cut can grow at most as z^2 for large z. Pick out the coefficients of $z^2, z, 1$ in the large z expansion.
- RATIONAL TERMS: Most troubling part of the amplitude in ordinary theory. Since nonplanar one loop graphs are UV convergent, there are no rational terms (no proof yet). This is quite fortuitous for UV-IR mixing would have made it very difficult to obtain these terms.

SUMMARY

- Given the on-shell three point amplitude in non-commutative gauge theories, we can reconstruct all amplitudes at tree level and one loop.
- At tree-level it was important that the same object — the color ordered subamplitude — was important in non-commutative perturbation theory and in the recursion relations.
- At loop-level we needed a larger basis of “master-integrals” than in ordinary theories. However, unitarity cuts gave us exactly enough information to determine the coefficients of this basis.

AN APPLICATION: N=4 NON-COMMUTATIVE SYM THEORY

- In ordinary theories, N=4 has a very simple S-matrix.
- One loop amplitudes may be written purely using scalar boxes.
- Non-commutative N=4 also has a very simple S-matrix. The two type of box diagrams appear together with a triangle whose coefficient is determined by the box diagrams.
- The absence of bubbles and tadpoles is enough to show absence of UV-IR mixing at one-loop in all amplitudes.

FUTURE DIRECTIONS

- Can we extend to higher loops.
- Can we construct more general non-local theories using this approach?