Phase Transition in R-charged Black Hole and Gauge Theory

Subir Mukhopadhyay

NISER, Institute of Physics Campus, Bhubaneswar, India.

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S. Mukherji, S. M., (work in progress)

K. Maeda, m. Natsumme, T. Okamura: 0809.4074; D. Son, A. starinets:hep-th 0601157

Outline

Introduction

- Spinning black D3 branes in IIB supergravity is thermodynamically stable only upto a critical value ($j_c \sim N^2 T^3$) of angular momentum density [Gubser '98]. At the boundary it undergoes phase transition.
- One expects similar phase transition to occur in the dual gauge theory. Since near critical point there are universal features which does not depend on details of microscopic theory it may be useful to study check/prediction of the duality.
- One of the universality features is critical exponents that parametrize divergences of various thermodynamic quantities near critical point.
- Our objective is to use informations from gravity theory and construct phenomenological models to compare with gauge theory near critical point.

Reduced Supergravity action for IIB on S^5

- S^5 reduction of type IIB supergravity gives $D = 5$ $\mathcal{N} = 2$ gauged supergravity with $U(1)^3$ gauge group is described by the following lagrangian

$$S = (1/2\kappa^2) \int \{ \sqrt{-g} \, [\, R - (1/2) \, G_{ij} \, F^i_{\mu\nu} \, F^{j\mu\nu} - G_{ij} \, (\partial_\mu X^i)(\partial^\mu X^j) + (4/l^2) \sum_{i=1}^{3} \frac{1}{X_i} \,] + \frac{1}{24} \, \epsilon_{ijk} \, F^i \wedge F^j \wedge A^k \}. \quad (1)$$

- A^i_μ and $F^i_{\mu\nu}$ are three $U(1)$ gauge fields and their respective field strength.
- X^i are three scalar fields constrained by $X_1 \, X_2 \, X_3 = 1$.
- Moduli space metric: $G_{ij} = \mathrm{diag}(\, 1/(X^1)^2, \, 1/(X^2)^2, \, 1/(X^3)^2 \,)$,

R-charged Black Hole Solution

- The above action admits black hole solutions with three $U(1)$ R-charges:

$$
\begin{aligned}
ds^2 &= L^2 \left[-\mathcal{H}(r)^{-2/3}\, r^2\, f(r) dt^2 + \frac{\mathcal{H}(r)^{1/3}}{f(r)}\, \frac{dr^2}{r^2} \right. \\
&\left. + \; \mathcal{H}(r)^{1/3}\, r^2\, (dx^2 + dy^2 + dz^2) \right] \qquad (2) \\
\mathcal{H}(r) &= (1 + \frac{q_1^2}{r^2})\, (1 + \frac{q_2^2}{r^2})\, (1 + \frac{q_3^2}{r^2}) \qquad (3) \\
f(r) &= \mathcal{H}(r) - \frac{r_0^4}{r^4}\, \mathcal{H}(r_0) \qquad (4) \\
X^i(r) &= \frac{\mathcal{H}(r)^{1/3}}{1 + (q_i/r^2)} \quad A_t^i(r) = \sqrt{2q_i}\, \frac{\mathcal{H}(r)^{1/3}}{1 + (q_i/r^2)}\, \frac{r_0^2}{r^2},
\end{aligned}
$$

where we scaled r by L^2 for convenience. In what follows, we will scale r and other parameters to absorb L. We will also set $G = (\pi/2N^2)$.

Thermodynamic Quantities

- Horizon radius of the Black Hole follows from largest root of $f(r) = 0$ and is given by $r = r_0$
- Hawking temperature is given by smoothness of the metric at horizon and is given by

$$\boxed{T = \frac{r_0}{2\pi}\frac{2 + (q_1 + q_2 + q_3)/r_0^2 - (q_1 q_2 q_3)/r_0^2}{\mathcal{H}(r_0)^{1/2}}} \quad (5)$$

- Entropy is calculated from the area of horizon and is given by

$$\boxed{S = \frac{N^2 r_0^3}{2\pi}\mathcal{H}(r_0)^{1/2}} \quad (6)$$

- The energy per unit area calculated using ADM Mass and it turns out to be:

$$\boxed{E = \frac{3N^2 r_0^4}{8\pi^2}\mathcal{H}(r_0)} \quad (7)$$

Thermodynamics Quantities

- Charge density, computed from the expression of gauge potential

$$\boxed{\rho_i = \frac{N^2 r_0^2}{8\pi^2}\sqrt{2q_i}\mathcal{H}(r_0)^{1/2}} \quad (8)$$

- Chemical potential computed using thermodynamic relation

$$dE = Tds + \sum_{i=1}^{3} \mu_i d\rho_i \Rightarrow \boxed{\mu_i = r_0 \frac{\sqrt{2q_i}}{1 + q_i/r_0^2} \mathcal{H}(r)^{1/2}} \quad (9)$$

- Pressure computed using thermodynamic relation

$$dP = sdT + \sum_{i=1}^{3} \rho_i d\mu_i \Rightarrow \boxed{P = \frac{N^2 r_0^4}{8\pi^2} \mathcal{H}(r_0)} \quad (10)$$

Thermodynamic Instability

- Gibbs free energy $\Omega = E - TS - \sum_{i=1}^{3} \mu_i \rho_i$ equals the negative of the pressure as expected.
- Stable Thermodynamic equilibrium requires

$$(\delta\Omega)_{T,\mu_i \text{fixed}} = 0, \qquad (\delta^2\Omega)_{T,\mu_i \text{fixed}} > 0, \tag{11}$$

which implies the following condition:

$$2r_0^6 - (q_1 + q_2 + q_3)\, r_0^4 + (q_1 q_2 q_3) > 0. \tag{12}$$

Once the above inequality is saturated the black hole undergoes phase transition.

Single R-charged Black Hole

- For the simplest case let us keep only a single gauge field (or setting $q_2 = q_3 = 0, q_1 = q$) the Black Hole solution simplifies into

$$
\begin{aligned}
ds^2 &= [-H(r)^{-2/3} f(r) r^2 dt^2 + \frac{H(r)^{1/3}}{f(r)} \frac{dr^2}{r^2} \\
&\quad + H(r)^{1/3} r^2 (dx^2 + dy^2 + dz^2)] \qquad (13) \\
H &= 1 + \frac{q}{r^2} \quad f(r) = 1 + \frac{q}{r^2} - \frac{r_0^2 (q + r_0^2)}{r^4} \qquad (14) \\
A &= \frac{r_0^2 \sqrt{2q(q + r_0^2)}}{q + r^2} dt \qquad (15)
\end{aligned}
$$

- Horizon radius remains r_0. Temperature and entropy become:

$$
T = \frac{1}{2\pi} \frac{q + 2r_0^2}{\sqrt{q + r_0^2}}, \qquad S = \frac{N^2}{2\pi} r_0^2 \sqrt{q + r_0^2} \qquad (16)
$$

Thermodynamic Quantities

- Energy and Charge densities are

$$E = \frac{3N^2}{8\pi^2} r_0^2 (q + r_0^2), \qquad \rho = \frac{N^2}{8\pi^2} r_0 \sqrt{2q(q + r_0^2)}, \tag{17}$$

- Chemical potential and free energy:

$$\mu = r_0 \sqrt{\frac{2q}{q + r_0^2}} \tag{18}$$

$$\Omega = -P = -\frac{N^2}{8\pi^2}[r_0^2(r_0^2 + q)]. \tag{19}$$

Susceptibility & Specific Heat

- R-charge susceptibility:

$$\chi_T = (\partial\rho/\partial\mu)_T = -(\partial^2\Omega/\partial\mu^2)_T = \frac{2r_0^4 - 5qr_0^2 - q^2}{16\pi(2r_0^2 - q)} \tag{20}$$

- Specific heat:

$$C_\mu = T(\partial S/\partial T)_\mu = -T(\partial^2\Omega/\partial T^2)_\mu = \frac{r_0^2(2r_0^2+q)(3r_0^2-q)}{4(2r_0^2-q)\sqrt{r_0^2+q}} \tag{21}$$

- We observe at $q = 2r_0^2$ the first derivatives of free energy, such as, entropy or charge density are regular while the above quantities which corresponds to second derivatives of free energy diverges. Thus this is associated with a second order phase transition.

Phase transition

- Consider ratio of temperature and chemical potential (associated with the BH solution) which can be written as

$$\frac{2\pi T}{\mu} = (k + \frac{1}{k}); \quad k = \sqrt{(r_0^2/2q)}. \tag{22}$$

- For a given temperature and chemical potential there are two BH solutions: $k > 1$ and $k < 1$ and they are related through a discrete transformation ($k \rightarrow 1/k$).
- The one with $k \leq 1$ has lower free energy and positive specific heat; so $k \leq 1$ is stable. Other one has higher free energy and negative specific heat showing its instability.
- At $T = T_c = \mu/\pi$ the two solutions merge. This merging corresponds to the phase transition. Instead of a single critical point we have a critical line in the $\mu \sim T$ plane.

Landau Function

- In order to investigate further properties we introduce following Landau function as a function of order parameter, which we choose to be horizon radius r_0:

$$8\pi F(r_0; T, \mu) = \frac{r_0^4(3r_0^2 - \mu^2)}{2r_0^2 - \mu^2} - T\frac{2\sqrt{2}\pi r_0^4}{\sqrt{2r_0^2 - \mu^2}}. \quad (23)$$

- At equlibrium this produces the expression of temperature:

$$(\partial F/\partial r_0) = 0 \Rightarrow T = \frac{1}{2\sqrt{2}\pi}\frac{4r_0^2 - \mu^2}{\sqrt{2r_0^2 - \mu^2}}. \quad (24)$$

- Substituting this back we get

$$8\pi F = -\frac{r_0^2(q + r_0^2)}{2}, \quad (25)$$

which is the free energy at equilibrium.

Static critical exponents

- Using the landau function we calculate critical exponents.
- Order parameter r_0 approaches r_c as $T \to T_c$ ($\epsilon_T = \frac{T - T_c}{T_c}$)

$$(r_0 - r_c) \sim \epsilon^{1/2} \quad \Rightarrow \quad \boxed{\beta = 1/2} \tag{26}$$

- Susceptibility approaches ∞ as $T \to T_c$

$$\chi \sim \epsilon^{-1/2} \quad \Rightarrow \quad \boxed{\gamma = 1/2} \tag{27}$$

- Specific heat approaches ∞ as $T \to T_c$

$$C_\mu \sim \epsilon^{-1/2} \quad \Rightarrow \quad \boxed{\alpha = 1/2} \tag{28}$$

  We have used $\epsilon = (T - T_c)/T_c$.
- At $T = T_c$ ($\epsilon_\mu = (\mu - \mu_c)/\mu_c$)

$$(r_0 - r_c) \sim \epsilon_\mu^{1/2} \quad \Rightarrow \quad \boxed{\delta = 2} \tag{29}$$

Landau function for gauge theory

- From Gravity-Gauge theory correspondence we expect a similar thermodynamic behaviour, in particular a similar phase transition in the dual gauge theory.
- In the gauge theory, a natural order parameter is the charge density ρ.
- We assume the same Landau function is valid for gauge theory and rewrite it in terms of the new order parameter:

$$\begin{aligned} F(\rho; T, \mu) &= -\mu\rho - \frac{6\pi\rho}{N^2\mu^2}\sqrt{4\pi^2\rho^2 - N^2\mu^3\rho} + \frac{12\pi^2\rho^2}{N^2\mu^2} \\ &- \frac{2\pi\rho}{\mu} + [-2\mu^2 + \frac{16\pi^2\rho}{\mu N^2} - \frac{8\pi}{\mu N^2}\sqrt{4\pi\rho^2 - N^2\mu^3\rho} \end{aligned} \quad (30)$$

- As $T \to T_c$ we have

$$\rho - \rho_c \sim \epsilon^{1/2} \quad \Rightarrow \quad \boxed{\beta = 1/2}. \quad (31)$$

Correlation Function

- In order to determine other exponents we follow the usual procedure and introduce spatial variation of charge density ρ over x. Landau function will have a term that costs energy due to variation from uniformity:

$$G = \int d^3x F(T, \mu, \langle\rho(x)\rangle) + (c/2)(\nabla\langle\rho(x)\rangle)^2. \qquad (32)$$

- The two point correlation function that follows is

$$\chi^{-1}(x, y) = \frac{\delta^2 G}{\delta\langle\rho(x)\rangle\delta\langle\rho(y)\rangle} = [(\partial^2 F/\partial\rho^2) - c\nabla^2]\delta(x - y). \qquad (33)$$

- Correlation length ξ is given in terms of Fourier transform

$$\tilde{\chi}(q) = (1/c)\frac{\xi^2}{1 + (q\xi)^2} \qquad (34)$$

Correlation Function

- Therefore the correlation length is

$$\xi = (\partial^2 G/\partial\rho^2)^{-1/2}. \tag{35}$$

- As $T \to T_c$ correlation length diverges as

$$\xi = \epsilon^{-1/2} \quad \Rightarrow \quad \boxed{\nu = 1/4} \tag{36}$$

- Similarly behaviour of the spatial correlation function $\chi(x,0)$ near $T = T_c$ diverges as $(1/c)|x|^{-(d-2)-\eta}$ and for our model we obtain

$$\boxed{\eta = 0} \tag{37}$$

Scaling Relations

- According to static scaling hypothesis that says near critical point singular part of Gibbs free energy is a generalized homogeneous function:

$$\Omega(\lambda^p \epsilon_T, \lambda^q \epsilon_\mu) = \lambda^d \Omega(\epsilon_T, \epsilon_\mu), \quad \epsilon_T = \frac{T - T_c}{T_c}, \quad \epsilon_\mu = \frac{\mu - \mu_c}{\mu_c}. \tag{38}$$

- This leads to following relations among critical exponents

$$\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1) = \nu(2 - \eta), \quad 2 - \alpha = \nu d. \tag{39}$$

The last relation is valid only if $d < d_c$, where d_c is called upper critical dimension.

- Critical exponents obtained from the Landau function satisfy all the relations except the last one.

Dynamic Equation

- In the dynamic case variables vary slowly over time. Consider dynamic variables for the present system to be energy density E and charge density ρ both of which satisfy continuity equations:

$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{j}^e = 0, \qquad \frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0, \tag{40}$$

  where $\vec{j}^e$ and $\vec{j}$ are associated currents. (We are keeping momentum density to be frozen).
- When T and μ are spatially uniform corresponding currents are zero. In linearized approximation, assuming currents are proportional to spatial gradient one can arrive

$$\frac{\partial E}{\partial t} = \kappa \nabla^2 T, \qquad \frac{\partial \rho}{\partial t} = \lambda \nabla^2 \mu, \tag{41}$$

  where κ and λ are transport coefficients called thermal and R-charge conductivity.

Dynamic Equation

- κ and λ can be calculated using Green-Kubo formula:

$$\lambda = \lim \omega \rightarrow 0 \frac{1}{\omega} \left[Im[\tilde{G}^{(R)}_{xx}(\omega, q = 0) \right] \tag{42}$$

  Using AdS/CFT this has been calculated in literature (hep-th/0601157, 0809.4074) and R-charge conductivity turns out to be

$$\lambda = \frac{N^2 r_0}{16\pi^2\sqrt{1+2k}}(k+1)^2 \tag{43}$$

  which shows λ is finite on critical line $k = (\mu/2r_0^2) = 1$.
- At $q \rightarrow 0$, writing $(\partial\mu/\partial\rho) = \chi^{-1}$, we get diffusion coefficient $D = \chi^{-1}\lambda$. Since this system has conserved order parameter and finite conductivity at critical point it corresponds to B-model Hohenberg-Halperin classification of universality classes.

Two R-charged Black Hole

- In the general R-charged solution if we set $q_3 = 0$ we obtain black hole solution with two R-charges.
- Thermodynamic quantities that follows are

$$S = \frac{N^2 r_0^3}{2\pi} \sqrt{(1 + q_1/r_0^2)(1 + q_2/r_0^2)}, \quad (44)$$

$$E = \frac{3N^2 r_0^4}{8\pi^2} (1 + q_1/r_0^2)(1 + q_2/r_0^2), \quad (45)$$

$$\rho_i = \frac{N^2 r_0^2}{8\pi} \sqrt{2q_i} (1 + q_1/r_0^2)(1 + q_2/r_0^2). \quad (46)$$

Landau Function

- A Landau function can be calculated using earlier method

$$F = \frac{N^2 r_0^4}{8\pi^2}[\,(k_1 + k_2 - k_1 k_2 + 3) - 4(\frac{\pi T}{r_0})\sqrt{(k_1+1)(k_2+1)}\,], \quad (47)$$

where we use $k_i = (q_i/r_0^2)$.

- In order to obtain critical exponents we evaluate derivativesnof F and then solve for r_0 in terms of T that turns out to be technically involved.
- we have checked a simple limit $\mu_1 = x\mu_2$ with x constant. For $x = 1$ we find the phase transition is of different type.
- We are yet to calculate critical exponents and conductivity for this case.

Summary

- We have considered phase transition associated with R-charged Black Hole.
- For single-charge BH we have constructed a Landau function. Critical exponents following from that obeys scaling relation (except the hyperscaling one).
- For more than one R-charge Black Hole the problem is technically more involved. We observe that for special points in μ-space phase transition may be of different type. It is interesting to study the dynamics of the two charged BH.
- It will be useful to get information from the gauge theory side (at least near criticality) and compare with gravity result.