# Spectrum of Dyons in $\mathcal{N} = 4$ String Theory

Shamik Banerjee (HRI, Allahabad)

Work done with: Ashoke Sen and
Yogesh Srivastava

## Introduction

- In this talk we shall focus our attention on a particular $N = 4$ string theory in four dimensions which is obtained by compactifying Heterotic string on $T^6$.

- Our aim is to calculate the microscopic degeneracy of 1/4 BPS dyons in this theory.

- Dyons carry both electric and magnetic charges-(Q,P)-under 28 U(1) gauge fields which are present in the theory.

- The charge vectors take value in the 28 dimensional Narain lattice Λ which is an even selfdual lattice of signature $(6, 22)$. Q and P are both 28 dimensional vectors.

- This theory also has a set of massless scalar fields:

  A complex scalar $\tau$ valued in the upper-half plane and a $28 \times 28$ dimensional matrix valued field M which satisfies:

  $M^T = M$ and $M^T L M = L$, where L is the metric on the Narain lattice.

- T and S duality symmetries of the theory act on the charges and the modulii fields in the following manner:

  $(Q,P) \to (\Omega Q, \Omega P)$, $\tau \to \tau$ and $M \to \Omega M \Omega^T$ where $\Omega \in$ T-duality group $O(6,22;Z)$.

  $(Q,P) \to (aQ + bP, cQ + dP)$, $M \to M$ and

  $\tau \to \dfrac{a\tau + b}{c\tau + d}$ where $a$,$b$,$c$,$d$ are integers

  satisfying $ad - bc = 1$.

- The degeneracy $d(Q,P)$ of a dyon carrying charge vectors (Q,P) is defined as the number of 1/4 BPS states with charges (Q,P) weighted by $(-1)^F(2h)^6/6!$

  F = Fermion number and $h$ = Helicity.

## Walls of Marginal Stability

- The degeneracy $d(Q,P)$ should be invariant under both T and S duality transformations, but it can vary with the asymptotic moduli in the following way.

- Since $d(Q,P)$ is an index, we do not expect continous variation with changes in the moduli.

  But there are Walls of Marginal Stability(WMS).

- WMS associated with a dyon carrying charge vectors (Q,P) is defined as the subspace of the moduli space where the following relation is satisfied :

$$\begin{aligned} m(Q,P;\phi) &= m(\alpha Q+\beta P,\gamma Q+\delta P;\phi) \\ &+ m(\delta Q-\beta P,-\gamma Q+\alpha P;\phi) \end{aligned}$$

$\alpha\delta = \beta\gamma, \qquad \alpha+\delta = 1$

- Here $m(Q, P)$ is the mass of the dyon carrying charges $(Q, P)$, $\phi$ denotes the asymptotic moduli and $\alpha$,$\beta$,$\gamma$,$\delta$ are rational numbers chosen such that the charge vectors belong to the Narain lattice.

- Additional constraints on $\alpha$,$\beta$,$\gamma$,$\delta$ force the decay products to be 1/2 BPS .

- In our case the releavent decays are those for which the decay products are 1/2 BPS dyons.

- So the dependence on $\phi$ is mild, in the sense that the degeneracy formula should be $\phi$ independent within a given domain in the moduli space bounded by walls of marginal stability.

## Constraints On the Degeneracy Function From T and S Dualities

- We use the T-duality invariance of the BPS mass formula to write

  $\rightarrow$

  $Q \to \Omega Q,\ P \to \Omega P,\ \phi \to \phi_\Omega$
  $m(\Omega Q, \Omega P; \phi_\Omega) = m(Q, P; \phi), \quad \Omega \in O(6, 22;\ \mathbb{Z})$

  $m(\Omega Q, \Omega P; \phi_\Omega) = m(\alpha\Omega Q + \beta\Omega P, \gamma\Omega Q + \delta\Omega P; \phi_\Omega) + m(\delta\Omega Q - \beta\Omega P, -\gamma\Omega Q + \alpha\Omega P; \phi_\Omega),$ .

- This shows that under a T-duality transformation on charges and moduli, the wall of marginal stability associated with the set $(\alpha, \beta, \gamma, \delta)$ gets mapped to the wall of marginal stability associated with the same $(\alpha, \beta, \gamma, \delta)$

- Thus if we consider a domain bounded by the walls of marginal stability associated with the sets $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ for $1 \leq i \leq n$ – collectively denoted by a set of discrete variables $\vec{c}$ – then under a simultaneous T-duality transformation on the charges and the moduli this domain gets mapped to a domain labelled by the same vector $\vec{c}$.

- The fact that the dependence of $d(Q,P)$ on the moduli $\phi$ comes only through the domain in which $\phi$ lies, i.e. the vector $\vec{c}$ and $\vec{c}$ remains unchanged under a T-duality transformation, leads to

$$d(Q,P;\vec{c}) = d(\Omega Q, \Omega P; \vec{c}), \quad \Omega \in O(6,22;\ \mathbb{Z}). \tag{1}$$

- This shows that $d(Q,P;\vec{c})$ must depend only on $(Q,P)$ via the T-duality invariants: $d(Q,P;\vec{c}) = f(I(Q,P);\vec{c})$ ,where $f$ is some function and $I(Q,P)$ denotes the set of all
<u>T-duality invariants built out of $Q$ and $P$ only.</u>

$\rightarrow$ we need to determine the complete set of T-duality invariants.

## T-Duality Invariants

- The continous T-duality group $O(6, 22; R)$ has invariants $Q^2$, $P^2$ and $Q.P$. These are also invariants of the discrete T-duality group.

- But discrete T-duality group has more invariants.

  For example, $g.c.d(Q_i P_j - Q_j P_i) = r$ is one of them.

  **(Dabholkar,Gaiotto,Nampuri)**

- We will state the result for other invariants.

- We shall assume that the dyon is primitive so that $(Q, P)$ cannot be written as an integer multiple of $(Q_0, P_0)$ with $Q_0, P_0 \in \Lambda$, but we shall not assume that $Q$ and $P$ themselves are primitive.

- Now consider the intersection of the two dimensional vector space spanned by $(Q, P)$ with the Narain lattice $\Lambda$. The result is a two dimensional lattice $\Lambda_0$. Let $(e_1, e_2)$ be a pair of basis elements whose integer linear combinations generate this lattice.

- We can always choose $(e_1, e_2)$ such that in this basis $Q = r_1 e_1, P = r_2(u_1 e_1 + r_3 e_2)$,
  $r_1, r_2, r_3, u_1 \in \mathbb{Z}^+$,
  gcd$(r_1, r_2) = 1$, gcd$(u_1, r_3) = 1, 1 \leq u_1 \leq r_3$

- It turns out that $r_1$,$r_2$,$r_3$ and $u_1$ together with $Q^2$,$P^2$ and $Q.P$ form a complete set of T-duality invariants.

  $\rightarrow I(Q, P) = Q^2,\ P^2,\ Q.P,\ r_1,\ r_2,\ r_3,\ u_1$

  Here $g.c.d(Q_i P_j - Q_j P_i) = r = r_1 r_2 r_3$

  **(Banerjee,Sen)**

- One can also show that there always exists an S-duality transformation which brings the T-duality invariants $(r_1, r_2, r_3, u_1)$ to
  $(r_1 r_2 r_3, 1, 1, 1)$ together with an appropriate transformation on $Q^2$, $P^2$ and $Q \cdot P$ induced by S-duality transformation.

- Subgroup of S-duality preserving $(r_1 r_2 r_3, 1, 1, 1)$ is $\Gamma^0(r)$.

## Form of $d(Q, P)$

- We shall now examine the consequences of these results for the formula expressing the degeneracy $d(Q, P)$

- under an S-duality transformation the wall associated with the parameters $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ gets mapped to the wall associated with $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$
$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$

- Thus S-duality invariance of the degeneracy formula now gives
$f(Q^2, P^2, Q \cdot P, r_1, r_2, r_3, u_1; \vec{c})$
$= f(Q'^2, P'^2, Q' \cdot P', r'_1, r'_2, r'_3, u'_1; \vec{c}')$ , where $\vec{c}'$ stands for the collection of the sets
$\{\alpha'_i, \beta'_i, \gamma'_i, \delta'_i\}$.

- We now use the result that there exists a special class of S-duality transformations under which $(r'_1, r'_2, r'_3, u'_1) = (r_1 r_2 r_3, 1, 1, 1)$ .

- Using this we can write $f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q'\cdot P', r_1 r_2 r_3, 1, 1, 1; \vec{c}\,')$ .

- Thus the complete information about the spectrum of quarter BPS dyons is contained in the set of functions $g(Q^2, P^2, Q\cdot P, r; \vec{c}) \equiv f(Q^2, P^2, Q\cdot P, r, 1, 1, 1; \vec{c})$ .

  **(Banerjee,Sen)**

- We shall focus our attention on this function during the rest of our analysis. Using the fact that $\Gamma^0(r)$ transformations leave the set $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ fixed, we see that $g(Q^2, P^2, Q\cdot P, r; \vec{c}) = g(Q'^2, P'^2, Q'\cdot P', r; \vec{c}\,')$ for

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(r)\,.$$

## Functional Form of The Degeneracy Function

- The answer for $g(Q^2, P^2, Q \cdot P, 1; \vec{c})$ is given by:

  $g(Q^2, P^2, Q \cdot P, 1; \vec{c}) =$
  $(-1)^{Q \cdot P+1} \int_{\mathcal{C}} d\rho \, d\sigma \, dv \, \frac{e^{-\pi i(\rho Q^2+\sigma P^2+2vQ\cdot P)}}{\Phi_{10}(\rho,\sigma,v)}$.

  **(Dijkgraaf,Verlinde,Verlinde)**

- Here $\rho, \sigma, v$ are three complex variables and $\Phi_{10}(\rho, \sigma, v)$ is the unique weight 10 Igusa cusp form of $Sp(2, Z)$.

- $\mathcal{C}$ is the contour of integration in $(\rho, \sigma, v)$ space, given by

$$\begin{aligned}
\Im(\rho) &= \Lambda \left( \frac{|\tau|^2}{\tau_2} + . \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right), \\
\Im(\sigma) &= \Lambda \left( \frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right), \\
\Im(v) &= -\Lambda \left( \frac{\tau_1}{\tau_2} + \frac{Q_R \cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right) \qquad (2)
\end{aligned}$$

  where $\Lambda$ is a large positive number, $\Im(z)$ denotes the imaginary part of $z$,

  $Q_R^2 = Q^T(M+L)Q, \quad P_R^2 = P^T(M+L)P,$
  $Q_R \cdot P_R = Q^T(M+L)P$, $\tau \equiv \tau_1 + i\tau_2$ denotes the asymptotic value of the axion-dilaton moduli.

  **(Cheng,Verlinde)**

## Proposal For The General Degeneracy Function

- We will propose a general formula for $g(Q^2, P^2, Q \cdot P, r; \vec{c})$ valid for all values of $r$. This will give the degeneracy of dyons with charge vectors of the form $(re_1, e_2)$ where $g.c.d(e_1 \wedge e_2) = 1$.

- Degeneracies for other values of charges can be generated by duality transformations.

- We define a partition function $\Phi(\rho, \sigma, v, r; \vec{c})$ as

  $\frac{1}{\Phi(\rho,\sigma,v,r;\vec{c})}$

  $= \sum_{Q^2,P^2,Q\cdot P} (-1)^{Q\cdot P+1} g(Q^2, P^2, Q\cdot P; r, \vec{c})$

  $e^{i\pi(\sigma Q^2+\rho P^2+2vQ\cdot P)}$ .

  Here $g(Q^2, P^2, Q \cdot P, r; \vec{c})$ is the degeneracy function .

- The sum runs over all possible values of $Q^2$, $P^2$ and $Q \cdot P$ . This in particular requires $Q^2/2 \in r^2\, \mathbb{Z}, \quad P^2/2 \in \mathbb{Z}, \quad Q \cdot P \in r\, \mathbb{Z}$ .

- The imaginary parts of $(\rho, \sigma, v)$ need to be adjusted to lie in a region where the sum is convergent.

- Although the degeneracy function depends on $\vec{c}$, the partition function turns out to be independent of the moduli.

  This is the lesson we learn from the $r = 1$ case and assume this to be true for all $r > 1$ cases.

- We extract the degeneracy function from the partition function in the following way :

$$d(Q,P) = g(Q^2, P^2, Q \cdot P, r; \vec{c}) = (\text{-}1)^{Q \cdot P+1} \quad r^3$$

$$\int_{iM_1-1/2}^{iM_1+1/2} d\rho \int_{iM_2-1/(2r^2)}^{iM_2+1/(2r^2)} d\sigma \int_{iM_3-1/(2r)}^{iM_3+1/(2r)} dv$$

$$e^{-i\pi(\sigma Q^2+\rho P^2+2vQ\cdot P)} \, \frac{1}{\Phi(\rho,\sigma,v)} \, ,$$

- The imaginary parts $M_1$, $M_2$ and $M_3$ of $\rho$, $\sigma$ and $v$ are fixed in a region where the original sum is convergent.

  This is how the information of the asymptotic moduli goes into the choice of contour in the $\rho, \sigma, v$ space. But the choice is not unique as long as we are moving inside the domain of convergence of the original sum.

- Our proposal for the partition function for general $r$ is

$$\Phi(\rho, \sigma, v)^{-1} = \sum_{\substack{s \in Z, s|r \\ \bar{s} \equiv r/s}} \frac{s}{\bar{s}^3} \sum_{k=0}^{\bar{s}^2-1} \sum_{l=0}^{\bar{s}-1} \Phi_{10}\left(\rho, s^2\sigma + \frac{k}{\bar{s}^2}, sv + \frac{l}{\bar{s}}\right) \tag{3}$$

  where $\Phi_{10}(\rho, \sigma, v)$ is the unique weight 10 Igusa cusp form of $Sp(2, Z)$.
  **(Banerjee,Sen,Srivastava)**

  A possible derivation has been suggested by **Dabholkar,Gomes,Murthy**

- The above formula satisfies various consistency checks.

  1. This function is manifestly invariant under $\Gamma^0(r)$ subgroup of the S-duality group.

  2. In the large charge regime one gets the correct black hole entropy.

  3. By going to appropriate regions in the moduli space where there is enhanced gauge symmetry, one can reproduce the correct degeneracy of gauge theory dyons.

## Wall Crossing Formula

- We shall only consider the decay into a pair of half-BPS dyons of the form

  $$(Q, P) \to (Q_1, P_1) + (Q_2, P_2) \ ,$$

  $$(Q_1, P_1) = (\alpha Q + \beta P, \gamma Q + \delta P) = N_1(e_1, f_1)$$

  $$(Q_2, P_2) = (\delta Q - \beta P, -\gamma Q + \alpha P) = N_2(e_2, f_2),$$

  $$\alpha\delta = \beta\gamma, \qquad \alpha + \delta = 1 \ .$$

  $(e_1, f_1)$ and $(e_2, f_2)$ are "coprime" pairs of vectors and $(N_1, N_2)$ are a pair of integers such that $g.c.d(N_1, N_2) = 1$.

- In the $r = 1$ case there is only one decay modulo $SL_2(Z) = \Gamma^0(1)$ transformation. It is given by

  $$(Q, P) \to (Q, 0) + (0, P)$$

- For general values of r there are more than one decays, counted modulo $\Gamma^0(r)$ transformation.

- For example, for $r = 2$ , the type of dyons we are considering will have charge vectors of the form $(2e_1, e_2)$. This can decay into, $(Q, P) \rightarrow (e_1, e_2) + (e_1, 0)$ and $(Q, P) \rightarrow (2e_1, 0) + (0, e_2)$.

- The first one is a primitive decay and the second one is semi-primitive. Decays of the second kind are not possible in the $r = 1$ case.

- We can use the partition function to write down a general Wall-Crossing formula in the following way.

- As the asymptotic moduli crosses the WMS associated with the decay, the contour of integration in the $(\rho,\ \sigma\ ,v)$ space crosses a pole of the partition function at $\rho\gamma - \sigma\beta + v(\alpha - \delta) = 0$ .

  Thus the change in the degeneracy can be calculated by evaluating the residue of the partition function at this pole.

- The answer is given by:

  $$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)$$
  $$\sum_{L_1|N_1}\sum_{L_2|N_2} d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\, d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right) \ .$$

  Where $d_h$ is the degeneracy of a 1/2 BPS dyon.

- It is easy to see that this formula gives us back the known result for primitive decay and also calculates the change in index for non-primitive decays.

THANK YOU