

On Non-Relativistic Limits

Of the AAdS-CFT Conjecture

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ISM08, Puducheri

Dec. 8, 2009

Based on

- ❖ Arjun Bagchi and Rajesh Gopakumar, (to appear)
- ❖ Arjun Bagchi, Turbasu Biswas and R. Gopakumar (work in progress).

Outline of the talk

- ◆ Motivation and General Remarks
 - ✧ Non-Relativistic Conformal Symmetry

- ◆ The Non-Relativistic Contraction
 - ✧ And a natural *infinite* dimensional extension

- ◆ Bulk Realisation of the Contraction
 - ✧ Galilean limits of Einstein's Equations

- ◆ "To Do" List

1 Motivation and General Remarks

Motivation

- ◆ Recent thrust towards understanding non-relativistic systems (e.g. in CMT) in the same spirit as the (relativistic) AdS/CFT correspondence.
- ◆ However, mostly in the context of the non-relativistic **Schrodinger Symmetry** (cf. talk by Kaushik).
- ◆ **Are there other non-relativistic symmetries? Yes.**
- ◆ One way is to take a direct non-relativistic limit of the relativistic theory.
- ◆ Also opens the possibility of studying a BMN like limit of the parent theory (e.g. $\mathcal{N} = 4$ Yang-Mills theory).
- ◆ Thus focus on a potentially simpler subsector of the full hilbert space.
- ◆ **Seems to be much more constrained by symmetry (Kac-Moody Algebra).**
- ◆ Report some preliminary progress along these lines.

General Remarks

- ◆ **Schrodinger** Symmetry $Sch(d, 1)$ is a non-relativistic analogue of conformal symmetry in $(d + 1)$ dimensions.
- ◆ It contains the usual **Galilean** group $G(d, 1)$ (with a central extension):

$$\begin{aligned} [J_{ij}, J_{rs}] &= so(d) \\ [J_{ij}, B_r] &= -(B_i \delta_{jr} - B_j \delta_{ir}) \\ [J_{ij}, P_r] &= -(P_i \delta_{jr} - P_j \delta_{ir}), \quad [J_{ij}, H] = 0 \\ [B_i, B_j] &= 0, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = m \delta_{ij} \\ [H, P_i] &= 0, \quad [H, B_i] = -P_i. \end{aligned} \tag{1}$$

- ◆ But also has *two* more generators \tilde{K}, \tilde{D} with some of its commutators given by

$$\begin{aligned} [\tilde{K}, P_i] &= B_i, \quad [\tilde{K}, B_i] = 0, \quad [\tilde{D}, B_i] = B_i \\ [\tilde{D}, \tilde{K}] &= 2\tilde{K}, \quad [\tilde{K}, H] = -\tilde{D}, \quad [\tilde{D}, H] = -2H. \end{aligned} \tag{2}$$

- ◆ The last line is an $SL(2, R)$ algebra.

General Remarks *continued*

- ✧ \tilde{K} is like a time component of special conformal transformations.
- ✧ No analogue of the spatial components K_i of special conformal transformations.
- ✧ Thus 12 parameter group (+ central term) in $d = 3$ as opposed to 15 parameters for $SO(4, 2)$.
- ✧ \tilde{D} is a dilatation operator which scales time and space *differently*

$$x_i \rightarrow \lambda x_i, t \rightarrow \lambda^2 t. \quad (3)$$

- ✧ $Sch(d, 1)$ is the symmetry of the free particle Schrodinger equation, i.e. generators that commute with the Schrodinger operator $S = i\partial_t + \frac{1}{2m}\partial_i^2$.
- ✧ This symmetry also believed to arise in interacting systems like fermions at unitarity.

2 The Non-Relativistic Contraction

Galilean Contraction

- ✧ A direct way to recover the Galilean group $G(d, 1)$ is to perform a contraction on the Poincare group $ISO(d, 1)$.
- ✧ Take $t \rightarrow \epsilon^r t$ and $x_i \rightarrow \epsilon^{r+1} x_i$ and scale $\epsilon \rightarrow 0$. (Thus $v_i \sim \epsilon$).
- ✧ The vector fields generating the Poincare algebra reduce to the Galilean vector fields (after appropriate rescaling)

$$\begin{aligned} H &= -\partial_t, & P_i &= \partial_i \\ B_i &= t\partial_i, & M_{ij} &= -(x_i\partial_j - x_j\partial_i). \end{aligned} \tag{4}$$

- ✧ One might therefore try and do a similar contraction on the relativistic conformal group $SO(d, 2)$ to obtain a non-relativistic conformal group.
- ✧ Since this is obtained as a parametric scaling limit, it is directly embedded within the relativistic theory.

Galilean Contraction *Conformal Generators*

◆ Carry out this scaling on the additional generators (D, K_0, K_i) present in the conformal group $SO(d, 2)$.

◆ Gives the contracted vector fields

$$\begin{aligned} D &= -(x_i \partial_i + t \partial_t) \\ K &= -(2tx_i \partial_i + t^2 \partial_t) \\ K_i &= t^2 \partial_i. \end{aligned} \tag{5}$$

◆ Note the dilatation generator D is the *same* as in the relativistic theory. It scales space and time in the same way.

◆ Therefore different from $\tilde{D} = -(2t\partial_t + x_i \partial_i)$.

◆ Spatial conformal transformation generators K_i are present and generate constant acceleration transformations.

◆ The temporal special conformal generator different from $\tilde{K} = -(tx_i \partial_i + t^2 \partial_t)$

Galilean Contraction *Conformal Generators*

- ❖ The algebra of these generators (together with that of the Galilean Algebra) is quite **different** from the Schrodinger group.
- ❖ However, a **subset** of these generators appear as symmetries of Navier-Stokes equations (cf. talk by Spenta).
- ❖ Galilean central extension in $[B_i, P_j]$ not admissible in this algebra – "**massless non-relativistic system**".
- ❖ However, one can have a **different** central extension of the form $[K_i, P_j] = N\delta_{ij}$. (Interpretation?)

Galilean Contraction *Conformal Algebra*

❖ Algebra of the contracted conformal group: Define

$$\begin{aligned} L^{(-1)} &= H, & L^{(0)} &= D, & L^{(+1)} &= K, \\ M_i^{(-1)} &= P_i, & M_i^{(0)} &= B_i, & M_i^{(+1)} &= K_i. \end{aligned} \quad (6)$$

❖ Then

$$\begin{aligned} [J_{ij}, L^{(n)}] &= 0, & [L^{(m)}, M_i^{(n)}] &= (m - n)M_i^{(m+n)} \\ [J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), & [M_i^{(m)}, M_j^{(n)}] &= 0, \\ [L^{(m)}, L^{(n)}] &= (m - n)L^{(m+n)}. \end{aligned} \quad (7)$$

❖ Note the $SL(2, R)$ algebra in the last line. *Different* from that in the Schrodinger group.

Galilean Contraction *Infinite dimensional extension*

- ❖ The most remarkable feature of this algebra is that it admits a very natural extension to an **infinite dimensional $SO(d)$ Current Algebra**.
- ❖ Define the vector fields **for arbitrary integer n**

$$\begin{aligned}L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1} \partial_t \\M_i^{(n)} &= t^{n+1} \partial_i \\J_{ij}^{(n)} &= -t^n (x_i \partial_j - x_j \partial_i)\end{aligned}\tag{8}$$

- ❖ They obey exactly the same commutation relations as the ones for $m, n = 0, \pm 1$.

$$\begin{aligned}[L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} & [L^{(m)}, J_a^{(n)}] &= nJ_a^{(m+n)} \\[J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} & [L^{(m)}, M_i^{(n)}] &= (m-n)M_i^{(m+n)}\end{aligned}\tag{9}$$

- ❖ The Virasoro and Kac-Moody algebra of the vector fields is, of course, without the central extension.

The Galilean Contraction *Interpretation*

- ❖ The $M_i^{(n)}$ act as generators of generalised time dependent but spatially homogeneous accelerations

$$x_i \rightarrow x_i + b_i(t). \quad (10)$$

- ❖ Similarly, the $J_{ij}^{(n)} \equiv J_a^{(n)}$ are generators of arbitrary time dependent rotations

$$x_i \rightarrow R_{ij}(t)x_j \quad (11)$$

- ❖ These two together generate what is sometimes called the **Coriolis group**: the biggest group of "isometries" of "flat" Galilean spacetime.

- ❖ $L^{(n)}$ seem to be generators of a **conformal "isometry"** of Galilean spacetime.

$$t \rightarrow f(t), \quad x_i \rightarrow \frac{df}{dt}x_i \quad (12)$$

- ❖ Galilean spacetime (in the absence of gravitation) is characterised by a degenerate metric: an absolute time (t) and a Euclidean spatial metric.

The Galilean Contraction *Realisations*

- ❖ How are these Galilean limits realised in gauge theories?
- ❖ One would expect an $SL(2, R)$ sector of the relativistic gauge theory in which the spatial rotation group behaves like an internal symmetry.
- ❖ Bulk dual provides hints as to how to look for this (see later).
- ❖ Another possibility is a partial realisation of a subset of these symmetries.
- ❖ For gauge theories at finite temperature in the hydrodynamic limit, one can study the non-relativistic limit (Fouxon and Oz, Bhattacharya, Minwalla and Wadia).
- ❖ Recover the Navier-Stokes equations. Has all the symmetries of the *finite contracted algebra* (except D which is broken by choice of temperature/scale T).
- ❖ Actually also has the whole of the boosts $M_i^{(n)}$ (Milne algebra).

3 Bulk Dual of the Galilean Conformal Algebra

The Bulk Dual

- ❖ What can we say about the bulk dual to a system with this Galilean Conformal Algebra?
- ❖ The $SL(2, R)$ suggests an AdS_2 part: in the t and z (radial directions).
- ❖ Boundary metric degenerating in the non-relativistic limit \Rightarrow some kind of Galilean limit in the $(d + 2)$ dimensional bulk.
- ❖ (Alternatively, realise the non-relativistic symmetry in a one higher dimensional bulk?)
- ❖ So expect the dual spacetime to have the structure $AdS_2 \times R^3$ with degenerate metric on the R^3 .
- ❖ Will first check how the (extended) Galilean Conformal Algebra is realised in this kind of bulk. And outline the geometric structure underlying it (analogous to Newtonian limit of Einstein's equations).

The Bulk Dual *Symmetries*

❖ AdS_{d+2} in Poincare coordinates:

$$ds^2 = R^2 \frac{dt^2 - dz^2 - dx_i^2}{z^2} \quad (13)$$

❖ In radially infalling coordinates for null geodesics ($t' = t + z, z' = z$)

$$ds^2 = \frac{R^2}{z'^2} (-2dt' dz' + dt'^2 - dx_i^2) = \frac{R^2}{z'^2} (-dt'(2dz' - dt') - dx_i^2). \quad (14)$$

❖ Take the generators of the AdS_{d+2} isometries and perform the contraction by taking $t', z' \rightarrow \epsilon^r, x_i \rightarrow \epsilon^{r+1} x_i$. Metric degenerates as expected.

❖ Contracted Killing vectors given by

$$\begin{aligned} P_i &= -\partial_i, & B_i &= -(t' - z')\partial_i, & K_i &= -(t'^2 - 2t'z')\partial_i \\ H &= \partial_{t'}, & D &= t'\partial_{t'} + z'\partial_{z'} + x_i\partial_i, & K &= t'^2\partial_{t'} + 2(t' - z')(z'\partial_{z'} + x_i\partial_i). \end{aligned} \quad (15)$$

The Bulk Dual *Symmetries*

★ More compactly (for $m, n = 0, \pm 1, l = 0$).

$$\begin{aligned} L^{(n)} &= t'^{n+1} \partial_{t'} + (n+1)(t'^n - nzt'^{n-1})(x_i \partial_i + z' \partial_{z'}) \\ M_i^{(m)} &= -(t'^{m+1} - (m+1)zt'^m) \partial_i \\ J_{ij}^{(l)} &= -t'^n (x_i \partial_j - x_j \partial_i) \end{aligned} \tag{16}$$

- ★ Reduces at the boundary ($z = 0$) to the generators of the contracted conformal algebra. *And satisfies the same algebra.*
- ★ *In fact, these bulk vector fields (for arbitrary m, n, l) reduce to that of the extended Kac-Moody algebra at the boundary.*
- ★ *What is the role of these vector fields in the bulk?*
- ★ The *Virasoro* generators act as the generators of asymptotic symmetries of the *AdS₂*.
- ★ *The others act only on the R^3 (like Galilean "isometries" on the boundary).*

The Bulk Dual *Geometric Structure*

- ★ Degenerate nature of the metric might seem to imply that the gravitational dynamics is singular.
- ★ However, similar situation in asymptotically flat space in recovering Newtonian gravity from Einstein gravity in the non-relativistic limit.
- ★ The answer: there is a well-defined geometric theory of newtonian gravitation - **Newton-Cartan theory** (see for e.g. MTW).
- ★ The ingredients: A space time endowed with absolute time function t , a non-dynamical *spatial* (Euclidean) metric, a dynamical *non-metric* connection $\Gamma_{00}^i \propto \partial_i \Phi$.
- ★ Einstein's equations reduce to $R_{00} \propto \rho$ which is Poisson's equation for Φ .
- ★ One can write all this in a more covariant form: A three dimensional fibre R^3 over a base R which is parametrised by the time t .
- ★ Nothing singular about this classical geometric description, only unusual.

The Bulk Dual *Geometric Structure*

- ★ **Proposal:** Proceed here in a similar manner.
- ★ Except that instead of base R we have AdS_2 and fibres are still Euclidean R^3 .
- ★ Separate metrics $g_{\alpha\beta}$ on AdS_2 and δ_{ij} on the spatial R^3 .
- ★ Dynamical affine connections $\Gamma_{\alpha\beta}^i$.
- ★ The non-relativistic scaling limit of Einstein's equations leave one with

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{17}$$

- ★ Thus the bulk boundary relation is some kind of an AdS_2/CFT_1 duality.
- ★ Presumably existing inside every relativistic conformal field theory in higher dimensions.

4 "To Do" List

"To Do" List

- ☆ Full realisation of galilean limit within relativistic gauge theories, (*a la* BMN) e.g. in $\mathcal{N} = 4$ SYM.
- ☆ Realisation/calculation of central charges in individual systems (*a la* Brown-Henneaux)
- ☆ How far can one exploit the presence of this Kac-Moody algebra sitting in higher dimensional gauge theories?
- ☆ Supersymmetric generalisation.
- ☆ Understand bulk geometric description better. Closed string description of the geometry?
- ☆ Develop the bulk-boundary dictionary.
- ☆ Is there a more standard geometric description in, say, $(d + 3)$ dimensions as in the bulk duals for Schrodinger symmetric theories?
- ☆ Are there intrinsically non-relativistic realisations of Galilean conformal algebra?

The end