Dimer Models and Quiver Gauge Theories

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Talk based on recent work

A note on dimer models and D-brane gauge theories

Work done with Prarit Agarwal (undergraduate student) and Tapobrata Sarkar (IIT Kanpur)

hep-th/0804.1902, JHEP06(2008)054.

Plan:

- Introduction
- Quiver Gauge Theories
- Non-compact Toric CY_3
- Quiver Gauge Theory $\leftrightarrow$ Toric CY_3

(i)Forward algorithm (*obtaining toric data*)

(ii) Inverse algorithm (*obtaining gauge theory data*)

- Dimer Models

(i) *Kasteleyn matrix* ; (ii)*Perfect Matchings*

- Power of dimer model approach

INTRODUCTION

- AdS-CFT:

$\mathcal{N} = 1$ quiver gauge theories arise on D-branes placed at the tip of the singular Calabi-Yau C and they are dual to string theory on $AdS_5 \times X_5$.

(X_5 : Sasaki-Einstein manifolds having $S^2 \times S^3$ topology).

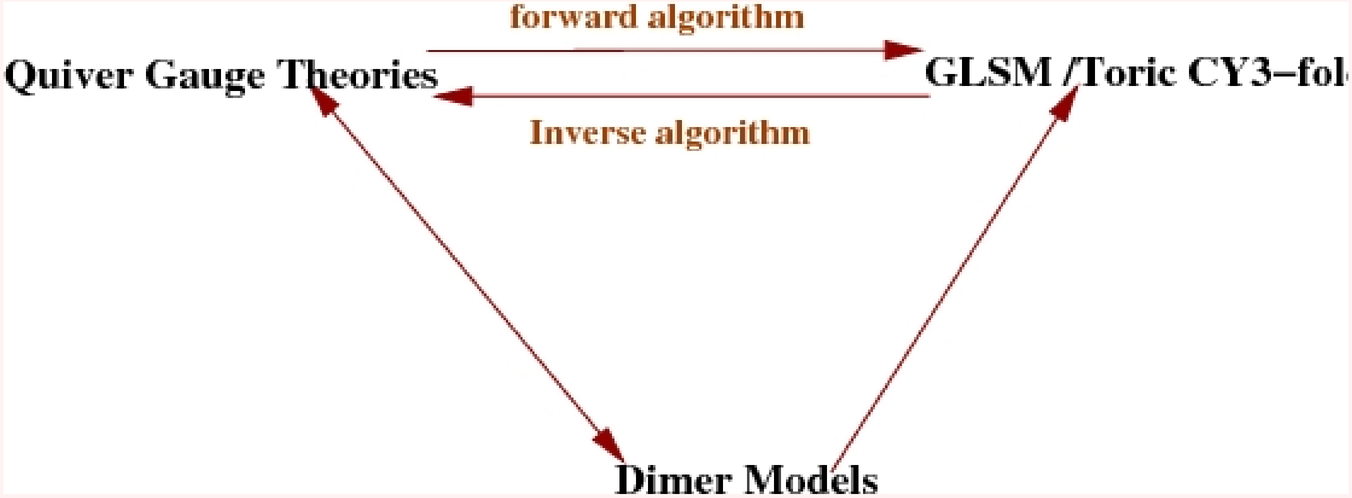
Mathematically, C is the cone over X_5:

$$ds^2(C) = dr^2 + r^2 dX_5^2 \ .$$

- Quiver gauge theory on a D-brane placed transverse to orbifold $\mathbb{C}^3/\Gamma$:

It is obtained from $U(r)$ gauge theory subjected to orbifolding action (r= rank of the orbifold group)

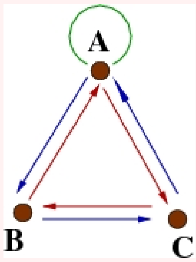
Results in breaking the gauge group $U(r) \rightarrow U(1)^r$ and appearance of bi-fundamental & adjoint fields.



QUIVER GAUGE THEORIES

• Matter content of quiver theories are represented by a quiver diagram:

nodes= # gauge groups=r , # edges= # adjoint & bi-fundamental fields=m



the directed edge from node A to node B implies that a bifundamental field X_{AB} charged -1 with respect to gauge group G_A and $+1$ with respect to gauge group G_B .

Diagram directly gives charge matrix d_{ai} where $a = (1, 2 \ldots r)$ is the gauge group index and $i = (1, 2 \ldots m)$ is the bi-fundamental field index.

These appear in the D-term equations:

$D_a = \sum_i d_{ai}|X_i|^2 - \xi_a$

One of the rows can always be eliminated to give a $(r-1) \times m$ matrix Δ.

• and the interaction is given by a superpotential $W[\{X\}]$

• Geometric data of a class of Calabi-Yau called **toric Calabi-Yau** can be obtained from the quiver gauge theory data- *we will see soon*

These quiver gauge theories are sometimes called **toric quiver gauge theories** where all the gauge groups have same rank and W must contain each bifundamental field in two terms with relative signs:

$$\begin{aligned} W &= X_{BA}X_{AB}X_{BC}X_{AC} - X_{CB}X_{BC}X_{CA}X_{AC} \\ &\quad + X_{BC}X_{CA}X_{AA} - X_{AB}X_{BA}X_{AA} \end{aligned}$$

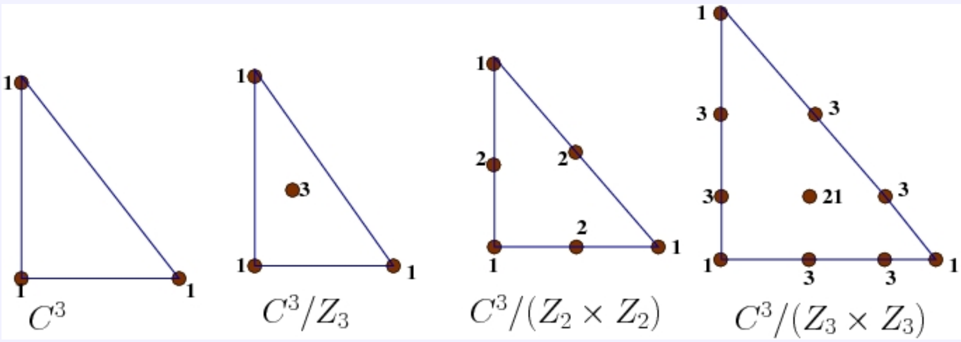
NON-COMPACT TORIC CALABI-YAU SPACES

• Spaces described as T^3 fibration over $\mathbb{R}^3$

(i) can be obtained from partial resolutions of orbifolds of $\mathbb{C}^3$

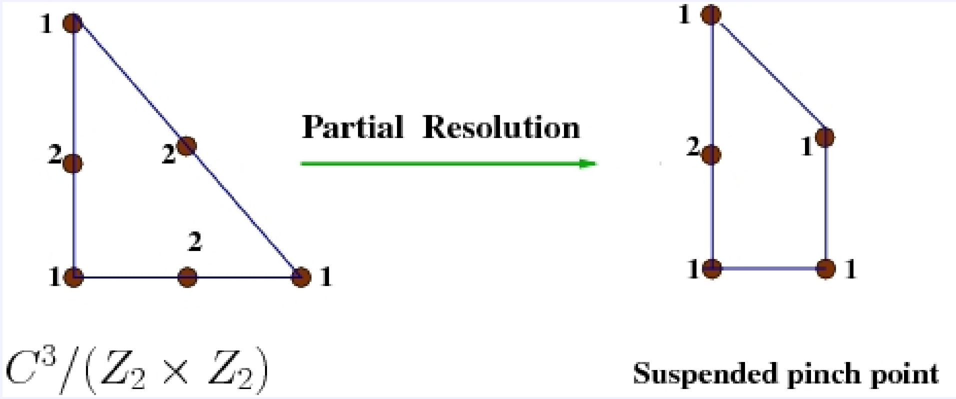
(ii) equivalent description in Witten's gauged linear sigma model

(iii) the toric Calabi-Yau spaces are represented by a toric diagram



numbers against every node i indicate multiplicity of the sigma model fields

partial resolutions of these spaces (**parent spaces**) gives non-orbifold toric Calabi-Yau spaces (**daughter spaces**)



QUIVER GAUGE THEORY $\rightarrow$ TORIC DIAGRAMS

• **Forward algorithm**: Given a gauge theory data (quiver diagram+W) (*Feng, Hanany et al*)
F-term equation

$$\frac{\partial W}{\partial X_i} = 0$$

are all not independent. Can be solved by introducting $r + 2$ new fields v_j as:

$$X_i = \prod_j v_j^{K_{ij}}$$

where $i \in (1, m)$ and $j \in (1, r + 2)$. For W

$$\begin{aligned} W &= X_{BA}X_{AB}X_{BC}X_{AC} - X_{CB}X_{BC}X_{CA}X_{AC} \\ &\quad + X_{BC}X_{CA}X_{AA} - X_{AB}X_{BA}X_{AA} \, , \end{aligned}$$

Calculation of K matrix (analogue of Δ):
Choosing $v_1 = X_{BA}, v_2 = X_{CB}, v_3 = X_{AA}, v_4 = X_{AB}, v_5 = X_{BC}$, F-term equations

$$\begin{aligned} \frac{\partial W}{\partial X_{AA}} = 0 \ \text{gives} X_{CA} = v_1 v_4 v_5^{-1} \, , \\ \frac{\partial W}{X_{BA}} = 0 \ \text{gives} X_{AC} = v_1 v_3 v_5^{-1} \, . \end{aligned}$$

$$K = \begin{array}{c} \\ X_{BA} \\ X_{CB} \\ X_{AA} \\ X_{AB} \\ X_{BC} \\ X_{CA} \\ X_{AC} \end{array} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Find a $(r+2) \times c$ matrix T satisfying $K.T \geq 0$. The $(r+2)$ v_j fields can be obtained in terms of c p_α fields using the T matrix:

$$v_j = \prod_\alpha p_\alpha^{T_{j\alpha}} .$$

We need to eliminate redundant $c - (r+2)$ fields using a $(c-r-2) \times c$ matrix Q_F such that $TQ_F^t = 0$. We say such a Q_F as the co-kernel of T.

Similarly Q_D matrix related to Δ. Rewriting X_i in terms of p_α:

$$X_i = \prod_\alpha p_\alpha^{\sum_j K_{ij} T_{j\alpha}} ,$$

we can see $(K.T)Q_D^t = \Delta^t$.

Concatenating Q_D and Q_F gives matrix Q.
Find G such that $Q.G^t = 0$.
Each column of G is a vector which can be plotted to give a toric diagram:

For this example, G is

$$G = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array}$$

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Converse problem: Given a toric diagram, can we determine the quiver gauge theory uniquely? No

Inverse algorithm: Starting from the data of G for parent space, giving vev's to some of the fields, quiver gauge theories on daughter spaces have been obtained- **does not give the adjoint fields.**

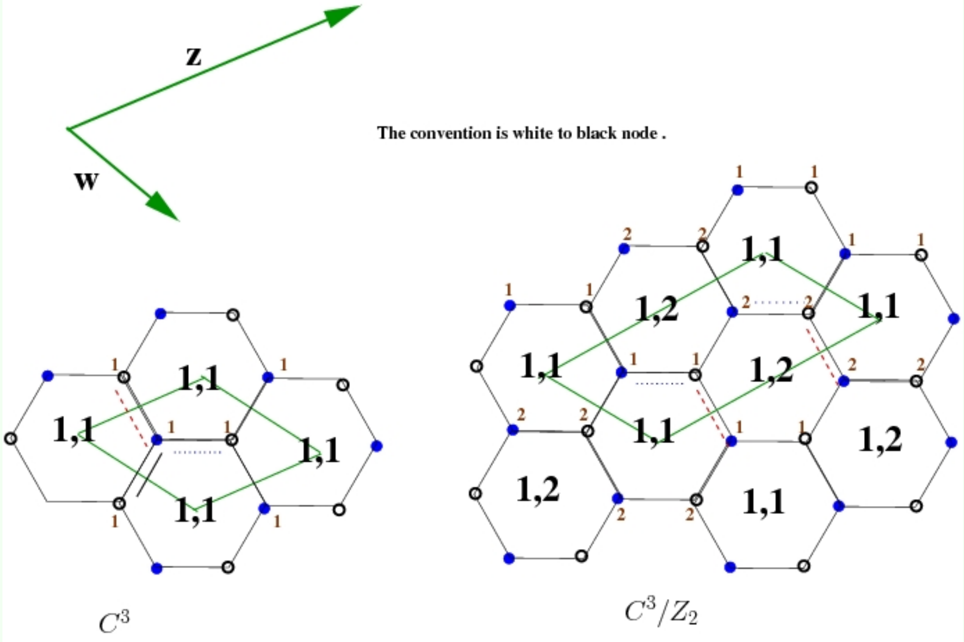
The algorithm not practical to implement for large toric diagrams!

- Is there a combinatorial approach?

Dimer models gives an elegant method of determining gauge-theory data as well as the geometry data.

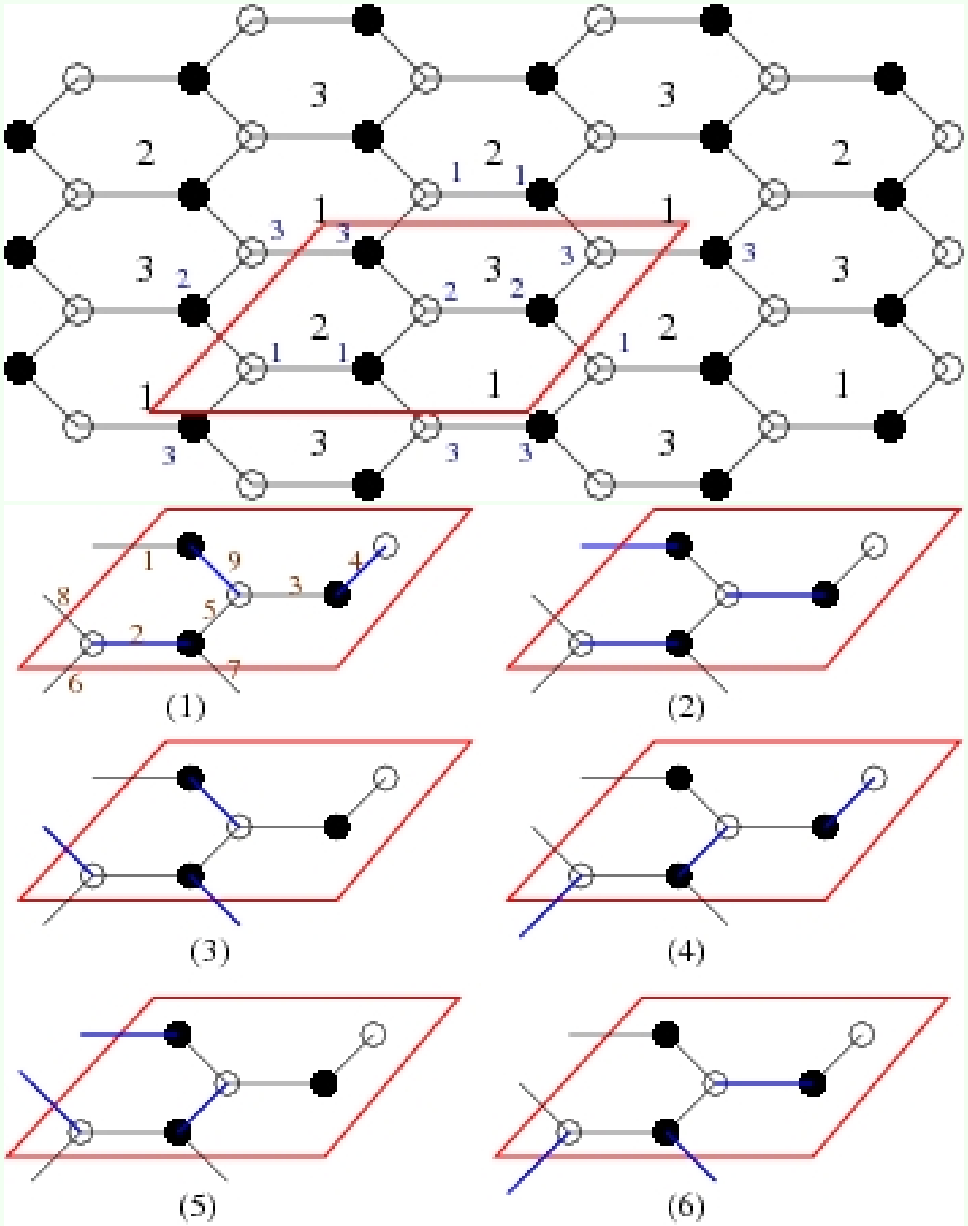
DIMER MODELS

A bipartite graph with vertices colored either black or white such that no two adjacent vertices have the same color. Each edge is called a dimer. These graphs are on a torus and hence periodic as indicated.



For such a graph, we can draw a **fundamental domain** as shown.

Perfect matching : consists of a subset of edges such that every vertex in the fundamental domain is an endpoint of one edge.



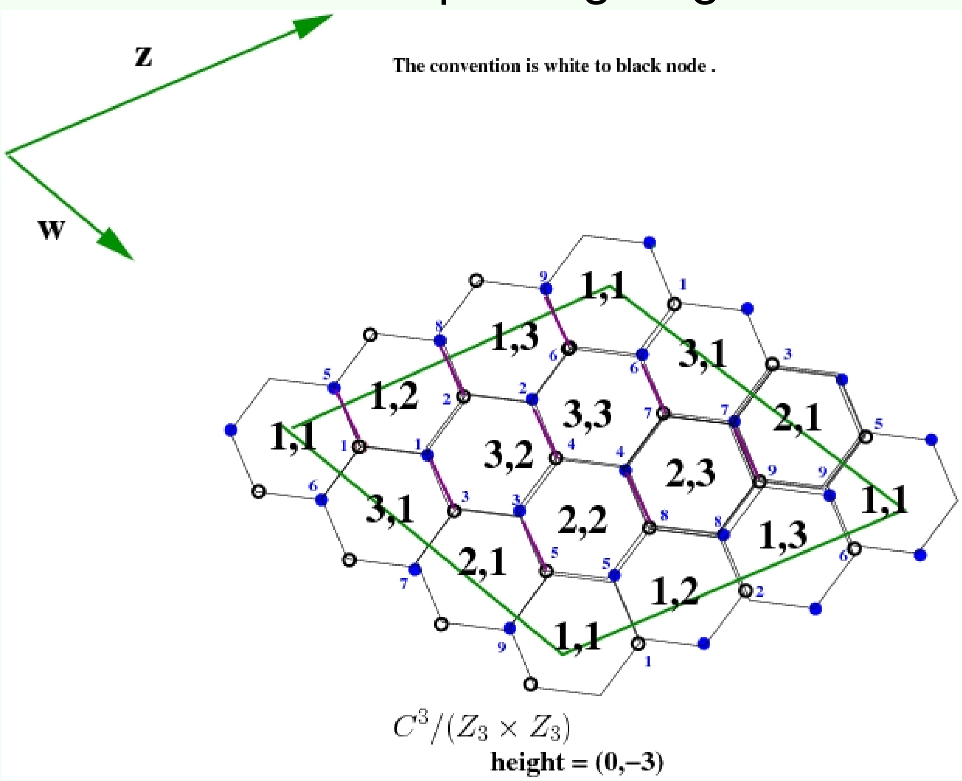
Difference between any two perfect matchings gives a **closed curve**- *faces for example*.

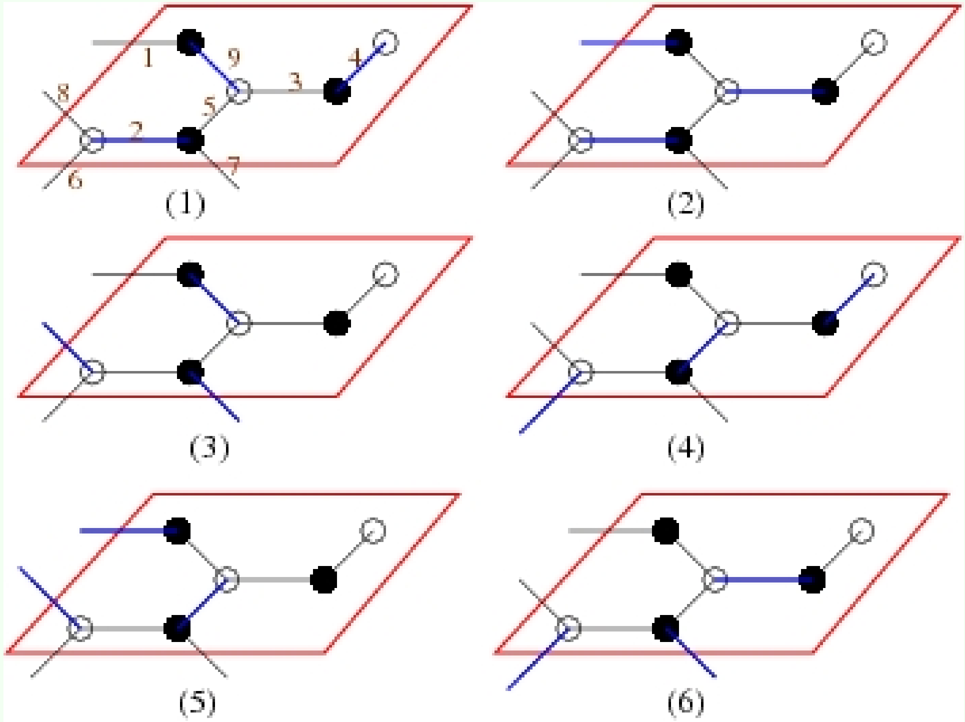
Total number of perfect matchings p_α's enumerated for a dimer diagram will give c GLSM fields automatically.

$\boxed{\text{Perfect matchings contains all information}}$

To determine the toric diagram associated with a a dimer diagram, we introduce what is called **height function** h

To each perfect matching of a dimer diagram, one associates a height function $h = (h_w, h_z)$ and these h's corresponding to all perfect matchings are plotted on a two dimensional plane giving a toric diagram.





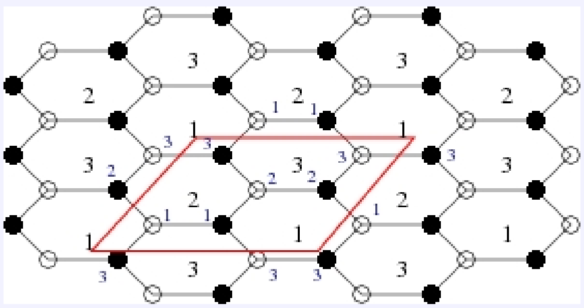
The height function for matchings (1),(5) and (6) are (0,0), and matchings (2), (3) and (4) will have weights $(-1,0)$, $(1,-1)$ and $(0,1)$ respectively.
The internal point in the toric diagram is $(0,0)$ and the multiplicity is 3.

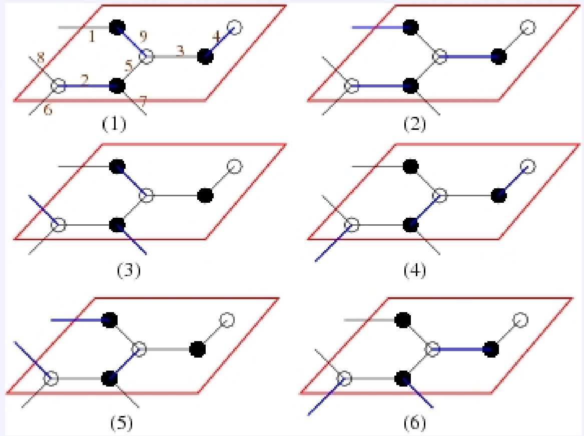
Closed string theory $\leftrightarrow$ dimer model allows giving R-charge vector to every p_α using basis vectors $(1/3,0,0)$, $(0,1/3,0)$,$(0,0,1/3)$. *-T. Sarkar*

The perfect matchings $1,5,6$ will have R-charge as $(1/3,1/3,1/3)$ which corresponds to marginal twisted sector - Here again, internal point corresponds to marginal twisted sector state.

Conjecture: All the face symmetries of the dimer diagram can be written in terms of **perfect matchings corresponding to an internal point in the toric diagram** whenever present- have verified for many examples.

For the dimer diagram and perfect matchings,





we see that the three faces $F_1 = p_5 - p6$, $F_2 = p_1 - p_5$ and $F_3 = p_6 - p_1$.

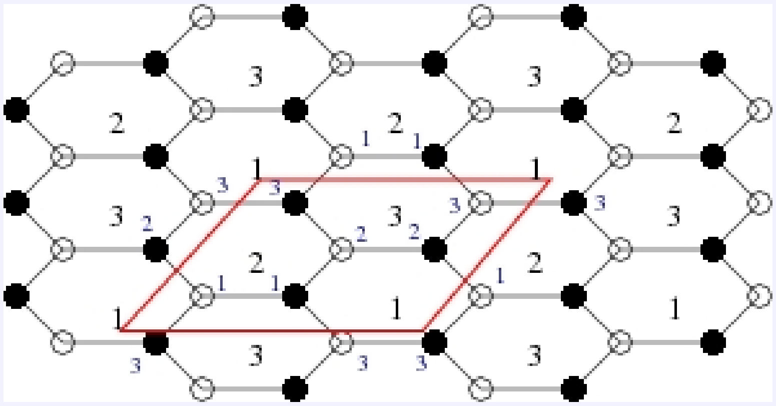
For a face/closed curve, we require the R-charge to be zero which is another proof for the conjecture .

Enumeration of the perfect matchings can be obtained algebraically from the determinant of a matrix called **Kastelyne matrix** $K(z, w)$- *useful for large toric diagrams*

Let a_{ij}^k used as edge-weight for edge k between white vertex i and black vertex j. Then

$$K_{ij}(z,w) = \sum_k a_{ij}^k z^{\langle k,\gamma_z \rangle} w^{\langle k,\gamma_w \rangle}$$

where $\langle k, \gamma_z \rangle$ gives the intersection number of edge k with the oriented γ_z cycle of the torus which will be $\pm 1, 0$.

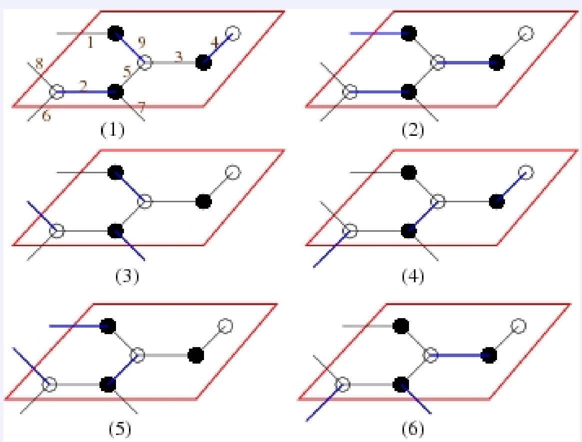


Taking the row index as white-vertex numbers and column index as black-vertex index, the matrix K will be:

$$K(Z,W) = \begin{pmatrix} a11 & a12W & a13Z \\ a21 & a22 & a23 \\ \frac{a31}{Z} & a32 & \frac{a33}{W} \end{pmatrix}$$

The determinant of $K(z,w)$ will give six terms corresponding to six perfect matchings:

$$
\begin{aligned}
det\ K &= -a13a22a31 + a12a23a31\frac{W}{Z} \\
&\quad -a11a23a32 - a12a21a33 \\
&\quad +a13a21a32Z + a11a22a33\frac{1}{W}\,.
\end{aligned}
$$

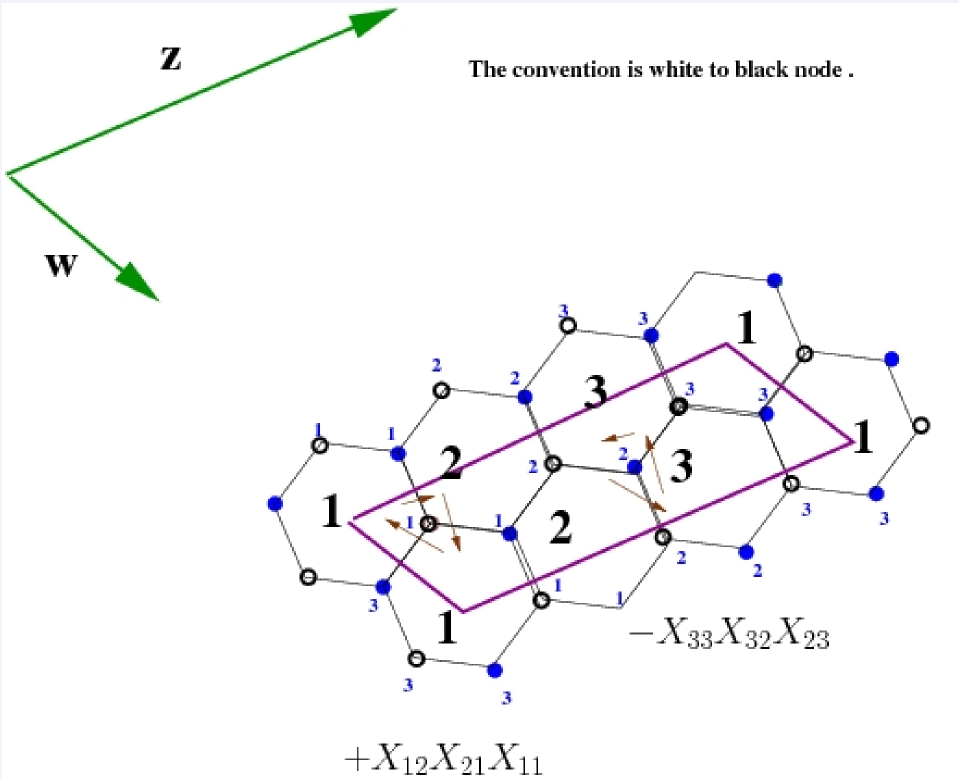


The last term, for example, is an algebraic representation of perfect matching numbered (2).

Quiver Gauge theory $\leftrightarrow$ dimer diagram

Gauge groups	Faces
bi-fundamental fields	edges
superpotential W	vertices

The terms in the superpotential is given by the bifundamental fields intersecting the vertices with a sign convention as follows:



Quiver, GLSM data

The charge matrix

$$d_{ai} = \langle F_a, e_i \rangle$$

$$(Q_D)_{a\alpha} = \langle F_a, p_\alpha \rangle$$

It is convenient to introduce matching matrix

$$\mathcal{M}_{i\alpha} = \langle e_i, p_\alpha \rangle$$

which is 1 if the edge e_i is in the perfect matching p_α. Q_F is given by

$$Q_F \mathcal{M}^t = 0 \ .$$

In terms of matching matrix, the matter fields X_i's can be written as

$$X_i = \prod_\alpha p_\alpha^{\mathcal{M}}{}_{i\alpha}$$

and

$$d = Q_D \mathcal{M}^t \ .$$

Now, we are in a position to do inverse algorithm!

Starting from the dimer diagram for the master space, we can remove one or more edges (this is called Higgsing), we get daughter theories and exotic theories. For these theories, we have verified the conjecture that the face symmetries are obtainable from internal perfect matchings.

Also, the number of edges partipating in any perfect matching are same and hence every row in the matching matrix will have equal number of 1's.

Higgsing procedure

Remove an edge from the diagram and delete the row representing the row from the matching matrix. Also, remove the columns whose entries are 1 for that row. Then find the Q_F.

From Q_F, we can determing T and K and construct $(KT)^t$ which will be the matching matrix for the daughter theory. If the matrix does not have equal number of 1's in every row, we know that it is not a correct theory. Then, we add columns to K such that we get a consistent $\mathcal{M}$. Thus we have a handle of when adjoint fields must be added.

Thank You