Asymptotic Expansion of $\mathcal{N} = 4$ Dyon Degeneracy

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References: (1) arXiv:0807.1314 [hep-th]

(2) arXiv:0810.3472 [hep-th]

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Plan of the talk

Motivation

- Black Holes are solutions of Einstein-Maxwell theory (low energy limit of string theory). They carry certain charges and quantum mechanically behave as thermodynamic objects.
- They can also be described in terms of specific configuration of states in the full string theory, carrying similar set of charges.

- We want to understand the statistical origin of Black Hole entropy, as the logarithm of the degeneracy of these states.

- Major success: for extremal Black Holes,

$$S_{BH} = S_{stat} \equiv \ln d(\vec{Q}), \quad \textit{in large charge}(\vec{Q})\ \textit{limit}.$$

- Can we go beyond large charge limit?

$$i.e. \qquad S_{BH} \stackrel{?}{=} \ln d(\vec{Q}),$$

for large but finite charges.

To do this, we need to take two way approach to the problem.

Gravity Side

- On gravity side, we need to consider all higher derivative and quantum corrections.
  - Higher derivative terms: Entropy function technology can be used.
  - Quantum corrections: Quantum entropy function can be used.
- Entropy corrections come as an expansion in inverse power of charges.

String Theory Side

- We need to compute the degeneracy of states more accurately.

GOAL

To understand these corrections to the entropy in the statistical side by doing systematic asymptotic expansion of the degeneracy function.

Setup

Theory

- We consider $\mathcal{N} = 4$ superstring theory with rank r gauge group.
- At a generic point in the moduli space,the unbroken gauge group is $U(1)^r$.
- The low energy SUGRA theory has a continuous $SO(6, r-6) \times SL(2, R)$ symmetry.
- We denote the $SO(6, r-6)$ invariant metric by L. All inner products are defined with respect to L.

Two Descriptions

First Description

- Type IIB string theory on $K3 \times S^1 \times \tilde{S}^1 / \mathbb{Z}_N$.
- A dyonic state in this theory is a particular brane configuration.

Second Description

- Equivalently heterotic string theory on $T^4 \times S^1 \times \hat{S}^1 / \mathbb{Z}_N$
- A dyonic state in this theory is described by a state carrying some electric and magnetic charges.

Duality

- The two descriptions of the theory are related by a chain of duality transformations as follows:

$$\begin{pmatrix}\textbf{IIB}\\ \mathbf{S^1\times S^1}\end{pmatrix} \overset{\mathbf{S}}{=} \begin{pmatrix}\textbf{IIB}\\ \mathbf{S^1\times S^1}\end{pmatrix} \overset{\mathbf{T}}{=} \begin{pmatrix}\textbf{IIA}\\ \mathbf{S^1\times \tilde{S}^1}\end{pmatrix} \overset{\textbf{st}-\textbf{st}}{=} \begin{pmatrix}\textbf{Heterotic}\\ \mathbf{T^6}\end{pmatrix}$$

Charge Vectors

- In general, any given state is characterized by r dimensional electric and magnetic charge vectors, $\vec{Q}$ and $\vec{P}$.
- The T-duality invariants are,

$$Q^2 = Q^T L Q \qquad P^2 = P^T L P \qquad Q.P = Q^T L P,$$

where L ,T-duality matrix.

Macroscopic Side

- We will consider quarter BPS dyonic Black Holes in the Heterotic theory.
- Restricting to Supergravity approximation, we can find the entropy carried by these Black Holes .

Microscopic Side

- By duality, we can also regard the states associated with this Black Holes as states of some particular quarter BPS D-brane configuration in the type IIB theory.
- Considering the dynamics of various fields in the D-brane configuration, the complete degeneracy function has been evaluated (JHEP 0611:072,2006).

Degeneracy Formula

- The microscopic degeneracy is,

$$d(\vec{Q},\vec{P}) = (-1)^{Q\cdot P+1} A \int_{\mathcal{C}} d\rho\, d\sigma\, dv \, \frac{e^{-\pi i(\rho Q^2+\sigma P^2+2vQ\cdot P)}}{\Phi(\rho,\sigma,v)}$$

- Contour $\mathcal{C}$ is a three real dimensional subspace of the complex dimensional space labeled by (ρ,σ,v).

- For $N=1$ theory,the function $\Phi(\rho,\sigma,v)$ is a modular form of weight 10.
- The analogous modular forms are also known for many other models . (Shamik's Talk)

Asymptotic Expansion

A .

- For a given set of charges, there are single centered and multi centered Black Hole solutions.
- We are Interested in single centered Black Hole entropy.
- We organize the integral such that the result can pick up the contribution from single centered Black holes. This is done by choosing the integration contour $\mathcal{C}$ in a specific way.
- In particular, we need to set the asymptotic values of the moduli fields equal to their attractor values.

B .

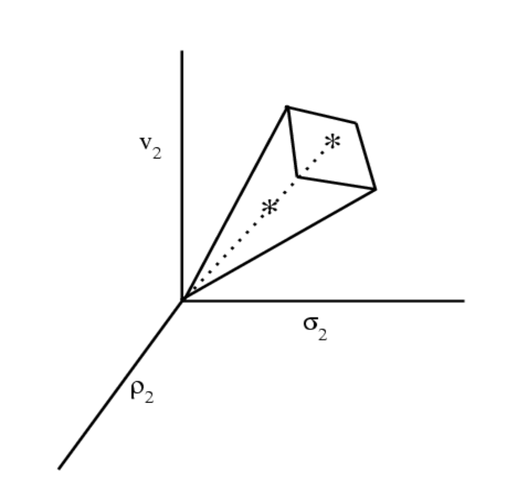
- We have to do three integrals, over (ρ, σ, v). For this,we need the pole of the integrand.
- The function has a second order zero at,

$$n_2(\sigma\rho - v^2) + jv + n_1\sigma - m_1\rho + m_2 = 0$$

for
$m_1, n_1, m_2, n_2 \in \mathbb{Z}, j \in 2\,\mathbb{Z} + 1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}$

- We consider cases with $n_2 \geq 1$.

Pole structure



B.1

- In saddle point approximation of the integral, the degeneracy is,

$$d(\vec{Q}, \vec{P}) = e^{\pi\sqrt{Q^2P^2-(Q.P)^2}/n_2}$$

- For $n_2 = 1$, we get the maximum contribution to the degeneracy.

- For $n_2 \geq 2$, the degeneracy is exponentially suppressed compared to the leading one. Hence, to compute exponentially suppressed contribution, we need to look at these sub-leading poles.

B.2

- For the leading pole, one integral can be done by residue method. The other two integral are done by saddle point analysis.
- The v integral is done by residue method. Near the pole, the function Φ behaves as.

$$\Phi(\rho, \sigma, v) \to v^2 g(\rho) g(\sigma) + O(v^4)$$

- The (ρ, σ) integral takes the form,

$$e^{S_{stat}(\vec{Q},\vec{P})} = d(\vec{Q},\vec{P}) \sim \int \frac{d^2\tau}{\tau_2^2} e^{F(\vec{\tau})}$$

  where, $\rho = \tau_1 + i\tau_2$ and $\sigma = -\tau_1 + i\tau_2$

- The function $F(\tau)$ can be easily computed after doing the v integral.

- This can be regarded as a zero dimensional field theory with fields (τ_1, τ_2) with action $F(\vec{\tau}) - 2\ln\tau_2$.

- The result for statistical entropy S_{stat} can be obtained by computing the possible diagrams of this field theory up to any desired order in charges.

- This will produce all sub-leading correction in inverse power of charges.

Up to $1/charge^2$ corrections

Statistical Entropy Function

- Leading result:

$$\Gamma_0(\vec{\tau}_B) = -\frac{\pi}{2\tau_{B_2}}|Q - \tau_B P|^2$$

- The $1/charge$ corection:

$$\Gamma_1(\vec{\tau}_B) = \ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k+2)\ln(2\tau_{B_2})$$

The $1/charge^2$ correction :

$$\Gamma_2(\vec{\tau}_B) = \ln d_2(\vec{Q}, \vec{P}) = -\frac{\tau_{2B}}{\pi|Q - \tau_B P|^2}(k+2)$$

Up to $1/charge^2$ corrections

Statistical Entropy

- Leading entropy: $S^{(0)} = \pi\sqrt{Q^2P^2 - (Q.P)^2}$
- The $1/charge$ correction:

$$S^{(1)} = -\ln g(\tau_{(0)}) - \ln g(-\bar{\tau}_{(0)}) - (k+2)\ln(2\tau_{(0)_2})$$

- The $1/charge^2$ correction :

$$S^{(2)} = \frac{2+k}{2\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}} + \frac{4\tau^3_{(0)2}}{\pi|Q - \tau_{(0)}P|^2}$$

$$\left[\left(\frac{g'(\tau_{(0)})}{g(\tau_{(0)})} + \frac{k+2}{\tau_{(0)} - \bar{\tau}_{(0)}}\right)\left(\frac{g'(-\bar{\tau}_{(0)})}{g(-\bar{\tau}_{(0)})} + \frac{k+2}{\tau_{(0)} - \bar{\tau}_{(0)}}\right)\right]$$

Exponential Suppressed Correction

- The zeros of the function Φ,

$$n_2(\sigma\rho - v^2) + jv + n_1\sigma - m_1\rho + m_2 = 0$$

- We get such corrections from the sub-leading poles, $n_2 \geq 2$.

- For this we define

$$\Omega = \begin{bmatrix} \rho & v \\ v & \sigma \end{bmatrix}$$

- We look for a symplectic transformation of the form:

$$\begin{bmatrix} \hat{\rho} & \hat{v} \\ \hat{v} & \hat{\sigma} \end{bmatrix} \equiv \hat{\Omega} = (A\,\Omega + B)(C\,\Omega + D)^{-1},$$

such that

$$\hat{v} = \frac{n_2(\sigma\rho - v^2) + jv + n_1\sigma - m_1\rho + m_2}{\det(C\Omega + D)}$$

- The behavior of Φ_k near the zero is,

$$\Phi_k(\rho, \sigma, v) = -\{\det(C\,\Omega + D)\}^{-k}\, 4\pi^2\, \hat{v}^2\, g(\hat{\rho})\, g(\hat{\sigma}) + O(v^4)$$

Degeneracy Formula

- The degeneracy formula for any sub-leading pole is,

$$\frac{\exp\left(\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/n_2\right)}{n_2}\left[\det(C\Omega+D)^{k+2}\,g(\rho)^{-1}\,g(\sigma)^{-1}\right]_{saddle}$$
$$(-1)^{Q\cdot P}\,\exp\left[i\pi(n_1P^2-m_1Q^2+jQ\cdot P)/n_2\right]$$

- To evaluate $\det(C\Omega+D)^{k+2}g(\rho)^{-1}\,g(\sigma)^{-1}$, we actually need the transformation matrix A,B,C,D.

- We will now compare the entropy with the leading pole results.

Comparison

Q^2, P^2	$Q.P$	$d(Q,P)$	S_{stat}	$S_{stat}^{(0)}$	$S_{stat}^{(1)}$	$S_{stat}^{(2)}$	Δd
2	0	5×10^4	10.82	6.28	10.62	11.58	34.6
4	0	3×10^7	17.31	12.57	16.90	17.38	480.6
6	0	1×10^{10}	23.51	18.85	23.19	23.51	18573
6	2	4×10^9	22.15	17.77	21.94	22.20	27652
6	-2	2×10^9	21.77	17.77	21.94	22.20	-

Macroscopic Understanding

Power suppressed correction

- Power suppressed corrections are identified to the α'/g_{string} correction to Black Hole macroscopic entropy.
- This Entropy function is just the value of the corresponding six derivative term in the Black Hole action computed on the $AdS_2 \times S^2$ background.
- We do not know a candidate for this term in the action. Our Analysis tells us that the term has to be duality invariant and puts a strong constraint to the possible terms.
- We have also been able to eliminate terms like R^3 as they gives zero result.

Exponentially Suppressed Correction

- These corrections naturally come from Quantum Entropy Function.

Quantum Entropy Function

- This is a proposal for computing the exact degeneracy of states of an extremal Black Holes (arXiv:0809.3304 [hep-th]).
- These Black Holes have the following near horizon geometry.

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad F_{rt}^{(i)} = -ie_i, \quad \cdots$$

v, e_i are constants, ... denotes near horizon values for other fields.

- The degeneracy is given as,

$$d(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

  - where $\langle\ \rangle_{AdS_2}$ denotes the unnormalized path integral over various fields of string theory on euclidean global AdS_2.
  - The superscript '*finite*' refers to the finite part of the amplitude. To get this, we need to put a cut of to regularize the AdS volume.

- Explicit computation shows that this proposal reproduces the right classical degeneracy of states for quarter BPS Black Holes in $\mathcal{N} = 4$ theories as,

$$d(q) \simeq \exp\left(\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}\right).$$

Possible Quantum Corrections

- There are two sources of quantum corrections.
  - From fluctuations of the string fields around AdS_2 background.
  - There can be different classical solutions with similar asymptotic configuration.

Fluctuation of the Background

- The degeneracy is actually given as the finite part of the amplitude in the AdS_2 background, and hence can only get power law corrections from fluctuation modes.

Different Solutions

- The different solution can come with a different action, and hence we can get a different exponential factor.
- Q. Can we identify such a different solution?

Solution

- We consider a $\mathbb{Z}_N$ quotient of the previous background by,

$$\theta \to \theta + \frac{2\pi}{N}, \quad \phi \to \phi - \frac{2\pi}{N} \tag{1}$$

- The new solution looks like,

$$ds^2 = v\left((\tilde{r}^2 - 1)d\tilde{\theta}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - 1}\right), \quad F^{(i)}_{\tilde{r}\tilde{\theta}} = -i\, e_i, \quad \cdots,$$

$$\tilde{\theta} \equiv \tilde{\theta} + \frac{2\pi}{N}$$

- In a new coordinate $r = \tilde{r}/N, \quad \theta = N\tilde{\theta}$, the solution looks as,

$$ds^2 = v\left((r^2 - N^{-2})d\theta^2 + \frac{dr^2}{r^2 - N^{-2}}\right), \quad F^{(i)}_{r\theta} = -i\,e_i, \quad \cdots$$

$$\theta \equiv \theta + 2\pi, \quad \phi \to \phi - \frac{2\pi}{N}$$

- This has the same asymptotic behavior as the original solution.

- The finite contribution to the quantum entropy function is,

$$d(\vec{Q}, \vec{P}) = \exp\left(\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/N\right)$$

- We recover the correct exponentially sub-leading correction identifying $N = n_2$.

Results

- We have shown that the degeneracy formula is valid for a generic quarter BPS dyonic Black Hole state in $\mathcal{N} = 4$ theory.
- We have explored possible power suppressed and exponentially suppressed corrections to the microscopic degeneracy formula.
- We have also identified the roots of these corrections in the Black Hole macroscopic entropy.
- The exponentially suppressed corrections to the black hole entropy is universal, does not depend on the particular kind of extremal Black Holes.

THANK YOU