DYON PARTITION FUNCTION
OF
THE STU MODEL

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INTRODUCTION

There is a good understanding of 1/4 BPS dyons in a class of $\mathcal{N} = 4$ theories.

- The partition function which counts the degeneracy of the dyons are given in terms of inverse of Siegel modular forms.

- There is an exact counting relying on string dualities which results in these Siegel modular forms.

- The asymptotic degeneracy for large charges evaluated from these partition functions agrees with the leading and sub-leading entropy of the corresponding black hole.
On the black hole side this requires taking into account the effect of the Gauss-Bonnet term.

- The degeneracy obeys the wall crossing formula at walls of marginal stability.

For $\mathcal{N} = 2$ theories at present we do not have such a detailed understanding.

- The question of writing down partition functions for black holes in a general $\mathcal{N} = 2$ theory might be difficult task.

There is a special class of $\mathcal{N} = 2$ theories which are closely related to $\mathcal{N} = 4$ theories in which this question might be tractable.

- These models are constructed by freely acting orbifolds of a parent $\mathcal{N} = 4$ theories.

The simplest example of such a theory is the STU model

Sen & Vafa(1995); Gregori, Kounnas & Petropoulos(1999)

- It is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of type IIB on $T^4 \times S^2 \times \tilde{S}^1$

g_1: $(-1)^{F_L}$ with $\frac{1}{2}$ shift on S^1

g_2: Inversion of T^4 with $\frac{1}{2}$ shift on $\tilde{S}^1$

- No of vector multiplets: S, T, U

- The vector multiplet moduli space is known to be exact:

$$\left|\frac{SU(1,1)}{U(1)}\right|_S \times \left|\frac{SU(1,1)}{U(1)}\right|_T \times \left|\frac{SU(1,1)}{U(1)}\right|_U$$

- The STU model has the following set of discrete symmetries:

$$\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$$

$$\Gamma(2): \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad ad - bc = 1; a, c \in 2\mathbb{Z}; a, d \in 2\mathbb{Z} + 1$$

with the action:

$$\Gamma(2)_S: \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

and tirality symmetry:

$$\tau \leftrightarrow y^+ \leftrightarrow -\frac{1}{y^-}$$

• The coefficient of the Gauss-Bonnet term is known:

Gregori, Kounnas, Petropoulos (1999)

$$\frac{1}{128\pi^2}\left(-4\log\left[\vartheta_2^2(\tau)\vartheta_2^2(-\bar{\tau})(\tau_2)\right]-4\log\left[\vartheta_2^2(y^+)\vartheta_2^2(-\bar{y}^+)(y_2^+)\right]\right.$$
$$\left.-4\log\left[\vartheta_2^2\left(-\frac{1}{y^-}\right)\vartheta_2^2\left(\frac{1}{\bar{y}^-}\right)\left(-\frac{1}{y^-}\right)_2\right]\right)$$

Note: The coefficient of the R^2 exhibits the discrete symmetries of the model.

In comparison with $\mathcal{N}=4$ models, this coefficient depends on all vector multiplet moduli.

We are interested in partition function of dyons with charges (Q,P) on $S^1\times\tilde{S}^1$

IN THIS TALK

- We propose a partition function of dyons in the STU model respecting all the discrete symmetries of the model

- It reproduces the degeneracy of black holes including the sub-leading term arising due to the Gauss-Bonnet term.

- It agrees with the OSV form of the partition function on performing the Laplace transform with respect to the electric charges.
The form of the measure in this mixed partition function agrees with that proposed by Cardoso, de Wit, Mahapatra (2008) for general $\mathcal{N} = 2$ models.

- We specify a region in the vector multiplet moduli space for which this partition function captures the degeneracy of dyons.

THE PARTITION FUNCTION

The partition function involves the product of inverses of three Sp(2, $\mathbb{Z}$) modular forms of weight zero.

$$
\begin{aligned}
d(Q,P) &= I(K_s,L_s,M_s)\times I(K_t,L_t,M_t)\times I(K_u,L_u,M_u) \\
I(K_s,L_s,M_s) &= \frac{A}{4}\exp\left(i\pi M_s\right)\int_{\mathcal{C}} d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v}\, \frac{1}{\widetilde{\Phi}_0(\tilde{\rho},\tilde{\sigma},\tilde{v})} \\
&\times \exp\left(-2\pi i\left[\frac{K_s}{2}\tilde{\rho}+\frac{L_s}{2}\tilde{\sigma}+M_s\tilde{v}\right]\right).
\end{aligned}
$$

and similar definitions for with $s \longrightarrow t$, and $s \longrightarrow u$.

We now define all quantities in the above formula:

- K_s, L_s, M_s are charge bi-linears which transform as vectors on S-duality:

$$3K_s = P \cdot P, \qquad 3L_s = Q \cdot Q, \qquad 3M_s = Q \cdot P$$

$\cdot$ refers to inner product with signature $(+,+,-,-)$
(K_t, L_t, M_t), (K_u, L_u, M_u) are obtained from by the action of the triality symmetry on the charge bilinears. K_s, L_s, M_s.
They transform as vectors under T-duality and U-duality respectively.

- Thus the integral is manifestly invariant under triality symmetry.

- The modular form of weight zero is given by:

$$\widetilde{\Phi}_0 = \widetilde{\Phi}_2 \sqrt{\frac{\widetilde{\Phi}'_2}{\widetilde{\Phi}_6}}.$$

The modular forms $\widetilde{\Phi}_2, \widetilde{\Phi}_6$ of weight $2, 6$ captures dyon degeneracy in $\mathcal{N} = 4$ models discussed earlier.

$\widetilde{\Phi}'_2$ is related to $\widetilde{\Phi}_2$ by an $Sp(2, \mathbb{Z})$ transformation.

$$\widetilde{\Phi}'_2(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \widetilde{\Phi}_2(\frac{\tilde{\sigma}}{2}, 2\tilde{\rho}, \tilde{v})$$

There is an explicit product representation of all these modular forms

- It can be shown that $\widetilde{\Phi}_0$ is an analytic function and has no branch cuts.

The complete list of its poles and zeros is known.

- A is the normalization -2^{-10} such that the coefficients of the Fourier

expansion are integers:

$$\frac{A}{\widetilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v})} = \sum_{\substack{m,n,p \\ m\geq -1/2, n\geq 1/2}} e^{2\pi i(m\tilde{\rho}+n\tilde{\sigma}+p\tilde{v})} g(m, n, p),$$

with $g(m,n,p)$ being integers, $p \in \mathbb{Z}$ while $m,n \in \mathbb{Z}/2$.

• Thus the Fourier expansion guarantees that the degeneracy is given by an integer. This also implies that the degeneracy formula is valid for

$$K_{s,t,u} \in \mathbb{Z}, \qquad L_{s,t,u} \in \mathbb{Z}, \qquad M_{s,t,u} \in \mathbb{Z}$$

• The 3-dimension contour $\mathcal{C}$ is given by

$$\mathrm{Im}\,\tilde{\rho} = M_1,\ \mathrm{Im}\,\tilde{\sigma} = M_2,\ \mathrm{Im}\,\tilde{v} = M_3,$$
$$0 \le \mathrm{Re}\,\tilde{\rho} \le 2,\ 0 \le \mathrm{Re}\,\tilde{\sigma} \le 2,\ 0 \le \mathrm{Re}\,\tilde{v} \le 1.$$

• It can be shown that the integrand in the degeneracy formula is invariant under the $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$ symmetry.

• The partition function for the STU model is given by product of 3 $SP(2, \mathbb{Z})$ modular forms of weight 0.

ASYMPTOTIC DEGENERACY

• The form of the partition function is same as that of the $\mathcal{N} = 4$ partition function, but with a product of three such integrals.
We can adopt the same procedure developed in the $\mathcal{N} = 4$ case for each of the integrals.

• Consider one of the integrals:

$$
\begin{aligned}
I(K_s, L_s, M_s) &= \frac{A}{4} \exp\left(i\pi M_s\right) \int_{\mathcal{C}} d\tilde{\rho} d\tilde{\sigma} d\tilde{v} \frac{1}{\widetilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v})} \\
&\times \exp\left(-2\pi i \left[\frac{K_s}{2}\tilde{\rho} + \frac{L_s}{2}\tilde{\sigma} + M_s \tilde{v}\right]\right).
\end{aligned}
$$

For large charges $K_s, L_s, M_s >> 0$ with $K_s L_s - (M_s)^2 >> 0$. the dominant contribution arises from the double pole at

$$\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v} = 0$$

At this pole the modular form factorizes as

$$\tilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = 4\pi^2 v^2 \vartheta_2^4(\rho) \vartheta_2^4(\sigma)$$

where

$$\rho = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2}{\tilde{\sigma}}, \qquad \sigma = \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v} - 1)^2}{\tilde{\sigma}}, \qquad v = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}}{\tilde{\sigma}},$$

Note: The factors $\vartheta_2^4(\rho)$ are precisely what is present in the coefficient of the Gauss-Bonnet term.

- After picking up the residue at the double pole, the remaining two integrals can be performed by a saddle point method.

This leads to minimizing the following function with respect to (γ_1, γ_2)

$$-\tilde{\Gamma}(\vec{\gamma}) = \frac{\pi}{2\gamma_2}\left(L_s + 2\gamma_1 M_s + (\gamma\bar{\gamma})K_s\right) - \ln\left[\vartheta_2(\gamma)^4 \vartheta_2(-\bar{\gamma})^4 (2\gamma_2)^2\right] + \text{constant} + \mathcal{O}(1/Q^2),$$

To $O(Q^0))$ it is sufficient to obtain the value of γ at the minimum by minimizing the $O(Q^2)$ term.

This is given by

$$\gamma_1|_*^s = -\frac{L_s}{M_s}, \qquad \gamma_2|_*^s = \frac{\sqrt{K_s L_s - M_s^2}}{K_s},$$

Substituting this value we obtain

$$-\tilde{\Gamma}(\vec{\gamma}^{\alpha})|_{*} \;=\; \pi\sqrt{K_s L_s - M_s^2} - \ln\left[\vartheta_2(\gamma)^4 \vartheta_2(-\bar{\gamma})^4 (2\gamma_2)^2\right]\Big|_{*} + \text{constant} + \mathcal{O}(1/Q^2),$$

● Performing the similar analysis for the remaining two integrals and summing their contributions we obtain

$$-\tilde{\Gamma}_B(\vec{\gamma}^{s}, \vec{\gamma}^{t}, \vec{\gamma}^{u})|_{*} \;=\; \pi\sqrt{Q^2 P^2 - (Q\cdot P)^2} - \sum_{\alpha=1}^{3} \ln\left[\vartheta_2(\gamma^{\alpha})^4 \vartheta_2(-\bar{\gamma}^{\alpha})^4 (2\gamma_2^{\alpha})^2\right]\Big|_{*} + \text{constant} + \mathcal{O}(1/Q^2).$$

Here $\alpha \in \{s, t, u\}$. $\gamma^t|_*, \gamma^u|_*$ are the values obtained by minimizing the

$O(Q^2)$ term in Γ_B.

They are given by similar equations as $\gamma^s|_*$.

$$\gamma_1|_*^t = -\frac{L_t}{M_t}, \qquad \gamma_2|_*^t = \frac{\sqrt{K_t L_t - M_t^2}}{K_t},$$

$$\gamma_1|_*^u = -\frac{L_u}{M_u}, \qquad \gamma_2|_*^u = \frac{\sqrt{K_u L_u - M_u^2}}{K_u},$$

- Note: The asymptotic degeneracy can be shown to be invariant under triality and $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$.

ENTROPY FUNCTION ANALYSIS

● Going through the entropy function analysis at the two derivative terms in the supergravity action one obtains the following function of the STU-moduli τ, y^+, y^- which has to be minimized.

$$F = \frac{\pi}{2}\left[\frac{|(Q+\tau P)\cdot w|^2}{\tau_2 Y} + \frac{|(Q+\bar{\tau} P)\cdot w|^2}{\tau_2 Y} - \frac{(Q+\tau P)\cdot(Q+\bar{\tau}P)}{\tau_2}\right],$$

where $\cdot$ is inner product with $(+,+,-,-)$, w is a four vector which depends on the T, U moduli.

$$w_1 = -1 + y^+ y^-, \qquad w_2 = y^+ + y^-,$$
$$w_3 = y^+ - y^-, \qquad w_4 = 1 + y^+ y^-.$$

• The critical values of the moduli at this 2-derivative level are given by

$$\begin{aligned} \tau_1\big|_* &= -\frac{L_s}{M_s}, \qquad \gamma_2\big|_* = \frac{\sqrt{K_s L_s - M_s^2}}{K_s}, \\ y_1^+\big|_* &= -\frac{L_t}{M_t}, \qquad y_2^-\big|_* = \frac{\sqrt{K_t L_t - M_t^2}}{K_t}, \\ -\frac{1}{y^-{}_1}\bigg|_* &= -\frac{L_u}{M_u}, \qquad -\frac{1}{y_2^-}\bigg|_* = \frac{\sqrt{K_u L_u - M_u^2}}{K_u} \end{aligned}$$

• Evaluating the entropy function at the 2-derivative level at this minimum we obtain

$$F_T = \pi\sqrt{Q^2 P^2 - (Q \cdot P)^2}$$

- We now incorporate the 4-derivative Gauss-Bonnet term.

To evaluate the entropy to $O(Q^0, P^0)$ it is sufficient to substitute the value of the moduli at the attractor values in the leading order. This results in

$$\begin{aligned} F &= \pi\sqrt{Q^2P^2-(Q\cdot P)^2} \\ &\quad -\ln\left[\vartheta_2(\tau)^4\vartheta_2(-\bar\tau)^4(\tau_2)^2\right]|_* - \ln\left[\vartheta_2(y^+)^4\vartheta_2(-\bar y^+)^4(\tau_2)^2\right]|_* \\ &\quad -\ln\left[\vartheta_2(-1/y^-)^4\vartheta_2(1/\bar y^-)^4((-1/y^-)_2)^2\right]|_* + \mathcal{O}(1/Q^0, 1/P^0). \end{aligned}$$

- Comparing with the entropy obtained from the partition function we see that they coincide to this order in charges.

THE MIXED PARTITION FUNCTION

• For general $\mathcal{N} = 2$ theories Ooguri, Strominger, Vafa (2004) suggested that the mixed partition function

$$Z(p,\phi) = \sum_{\{q\}} d(p,q)\, \mathrm{e}^{\pi\, q_I \phi^I}\ ,$$

a Laplace transform with respect to the electric charges is related to the topological partition function.

$$Z(p,\phi) \sim \mathrm{e}^{\pi\, \mathcal{F}_{\mathrm{E}}(p,\phi)} = |Z_{\mathrm{top}}(p,\phi)|^2\ .\ ,$$

• $\mathcal{F}_{\rm E}(p,\phi)$ for the STU model is

$$\begin{aligned}\mathcal{F}_{\rm E}(p,\phi) &= 4{\rm Im}(F(Y)) \\ &\quad -\frac{1}{\pi}\left[\ln\omega(p^0,p^1,\phi^0,\phi^1)+\ln\omega(p^0,p^2,\phi^0,\phi^2)\right. \\ &\quad \left.+\ln\omega(p^0,p^3,\phi^0,\phi^3)\right] \ .\end{aligned}$$

where

$$F(Y) = -\frac{Y^1Y^2Y^3}{Y^0}, \qquad Y^I = \frac{1}{2}(\phi^I + ip^I)$$

$I = 0,1,2,3.$ Just to recall the S,T,U moduli are given by

$$iS = \frac{Y^1}{Y^0}, \qquad iT = \frac{Y^2}{Y^0}, \qquad iU = \frac{Y^3}{Y^0}$$

and

$$\omega(p^0, p^1, \phi^0, \phi^1) = \vartheta_2^4(iS)\vartheta_2^4(i\bar{S}),$$
$$\omega(p^0, p^2, \phi^0, \phi^2) = \vartheta_2^4(iT)\vartheta_2^4(i\bar{T})$$
$$\omega(p^0, p^1, \phi^0, \phi^3) = \vartheta_2^4(iU)\vartheta_2^4(i\bar{U})$$

- We would like to see if degeneracies given by the partition function earlier agrees with this expectation.

• We can show that the mixed partition function for the STU model for large p^I and potentials ϕ^I is given by

$$
\begin{aligned}
Z(p,\phi) &\sim e^{\pi \mathcal{F}_E(p,\phi)+\mathcal{K}} \\
\mathcal{K} &= -\ln[(S+\bar{S})(T+\bar{T})(U+\bar{U})] \\
&= -\ln\left[\frac{\mathrm{i}(\bar{Y}^I\partial_I F - Y^I\bar{\partial}_I\bar{F})}{|Y^0|^2}\right],
\end{aligned}
$$

This form of the mixed partition function agrees precisely with that suggested by Cardoso, de Wit and Mahapatra recently.

Outline of the strategy to obtain the mixed partition function

- The Laplace transform

$$Z_s(p,\phi) = \sum_q d(K_s, L_s, M_s) e^{\pi q_I \phi^I}$$

where $d(K,L,M)$ are degeneracies of $\mathcal{N}=4$ models given in terms of inverse $Sp(2,\mathbb{Z})$ modular forms has been done earlier.

Cardoso, de Wit, Kappeli and Mohaupt (2004, 2006); Shih and Yin (2005).

- The proposed degeneracy formula for the STU model is the product of three such terms

$$d(K_s, L_s, M_s) d(K_t, L_t, M_t) d(K_u, L_u, M_u)$$

We need to evaluate

$$Z_{\rm STU}(p,\phi) = \sum_{\{q\}} d(K_s, L_s, M_s) d(K_t, L_t, M_t) d(K_u, L_u, M_u)\, \mathrm{e}^{\pi q_I\, \phi^I}$$

- All the charge bi-linears in the above formula are functions of (q,p).

$$\begin{aligned} 3\,K_s &= \langle P, P\rangle_s = -\,2(p^0 q_1 + p^2 p^3)\ , \\ 3\,L_s &= \langle Q, Q\rangle_s = 2(q_0 p^1 - q_2 q_3)\ , \\ 3\,M_s &= \langle P, Q\rangle_s = q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3\ . \end{aligned} \qquad (1)$$

From the fact that $\{K_s, L_s, T_s\} \in \mathbb{Z}$ we see that

$$p^{1,2,3}, q_0 \in \lambda^{-1}\,\mathbb{Z}\,, \qquad p^0, q_{1,2,3} \in \lambda\,\mathbb{Z}\,,$$

where $\lambda = \sqrt{2}$ or $\frac{1}{\sqrt{2}}$.

•Therefore we have to decouple the sum over q occurring in the Laplace transform to use the earlier results.

• This is done by introducing the delta function.

$$
\begin{aligned}
Z_{\mathrm{STU}}(p,\phi) \;=\; & \sum_{\{q,q',q''\}} \delta_{q,q'}\,\delta_{q',q''} \\
& \times d_0(K_s,L_s,M_s)\,d_0(K_t,L_t,M_t)\,d_0(K_u,L_u,M_u) \\
& \times \mathrm{e}^{\frac{\pi}{3}\left[\left(q_0+q'_0+q''_0\right)\phi^0+\left(q_1+q'_1+q''_1\right)\phi^1+\left(q_2+q'_2+q''_2\right)\phi^2+\left(q_3+q'_3+q''_3\right)\phi^3\right]}
\end{aligned}
$$

Then rewrite the delta functions in terms of their integral representation

$$
\delta_{mn} = \int_0^1 \mathrm{d}\theta\; \mathrm{e}^{2\mathrm{i}\pi(m-n)\theta}\,,
$$

This leads to

$$Z_{\mathrm{STU}}(p,\phi) = \int_0^1 \mathrm{d}^4\theta \, \mathrm{d}^4\varphi \; Z_s(p,\phi_s)\, Z_t(p,\phi_t)\, Z_u(p,\phi_u) \ ,$$

where

$$\begin{aligned} {\phi_s}^0 &= \phi^0 + 6\mathrm{i}\lambda\, \theta^0 \ , & {\phi_s}^{1,2,3} &= \phi^{1,2,3} + 6\mathrm{i}\lambda^{-1}\, \theta^{1,2,3} \ , \\ {\phi_t}^0 &= \phi^0 + 6\mathrm{i}\lambda\, (\varphi^0 - \theta^0) \ , & {\phi_t}^{1,2,3} &= \phi^{1,2,3} + 6\mathrm{i}\lambda^{-1}\, (\varphi^{1,2,3} - \theta^{1,2,3}) \\ {\phi_u}^0 &= \phi^0 - 6\mathrm{i}\lambda\, \varphi^0 \ , & {\phi_u}^{1,2,3} &= \phi^{1,2,3} - 6\mathrm{i}\lambda^{-1}\, \varphi^{1,2,3} \ . \end{aligned}$$

Observe that

$${\phi_s}^I + {\phi_t}^I + {\phi_u}^I = 3\,\phi^I \ .$$

- The Laplace transform of each of the components is given by

$$
\begin{aligned}
& Z_s(p,\phi_s) \approx \\
& \quad = \exp\left\{ \tfrac{1}{3}\pi[\mathcal{F}_0(p,\phi_s)] - [\ln\omega(p^0,p^1,\phi_s^0,\phi_s^1)] \right. \\
& \qquad\qquad \left. -[\mu_s(p,\phi_s)] \right\} \ ,
\end{aligned}
$$

where in the limit of where p^I, ϕ^I are large

$$
\mathrm{e}^{-\mu_s(p,\phi)} \equiv \frac{(p^2\phi^0 - p^0\phi^2)(p^3\phi^0 - p^0\phi^3)}{((\phi^0)^2 + (p^0)^2)(\phi^0 p^1 - \phi^1 p^0)} = \frac{(T+\bar{T})(U+\bar{U})}{2\,(S+\bar{S})} \,,
$$

And similar cyclic relations for with $s \longrightarrow t, s \longrightarrow u$

• The final integral over θ, φ can be performed by saddle point approximation.

It can be shown that the the leading contribution for large p and ϕ occurs at $\theta^I = \varphi^I = 0$.

• Performing 8-dimensional Gaussian integral one obtains the determinant which is given by $\exp(2\mathcal{K})$.

• We also have

$$\exp[-\mu_s(p, \phi) - \mu_t(p, \phi) - \mu_u(p, \phi)] = \frac{1}{8} \exp[-\mathcal{K}]$$

- Combining all this results in

$$Z_{\rm STU}(p,\phi) \approx \, {\rm e}^{\pi\, \mathcal{F}_{\rm E}(p,\phi)+\mathcal{K}}\,,$$

- This agrees with the proposed form of the mixed partition function given by Cardoso, de Witt and Mahapatra.

DOMAIN OF VALIDITY OF THE DEGENERACY FORMULA

• A remarkable feature of the dyonic degeneracy formula for $\mathcal{N} = 4$ models is that its from remains the same across walls of marginal stability (determined by asymptotic moduli), though the number of BPS states jumps across such walls.

• Consider the decay

$$(Q, P) \to (Q, 0) + (0, P)$$

The wall crossing formula states that the jump across such a wall is given by

$$d_>(Q, P) - d_<(Q, P) = (Q \cdot P)\,(-1)^{(Q \cdot P)+1}\, d_{\rm el}(Q)\, d_{\rm mag}(P)\,,$$

• The reason the $\mathcal{N} = 4$ degeneracy formula is able to reproduce these jumps is due to the factorization property of the modular form across the divisor $v = 0$.

$$\Phi_k(\rho, \sigma, v) \sim 4\pi^2 \, v^2 \, g_{\rm mag}^{(k)}(\rho) \, g_{\rm el}^{(k)}(\sigma) \, ,$$

- The STU model also admits the same wall of marginal stability at which the dyon decays to purely electric and purely magnetic states.

Does the partition function factorize in similar form?

- The partition function is a product of three modular forms.

There are three divisors $v_s = 0, v_t = 0, v_u = 0$. At say divisor $v_s = 0$ and

$v_t, v_u \neq 0$ the degeneracy formula factorizes as

$$\Phi_0(\rho_s, \sigma_s, v_s)\ \Phi_0(\rho_t, \sigma_t, v_t)\ \Phi_0(\rho_u, \sigma_u, v_u)$$
$$\sim 4\pi^2\, {v_s}^2\ \frac{\eta^8(2\rho_s)}{\eta^4(\rho_s)}\ \frac{\eta^8(\sigma_s/2)}{\eta^4(\sigma_s)}\ \Phi_0(\rho_t, \sigma_t, v_t)\ \Phi_0(\rho_u, \sigma_u, v_u)\ .$$

The contribution of this double pole to the degeneracy is of the form

$$M_s\, (-1)^{M_s+1}\ d_1(K_s)\ d_2(L_s)\ I_0(K_t, L_t, M_t)\ I_0(K_u, L_u, M_u)\ ,$$

where

$$d_1(K_s) = \oint \mathrm{d}\rho\ \frac{\mathrm{e}^{\mathrm{i}\pi K_s \rho}}{\eta^8(2\rho)\, \eta^{-4}(\rho)}\,, \qquad d_2(L_s) = \oint \mathrm{d}\sigma \frac{\mathrm{e}^{\mathrm{i}\pi L_s \sigma}}{\eta^8(\sigma/2)\, \eta^{-4}(\sigma)}\ .$$

This does not obey the expected wall crossing formula.

The same conclusion holds at other divisors $v_t = 0, v_u = 0$, or combinations.

- This suggests that the degeneracy formula is valid only in the region of asymptotic moduli where the single centered black hole is stable.

- The region in the neighbourhood of the point where the asymptotic moduli is tuned to the attractor moduli is one such region.