# BPS Black Holes Effective Actions and the Topological String

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## 1: N=2 BPS black holes

effective action

attractor phenomena

entropy function & free energy

subleading & non-holomorphic corrections

## 2: Partition functions and OSV

mixed partition function

topological string versus the effective action

## 3: Non-holomorphic deformation

equivalence classes

## 4: Summary / conclusion

# *N=2 BPS* black holes

*N=2 supergravity:* *vector multiplet sector*

vector multiplets $\longrightarrow$ scalars $X^I$ (Wilsonian effective action)

projectively defined: $X^I \longrightarrow Y^I$

Lagrangian encoded in a holomorphic homogeneous function $F(\lambda Y) = \lambda^2 F(Y)$

**BPS** : attractor phenomena

full supersymmetry enhancement at the horizon

$\longrightarrow$ *extremal*

versus extremal non-supersymmetric black holes

*We don't have to work in terms of the (complicated) effective actions!*

# Attractor equations *(horizon behaviour)*

$Y^I - \bar{Y}^I = \mathrm{i}p^I$ magnetic charges

$F_I - \bar{F}_I = \mathrm{i}q_I$ electric charges

Ferrara, Kallosh, Strominger, 1996
Cardoso, dW, Käppeli, Mohaupt, 2000

homogeneity: entropy and area are proportional to $Q^2$

$$\frac{R_{\mathrm{hor}}}{l_{\mathrm{s}}} \sim g_{\mathrm{s}}\, Q$$

large $Q \longrightarrow$ macroscopic black hole

"large black hole"

and consistent with E/M duality

duality: equivalence classes or invariances

# BPS entropy function

$$\Sigma(Y,\bar{Y},p,q) = \mathcal{F}(Y,\bar{Y}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I)$$

$X + \bar{X}^I$ and $F_I + \bar{F}_I$ play the role of electro- and magnetostatic potentials at the horizon

$\mathcal{F}(Y,\bar{Y})$ 'free energy'

$\delta\Sigma = 0 \Leftrightarrow$ attractor equations $\det[\mathrm{ImF_{IJ}}] \neq 0$

$\pi\Sigma\big|_* = \mathcal{S}_{\mathrm{macro}}(p,q)$ entropy, like the area,
scales quadratically in the charges

*subleading corrections ??*

# Higher-derivative interactions

chiral class: Weyl background $F(Y) \longrightarrow F(Y,\Upsilon)$

Weyl background

homogeneity: $\boxed{F(\lambda Y, \lambda^2\Upsilon) = \lambda^2\, F(Y,\Upsilon)}$

attractor equations remain valid and $\boxed{\Upsilon = -64}$

$\Upsilon$ dependence induces $R^2$-terms in the action and subleading corrections in area and entropy

$$\boxed{F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1}(Y^0)^{2-2g}\Upsilon^g\, F^{(g)}(t)}$$

subleading corrections

$t^A = Y^A/Y^0$

# Free energy :

$$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = -\mathrm{i}\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right) - 2\mathrm{i}\left(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_\Upsilon\right)$$

Cardoso, dW, Käppeli, Mohaupt, 2006

example :

$$F(Y,\Upsilon) = -\frac{1}{6}\,\frac{C_{ABC}\,Y^A Y^B Y^C}{Y^0} - \frac{c_{2A}\,Y^A}{24\cdot 64\,Y^0}\,\Upsilon$$

leads indeed to the microscopic result

Cardoso, dW, Mohaupt, 1998

$$S_{\mathrm{macro}} = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\left(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A\right)}$$

triple intersection number

second Chern class

$c_{2A}$ subleading correction !

Maldacena, Strominger, Witten, 1997
Vafa, 1997

membrane charges :

$$\hat{q}_0 = q_0 - \tfrac{1}{2}C^{AB}q_A q_B \qquad C_{AB} = C_{ABC}\,p^C$$

$p^0 = 0$

dictated by symmetry

area/entropy $\begin{cases} \sim Q^2\Big\{1+\mathcal{O}(\Upsilon/Q^2)\Big\} \\ \sim Q\sqrt{\Upsilon}\Big\{1+\mathcal{O}(\Upsilon/Q^2)\Big\} \end{cases}$

$\dfrac{R_{\rm hor}}{l_{\rm s}} \sim g_{\rm s}\, Q \;\; \gg 1$ large/macroscopic

$\dfrac{R_{\rm hor}}{l_{\rm s}} \sim g_{\rm s}\, \sqrt{Q} \;\; \approx 1$ small/microscopic ⇒ elementary string states

*Tested extensively for N=4 supersymmetric string compactifications*

*(in N=2 formulation)*

# ***Problem*** : Non-holomorphic corrections

So far: holomorphicity ⇒ 'standard' SG Lagrangians

Wilsonian effective action (integrated out modes with cutoffs)

**integrating out massless modes leads to non-local terms**
**holomorphicity is lost !**

non-holomorphic corrections are required:

- to realize certain symmetries  Dixon, Kaplunovsky, Louis, 1991
- background dependence of topological string

Bershadsky, Cecotti, Ooguri, Vafa, 1994

*The full non-local action is not known !*

# Early example : *N=4 supersymmetry with S-duality*

Cardoso, dW, Mohaupt, 1999

$$F = -\frac{Y^1}{Y^0} Y^a \eta_{ab} Y^b + \frac{\mathrm{i}}{256\,\pi}\Big[\Upsilon \log \eta^{12}(S) + \boxed{\bar\Upsilon \log \eta^{12}(\bar S)} + \tfrac{1}{2}(\Upsilon + \bar\Upsilon)\, \boxed{\log(S+\bar S)^6}\Big]$$

$$\mathrm{i}S = \frac{Y^1}{Y^0}$$

harmonic

non-holomorphic

required by S-duality

related to threshold correction

Harvey, Moore, 1996

More general decomposition:

$$F = -\tfrac{Y^1}{Y^0} Y^a \eta_{ab} Y^b + 2\mathrm{i}\Omega$$

real, homogeneous

$\begin{pmatrix} Y^I \\ F_I \end{pmatrix}$ transforms as $\begin{pmatrix} p^I \\ q_I \end{pmatrix}$ under duality rotations (monodromies)

This determines the transformation of $\Omega$

The function $F$ is not invariant!

# Furthermore microscopic results *1/4* BPS states

dyonic degeneracies

$$d_k(p,q) = \oint_{3-\mathrm{cycle}} \mathrm{d}\Omega \, \frac{\mathrm{e}^{\mathrm{i}\pi[\rho\, p^2 + \sigma\, q^2 + (2\upsilon - 1) p\cdot q]}}{\Phi_k(\Omega)}$$

$k = 10, 6, 4, 2, 1$

$\Omega = \begin{pmatrix} \rho & \upsilon \\ \upsilon & \sigma \end{pmatrix}$ period matrix of g=2 Riemann surface

formally S-duality invariant

Dijkgraaf, Verlinde, Verlinde, 1997
Shih, Strominger, Yin, 2005
Jatkar, Sen, 2005

Leading degeneracy for large charges :

make saddle-point approximation on a leading divisor

The result is identical as that obtained on the basis of :

$$\Sigma(S,\bar S,p,q) = -\frac{q^2 - \mathrm{i}p\cdot q(S-\bar S) + p^2|S|^2}{S+\bar S} + 4\,\Omega(S,\bar S,\Upsilon,\bar\Upsilon)$$

Cardoso, dW, Käppeli, Mohaupt, 2004

# Partition functions and OSV

$$Y^I = \frac{\phi^I + \mathrm{i}p^I}{2} \qquad \begin{cases} \phi^I & \text{electrostatic potentials} \\ p^I & \text{magnetic charges} \end{cases}$$

(mixed ensemble)

⇒ reduced entropy functions $\Sigma = \mathcal{F}_{\mathrm{E}} - q_I \phi^I$

where

$$\mathcal{F}_{\mathrm{E}}(p,\phi) = 4\left[\mathrm{Im}\, F(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - \Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\right]_{Y^I=(\phi^I+\mathrm{i}p^I)/2}$$

***Topological string:***

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1} (Y^0)^{2-2g}\,\Upsilon^g\, F^{(g)}(t)$$

$(Y^0)^2 F^{(0)}(t)$

genus-*g* partition function of a twisted non-linear sigma model with CY target space

$t^A = Y^A/Y^0$

$Y^0$ loop-counting parameter

$$\left.\begin{aligned} Z_{\rm BH}(p,\phi) &= {\rm e}^{\mathcal{F}_{\rm E}} \\ Z_{\rm top}(p,\phi) &= {\rm e}^{-2{\rm i}F} \end{aligned}\right\} \qquad Z_{\rm BH}(p,\phi) \approx |Z_{\rm top}(p,\phi)|^2$$

Strominger, Ooguri, Vafa, 2004

*Topological string:*

Bershadsky, Cecotti, Ooguri, Vafa, 1994

- Holomorphic anomaly $\partial_{\bar t} F^{(g)} \neq 0 \Rightarrow F^{(g)}(t,\bar t)$
- Topological string coupling : $Y^0 = g_{\rm top}^{-1}$
- Duality invariant sections $F^{(g)}$
- $F^{(g)}$ captures certain string amplitudes

Antoniadis, Gava, Narain, Taylor, 1993

Note : identification with the effective action !

Non-holomorphic extension ?

Cardoso, dW, Käppeli, Mohaupt, 2006

# The mixed partition function

$$Z_{\mathrm{BH}}(p,\phi) = \sum_{\{q\}} d(p,q)\, \mathrm{e}^{\pi\, q_I \phi^I} \sim \mathrm{e}^{\pi\, \mathcal{F}_{\mathrm{E}}(p,\phi)}$$

inverse Laplace transform :

$$d(p,q) \propto \int \mathrm{d}\phi\; \mathrm{e}^{\pi\, (\mathcal{F}_{\mathrm{E}} - q_I \phi^I)} = \int \mathrm{d}\phi\; \mathrm{e}^{\pi\, \Sigma(\phi,p,q)}$$

saddle-point approximation

$$\delta(\mathcal{F}_{\mathrm{E}} - q_I \phi^I) = 0 \qquad\qquad q_I = \frac{\partial \mathcal{F}_{\mathrm{E}}}{\partial \phi^I}$$

$$\mathcal{S}_{\mathrm{macro}} = \pi\, \Sigma\Big|_{*}$$

❖ integrals ill-defined (contour, convergence)

❖ E/M duality problematic

## improving :

define a duality invariant canonical partition function

$$Z(\phi,\chi) = \sum_{\{p,q\}} d(p,q)\,\mathrm{e}^{\pi[q_I\phi^I - p^I\chi_I]}$$

defines a free energy (naturally formulated as a function of the electro- and magnetostatic potentials $\phi$ and $\chi$ )

## inverse Laplace transform:

$$d(p,q) \propto \int \mathrm{d}\chi_I\,\mathrm{d}\phi^I\;Z(\phi,\chi)\,\mathrm{e}^{\pi[-q_I\phi^I + p^I\chi_I]}$$

over periodicity intervals $(\phi-\mathrm{i},\phi+\mathrm{i})$ $(\chi-\mathrm{i},\chi+\mathrm{i})$

## identify with the field-theoretic data:

$$Z(\phi, \chi) \sim \mathrm{e}^{2\pi\,\mathcal{H}(\phi/2,\chi/2,\Upsilon,\bar{\Upsilon})}$$

**Hesse potential** : Legendre transform of $\mathrm{Im}[F]$ with respect to $(Y-\bar{Y})^I$

equal to $\frac{1}{2}\mathcal{F}$ as defined previously, including the $\Upsilon$-dependence

$$\sum_{\{p,q\}} d(p,q)\,\mathrm{e}^{\pi[q_I\phi^I - p^I\chi_I]} \sim \sum_{\text{shifts}} \mathrm{e}^{2\pi\,\mathcal{H}(\phi/2,\chi/2,\Upsilon,\bar{\Upsilon})}$$

## complex formulation:

$$\sum_{\{p,q\}} d(p,q)\,\mathrm{e}^{\pi[q_I(Y+\bar{Y})^I - p^I(\hat{F}+\bar{\hat{F}})_I]} \sim \sum_{\text{shifts}} \mathrm{e}^{\pi\,\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon})}$$

## inverse Laplace transform:

$$\begin{aligned} d(p,q) &\propto \int \mathrm{d}(Y+\bar{Y})^I\,\mathrm{d}(\hat{F}+\bar{\hat{F}})_I\;\mathrm{e}^{\pi\,\Sigma(Y,\bar{Y},p,q)} \\ &\propto \int \mathrm{d}Y\,\mathrm{d}\bar{Y}\,\boxed{\Delta^-(Y,\bar{Y})}\,\mathrm{e}^{\pi\,\Sigma(Y,\bar{Y},p,q)} \end{aligned}$$

measure factor : implied by duality!

$$\Delta^{\pm}(Y,\bar{Y}) = \left|\det\left[\mathrm{Im}\,[2\,F_{KL} \pm 2\,F_{K\bar{L}}]\right]\right|$$

Cardoso, dW, Käppeli, Mohaupt, 2006

# Saddle-point approximation

$$d(p,q) = \sqrt{\left|\frac{\Delta^-(Y,\bar{Y})}{\Delta^+(Y,\bar{Y})}\right|_{\text{attractor}}} \quad \mathrm{e}^{\mathcal{S}_{\text{macro}}(p,q)}$$

(semiclassical approximation)

$$\left.\frac{\Delta^-(Y,\bar{Y})}{\Delta^+(Y,\bar{Y})}\right|_{\text{attractor}} \approx 1$$

required by duality

mixed partition function :

$$Z(p,\phi) = \sum_{\{q\}} d(p,q)\, \mathrm{e}^{\pi\, q_I \phi^I} \sim \sum_{\text{shifts}} \sqrt{\Delta^-(p,\phi)}\; \mathrm{e}^{\pi\, \mathcal{F}_{\mathrm{E}}(p,\phi)}$$

(and higher-order corrections!)

modification of OSV ⇒ predictive power is lost !

These results have been confirmed in a variety of applications, e.g.,

Shih, Yin, 2005
Cardoso, dW, Käppeli, Mohaupt, 2006
Denef, Moore, 2007
Cardoso, David, de Wit, Mahapatra, 2008

# A more subtle question :

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1}(Y^0)^{2-2g}\Upsilon^g\, F^{(g)}(t)$$

this same expansion is applied to
(a) the topological string and (b) the effective action !
*But are they identical functions ?*

**NO!** (and still agreement with string amplitudes?)

use duality arguments :

*effective action*

the $F^{(g)}$ are NOT invariant
the periods transform correctly under monodromies
the duality transformations are $\Upsilon$-dependent

*topological string*

the $F^{(g)}$ are INVARIANT sections
the periods refer to $F^{(0)}$
the duality transformations are $\Upsilon$-independent

***difference has been confirmed:***

- $\mathcal{F}^{(1)}$ is still invariant
- for $g \geq 2$ there are differences

explicit evaluation and comparison of the non-holomorphic corrections for the FHSV model supports this conclusion.

consistent with the reality of $\Omega$

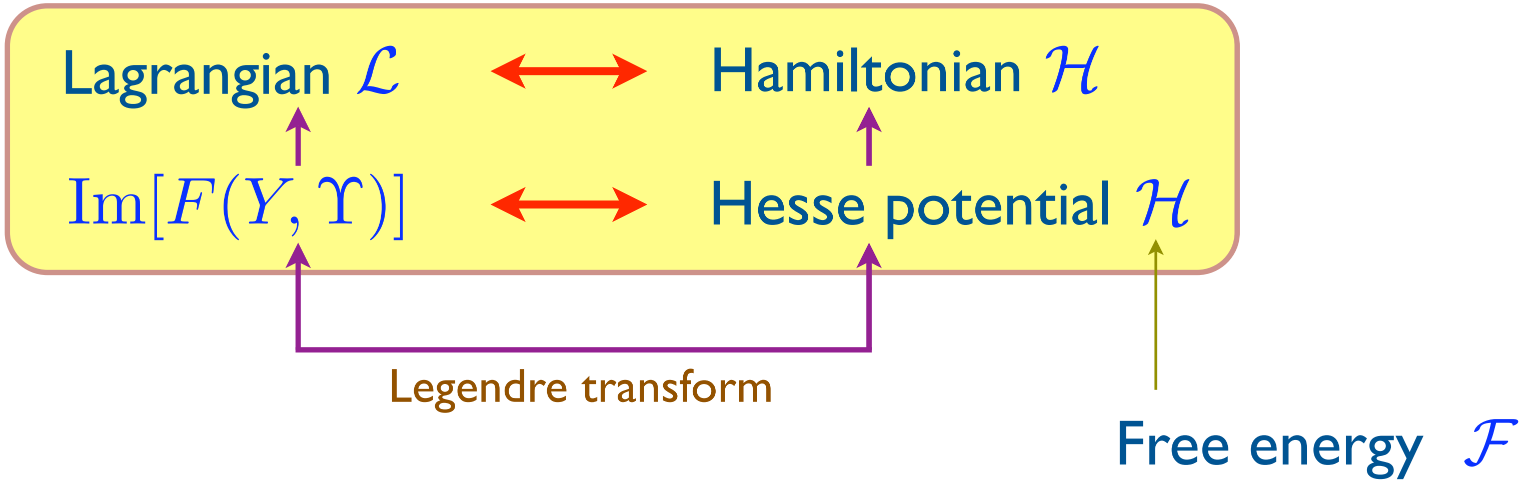
Grimm, Klemm, Marino, Weiss, 2007

Cardoso, dW, Mahapatra, 2008

What then is the precise relation?

☛ Recall : amplitudes $\Leftrightarrow$ connected graphs $\nLeftrightarrow$ ***1PI*** graphs

☛ Then : compare with E/M duality properties of



$$\begin{tikzcd}
\text{Lagrangian } \mathcal{L} \arrow[r, leftrightarrow] & \text{Hamiltonian } \mathcal{H} \\
\operatorname{Im}[F(Y,\Upsilon)] \arrow[u] \arrow[r, leftrightarrow] \arrow[r, bend right, leftrightarrow, "\text{Legendre transform}"'] & \text{Hesse potential } \mathcal{H} \arrow[u] \\
& \text{Free energy } \mathcal{F} \arrow[u]
\end{tikzcd}$$

Legendre transform

Free energy $\mathcal{F}$

Compare : electromagnetism $\mathcal{L}(E, B)$

$$\boxed{\begin{matrix}(E \leftrightarrow H) \\ (B \leftrightarrow D)\end{matrix}}$$

under E/M duality

transformations depend on the details of the Lagrangian (which is not invariant)

$$\left.\begin{aligned} D &= \frac{\partial \mathcal{L}}{\partial E} \\ H &= \frac{\partial \mathcal{L}}{\partial B} \end{aligned}\right\} \text{depend on the details of the Lagrangian}$$

$\mathcal{H}(D, B)$ invariant under monodromies

transformations do not depend on the details of the Lagrangian

# Example : Born-Infeld Lagrangian

$$\mathcal{L} = -g^{-2}\sqrt{\det[\eta_{\mu\nu} + g\,F_{\mu\nu}]} + g^{-2}$$

spherical symmetry

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\sin^2\theta\,\mathrm{d}\varphi^2 + \mathrm{d}\theta^2)$$

$$F_{rt} = e \qquad F_{\varphi\theta} = p\,\sin\theta$$

$$\begin{aligned} \mathcal{L}_{\mathrm{red}} &= \int \mathrm{d}\varphi\,\mathrm{d}\theta\,\mathcal{L} \\ &= 4\pi r^2 g^{-2}\left[\sqrt{1-g^2e^2}\sqrt{1+g^2p^2r^{-2}} - 1\right] \end{aligned}$$

symmetry:

$$\left\{ \begin{aligned} \delta e &= p\sqrt{\frac{1-g^2e^2}{1+g^2p^2}} \\ \delta p &= -e\sqrt{\frac{1+g^2e^2}{1-g^2p^2}} \end{aligned} \right.$$

(suppress: $4\pi, r$)

define electric charge $q = \frac{\partial \mathcal{L}_{\mathrm{red}}}{\partial e}$

Hamiltonian $\mathcal{H} = g^{-2}\left[\sqrt{1 + g^2(p^2 + q^2)} - 1\right]$

symmetry: $\begin{cases} \delta q & = & p \\ \delta p & = & -q \end{cases}$

Schrödinger, 1935

independent of the coupling constant $g$ !

# Non-holomorphic deformation

$$F \longrightarrow F + 2\mathrm{i}\Omega$$

***Special geometry :***

N=2 supersymmetric gauge theory encoded in function $F(X)$

complex scalar fields $X^I$ and $F_I = \dfrac{\partial F(X)}{\partial X^I}$

period vector $\begin{pmatrix} X^I \\ F_I \end{pmatrix} \Longleftrightarrow \displaystyle\oint_{A^I, B_J} \Omega$

electric/magnetic duality (monodromies):

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U^I{}_J & Z^{IJ} \\ V_I{}^J & W_{IJ} \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix}$$

$\mathrm{Sp}(2n, \mathbb{R})$

integrable :

$$\tilde{F}(\tilde{X}) - \tfrac{1}{2}\tilde{X}^I \tilde{F}_I(\tilde{X}) = F(X) - \tfrac{1}{2}X^I F_I(X) + \cdots$$

with non-holomorphic deformation :

The starting point: monodromies

$$\begin{array}{rcl} X^I & \rightarrow & \tilde{X}^I = U^I{}_J X^J + Z^{IJ} F_J(X,\bar{X}) \\ F_I(X,\bar{X}) & \rightarrow & \tilde{F}_I(\tilde{X},\bar{\tilde{X}}) = V_I{}^J F_J(X,\bar{X}) + W_{IJ} X^J \end{array}$$

so that

$$\frac{\partial \tilde{X}^I}{\partial X^J} \equiv \mathcal{S}^I{}_J = U^I{}_J + Z^{IK} F_{KJ} \qquad \frac{\partial \tilde{X}^I}{\partial \bar{X}^J} = Z^{IK} F_{K\bar{J}}$$

As a first result we derive ($F_{IJ} \equiv \partial_J F_I$) :

$$F_{IJ} \rightarrow \tilde{F}_{IJ} = (V_I{}^L \hat{F}_{LK} + W_{IK})\, [\hat{\mathcal{S}}^{-1}]^K{}_J$$

where

$$\begin{array}{rcl} \hat{F}_{IJ} & = & F_{IJ} - F_{I\bar{K}}\, \bar{\mathcal{Z}}^{\bar{K}\bar{L}}\, \bar{F}_{\bar{L}J} \\ \hat{\mathcal{S}}^I{}_J & = & U^I{}_J + Z^{IK} \hat{F}_{KJ} \\ \mathcal{Z}^{IJ} & = & [\mathcal{S}^{-1}]^I{}_K\, Z^{KJ} \end{array}$$

Assume: $F_{I\bar J} = \mathrm{e}^{\mathrm{i}\alpha}\,\bar F_{\bar J I}$

so that $\hat F_{IJ} = \hat F_{JI}$ provided that $\boxed{F_{IJ} = F_{JI}}$

In that case: $\boxed{\tilde F_{IJ} = \tilde F_{JI}}$

so that both $F_I$ and $\tilde F_I$ can be integrated:

$$F_I = \frac{\partial F}{\partial X^I} \qquad\qquad \tilde F_I = \frac{\partial \tilde F}{\partial \tilde X^I}$$

One may also derive

$$F_{I\bar J} \to \tilde F_{I\bar J} = [\hat{\mathcal S}^{-1}]^K{}_I\,[\bar{\mathcal S}^{-1}]^{\bar L}{}_{\bar J}\,F_{K\bar L} = [\mathcal S^{-1}]^K{}_I\,[\bar{\hat{\mathcal S}}^{-1}]^{\bar L}{}_{\bar J}\,F_{K\bar L}$$

as well as similar formulae for higher derivatives !

Furthermore, with external parameter dependence $\eta$ :

$$F(X,\bar X;\eta) = F^{(0)}(X) + 2\mathrm{i}\Omega(X,\bar X;\eta)$$

one derives $\boxed{\partial_\eta \tilde F(\tilde X, \tilde{\bar X};\eta) = \partial_\eta F(X,\bar X;\eta)}$

i.e. transforms as a function !

# Summary / conclusions

$\boxed{F \longrightarrow F + 2\mathrm{i}\Omega}$ *seems consistent with special geometry*

- *free energy $\mathcal{F}$ duality invariant*
- *BPS attractor equations with non-holomorphic terms*
- *E/M equivalence classes seem to be realized*

  ➞ *$\mathcal{F}$ transforms as a function*

- *confirmed by explicit results for FHSV and STU models*
- *prediction for measure factor in class of N=2 models*
- *the measure factor for the STU model → Justin David's talk*

*Precise relation 'effective action ⇔ topological string' remains open*

***Many open questions !***