Extremal Black Hole Entropy

Most of the material and references can be found in

A.S. arXiv:0708.1270, arXiv: 0809.3304

Collaborators: Dumitru Astefanesei, Nabamita Banerjee, Shamik Banerjee, Atish Dabholkar, Justin David, Kevin Goldstein, Rajesh Gupta, Norihiro Iizuka, Ashik Iqubal, Dileep Jatkar, Rudra Jena, Bindusar Sahoo, Masaki Shigemori, Yogesh Srivastava

One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states.

$$S_{BH} = \ln d_{micro}$$

Strominger, Vafa

$S_{BH} = A/4G_N$, $\quad A =$ Area of event horizon

d_{micro}: degeneracy of microstates carrying a given set of charges

Originally the comparison between black hole and statistical entropy was carried out in the limit of large charges.

Can we go beyond this limit?

In order to study this problem we need to address two separate issues.

1. We need to learn how to take into account the effect of the higher derivative terms / quantum corrections on the computation of black hole entropy.

2. We also need to know how to calculate the microscopic degeneracy to greater accuracy.

In this talk I shall try to address both these issues.

Although supersymmetry may not be essential for our discussion, in string theories with sufficient amount of supersymmetry we often have good control on both sides of the story.

For this reason our explicit analysis will focus on BPS black holes with sufficient amount of unbroken supersymmetries.

Plan:

- Higher derivative corrections (concrete)
- Quantum corrections (postulate)
- Microstate counting (concrete)
- Comparison (concrete / speculative)

Higher derivative corrections

A general frameork for computing higher derivative corrections to classical black hole entropy has been developed by Wald.

This has been applied to compute the entropy of stringy black holes in many examples.

Cardoso, de Wit, Mohupt, . . .

More generally, for extremal black holes Wald's formula can be encoded in the entropy function formalism.

We shall begin with a lightening review of the results of the entropy function formalism.

Q. How do we define an extremal black hole in a higher derivative theory of gravity?

Consider Reissner-Nordstrom metric in D=4:

$$\begin{aligned} ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\ &\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\ &\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Define

$$2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\,\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}$$

and take $\lambda \to 0$ limit.

$$ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$ds^2 = \rho_+^2 \left[-(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right] + \rho_+^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$\rightarrow$ an $AdS_2 \times S^2$ space.

All known extremal black holes have an AdS_2 factor in their near horizon geometry.

$\leftrightarrow$ an $SO(2,1)$ isometry of the near horizon geometry.

Postulate: An extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry even in higher derivative theories of gravity + other fields.

Regarding all other directions (including angular coordinates) as compact we can regard the near horizon geometry of an extremal black hole as

$AdS_2 \times$ a compact space (fibered over AdS_2)

Consider string theory in such a background containing two dimensional metric $g_{\mu\nu}$ and $U(1)$ gauge fields $A_\mu^{(i)}$ among other fields.

The most general field configuration consistent with $SO(2,1)$ isometry:

$$ds^2 \equiv g^{(2)}_{\mu\nu} dx^\mu dx^\nu = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F^{(i)}_{rt} = e_i, \quad \cdots\cdots\cdots$$

$\mathcal{L}^{(2)}(v, \vec{e}, \cdots)$: The dimensionally reduced two dimensional Lagrangian density evaluated in this background.

Define the classical entropy function

$$\mathcal{E}(\vec{q}, v, \vec{e}, \cdots) \equiv 2\pi \left(e_i \, q_i - v \, \mathcal{L}^{(2)} \right)$$

One finds that for a black hole of charge $\vec{q}$

1. All the near horizon parameters are obtained by extremizing $\mathcal{E}$ with respect to v, e_i and the other near horizon parameters.

2. $S_{wald}(\vec{q}) = \mathcal{E}$ at this extremum.

This reduces the computation of Wald entropy of an extremal black hole into an algebraic problem once the higher derivative corrections to the action are known.

Q. What is the generalization of Wald's entropy for extremal black holes in the full quantum theory?

On general grounds one expects that this will involve some quantum computation in the near horizon geometry of the black hole.

Quantum entropy function $d(\vec{q})$ is a proposal for this generalization of $e^{S_{wald}(\vec{q})}$.

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F^{(i)}_{rt} = e_i$$

Euclidean continuation:

$t = -i\theta, \quad r = \cosh\eta, \quad \theta \equiv \theta + 2\pi, \quad 0 \leq \eta < \infty$

This gives

$$\begin{aligned} ds^2 &= v\left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \\ F^{(i)}_{\theta\eta} &= ie_i \sinh\eta \end{aligned}$$

$$\rightarrow \quad A^{(i)}_\theta = -i\,e_i\,(\cosh\eta - 1) = -i\,e_i\,(r-1)\,.$$

Proposal for the quantum entropy function $d(\vec{q})$

$$d(\vec{q}) = \left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

$\langle\ \rangle_{AdS_2}$: unnormalized path integral over various fields of string theory on euclidean global AdS_2.

$\oint$: a closed contour at the boundary of AdS_2.

'finite': Infrared finite part of the amplitude.

We need to regularize the infinite volume of AdS_2 by putting a cut-off $r \le r_0$.

The superscript '$finite$' refers to the finite part of the amplitude defined by expressing it as

$$e^{CL} \times \text{finite part}$$

L: length of the boundary of AdS_2.

C: A constant

The definition is independent of the details of the cut-off, i.e. a cut-off $r \le r_0 f(\theta)$ for any smooth function $f(\theta)$ gives the same result for C and the 'finite part'.

We shall try to justify this proposal by showing that

1. In the classical limit

$$\mathcal{L}^{(2)} \to \lambda \mathcal{L}^{(2)}, \qquad \vec{q} \to \lambda \vec{q}$$

with λ large,

$$\ln d(\vec{q}) \to S_{wald}(\vec{q})$$

2. This fits in with the usual rules of AdS/CFT correspondence.

$$d(\vec{q}) = \left\langle \mathrm{exp}[-iq_i \oint d\theta \, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

In the classical limit this reduces to

$$d(\vec{q}) \simeq \left[e^{-\mathcal{S}} \, \mathrm{exp}[-iq_i \oint d\theta \, A_\theta^{(i)cl}] \right]^{finite}$$

$$A_\theta^{(i)cl} = -i \, e_i \, (r_0 - 1)$$

$$\mathcal{S} = \text{Euclidean action} = \mathcal{S}_{bulk} + \mathcal{S}_{boundary}$$

$$\mathcal{S}_{bulk} = -\int_1^{r_0} dr\sqrt{\det g}\, d\theta\, \mathcal{L}^{(2)} = -(r_0 - 1)\, 2\pi v\, \mathcal{L}^{(2)}$$

One can also show that

$$\mathcal{S}_{boundary} = -K r_0 + \mathcal{O}(r_0^{-1})$$

K: some constant which depends on the details of the boundary terms.

The length of the boundary is

$$L = 2\pi\sqrt{v} r_0 + \mathcal{O}(r_0^{-1})\,.$$

This gives

$$d(\vec{q}) \simeq \left[e^{r_0(2\pi v\,\mathcal{L}^{(2)} + K - \vec{e}\cdot\vec{q}) + 2\pi(\vec{e}\cdot\vec{q} - v\,\mathcal{L}^{(2)}) + \mathcal{O}(r_0^{-1})} \right]^{finite}$$

$$\rightarrow \ln d(\vec{q}) \simeq 2\pi(\vec{e}\cdot\vec{q} - v\,\mathcal{L}^{(2)}) = S_{wald}(\vec{q})$$

In principle quantum corrections to $d(\vec{q})$ can be calculated using perturbation theory / non-peturbative effects.

AdS_2/CFT_1 **correspondence**

Euclidean AdS_2 is the Poincare disk.

$\to$ its boundary is a circle of circumference L.

Thus AdS/CFT correspondence $\to$

$$\left\langle \text{exp}[-iq_i \oint d\theta\, A^{(i)}_\theta] \right\rangle_{AdS_2} = Tr_{\vec{q}}\, e^{-LH}$$

$Tr_{\vec{q}}$: trace over states of charge $\vec{q}$ in CFT_1

H: Hamiltonian of dual CFT_1

What can we say about CFT_1?

It should be identified as the infrared limit of the quantum mechanics that describes the black hole solution.

In all stringy black holes the microscopic theory has a gap that separates the ground states from the excited states.

Thus in the infrared limit only the ground states remain and we have a quantum mechanics with a finite dimensional Hilbert space.

$$\to Tr_{\vec{q}} e^{-LH} = d_{micro}(\vec{q}) e^{-E_0 L}$$

This gives

$$
\begin{aligned}
d(\vec{q}) &\equiv \left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite} \\
&= \left[Tr_{\vec{q}} e^{-LH} \right]^{finite} \\
&= \left[d_{micro}(\vec{q}) e^{-E_0 L} \right]^{finite} . \\
&= d_{micro}(\vec{q}) .
\end{aligned}
$$

Thus our generalization of the Wald entropy is directly related to the microscopic degeneracies of the black hole via AdS/CFT correspondence.

Microscopic degeneracy

$d(\vec{q})$ can in principle be calculated by analyzing string theory in the near horizon geometry of the black hole.

How can we calculate $d_{micro}(\vec{q})$?

We shall now describe the results for $d_{micro}(\vec{q})$ is a class of

$\mathcal{N} = 4$ supersymmetric string theories obtained by taking an appropriate quotient of heterotic ot type II string theory compactified on T^6.

A generic $\mathcal{N}=4$ supersymmetric string theory in $D=4$ has R U(1) gauge fields. $(R \geq 6)$

6 graviphotons $+$ $(R-6)$ matter multiplets

There are also two sets of moduli scalar fields:

a complex scalar modulus τ: $\Im(\tau) > 0$

$6(R-6)$ real scalars labelled by $R \times R$ matrix M subject to the constraint

$$M^T = M, \quad M^T L M = L$$

L: a matrix with 6 eigenvalues 1 and $(R-6)$ eigenvalues -1

Dyon: carries (electric,magnetic) charges (Q, P)

Q, P: R-dimensional vectors

$d_{micro}(Q, P)$: number of quarter BPS states with charge (Q, P) weighted by $(-1)^F (2h)^6/6!$

F: fermion number, h: helicity

$\rightarrow$ a non-vanishing and protected index Kiritsis

We calculate the index in the weak coupling limit.

When we take into account the interaction, we expect all states whose masses are not protected to become non-BPS.

$\rightarrow$ only the index worth of states will remain BPS.

Thus we shall interpret $d_{micro}(Q,P)$ as the microscopic degeneracy of BPS states carrying charges (Q,P).

$d_{micro}(Q,P)$ can also depend on the asymptotic values (M_∞, τ_∞) of the moduli fields.

However since it is an index we expect it to remain unchanged under a continuous deformation of (M_∞, τ_∞) around a generic point.

It can jump across a wall of marginal stability at which a quarter BPS dyon breaks into a pair of half-BPS dyons.

$$(Q,P) \Rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + (\delta Q - \beta P, -\gamma Q + \alpha P)$$

$\alpha\delta = \beta\gamma, \quad \alpha + \delta = 1, \quad (\alpha Q + \beta P, \gamma Q + \delta P) \in$ charge lattice

T-duality transformation:

$$Q \to \Omega Q, \quad P \to \Omega P, \quad M \to \Omega M \Omega^T$$

$\Omega \in$ a discrete subgroup of $O(6, R-6)$.

S-duality transformation:

$$Q \to aQ + bP, \quad P \to cQ + dP$$

$$\tau \to (a\tau + b)/(c\tau + d)$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ a discrete subgroup of SL(2,R).

T-duality invariants constructed from (Q, P)

$$Q^2 = Q^T L P, \quad P^2 = P^T L P, \quad Q \cdot P = Q^T L P$$

Besides these there can be an additional set of T-duality invariants $\vec{u}$ which are not invariants of the continuous group $O(6, R-6)$, but are invariants of the discrete subgroup that corresponds to the T-duality group.

Result

$$
\begin{aligned}
& d_{micro}(Q,P) \\
= \; & (-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\rho d\sigma dv\, \Psi(\rho,\sigma,v;\vec{u}) \\
& \qquad\qquad \exp\left[-i\pi(\sigma Q^2+\rho P^2+2vQ\cdot P)\right] .
\end{aligned}
$$

$(\rho = \rho_1 + i\rho_2, \sigma = \sigma_1 + i\sigma_2, v = v_1 + iv_2)$ are three complex variables.

$\mathcal{C}$: a three real dimensional contour in the (ρ,σ,v) space defined by fixed values of (ρ_2,σ_2,v_2) and (ρ_1,σ_1,v_1) integrated over their periods.

$\Psi(\rho,\sigma,v;\vec{u})$ can be explicitly calculated in many cases.

The choice of $\mathcal{C}$, i.e. of (ρ_2, σ_2, v_2) depends on the values of (M_∞, τ_∞).

As we vary (M_∞, τ_∞) the contour changes, but as long as we do not encounter a pole of the integrand the result for $d_{micro}(Q,P)$ remains unchanged.

As (M_∞, τ_∞) crosses a wall of marginal stability the contour hits a pole.

Thus the jump in $d_{micro}(Q,P)$ as we cross the walls of marginal stability can be encoded in the residue of the integrand at the corresponding pole.

The jump across the wall associated with the decay

$$(Q,P) \Rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + (\delta Q - \beta P, -\gamma Q + \alpha P)$$

is controlled by the pole at

$$\rho\gamma - \sigma\beta + v(\alpha - \delta) = 0\,.$$

On the black hole side these jumps can be explained by the (dis-)appearance of multicentered black hole solutions as (M_∞, τ_∞) crosses a wall of marginal stability.

Thus the residues at these poles capture the degeneracies of the multicentered black hole solutions.

The quantum entropy function captures information about the degeneracies of single centered black holes only.

Thus in order to compute the relevant $d_{micro}(Q,P)$ we must adjust (M_∞, τ_∞) so that only single centered black hole solutions exist.

Requires choosing Cheng, Verlinde

$$\begin{aligned}
\rho_2 &= \Lambda \frac{Q^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, \\
\sigma_2 &= \Lambda \frac{P^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, \\
v_2 &= -\Lambda \frac{Q\cdot P}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, \qquad \Lambda >> 1\,.
\end{aligned}$$

Given the exact expression for $d(Q, P)$ we can try to extract its behaviour for large charges.

1. Do the v integral by picking up residues at various poles.

2. Do the (ρ, σ) integral by saddle point method.

Example: For heterotic string theory on T^6 the relevant poles are at

$$n_2(\sigma\rho - v^2) + jv + n_1\sigma - m_1\rho + m_2 = 0,$$
$$\text{for} \quad m_1, n_1, m_2, n_2 \in \mathbb{Z},\ j \in 2\mathbb{Z} + 1,$$
$$m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}.$$

The contribution from this pole to $d_{micro}(Q,P)$ has the form

$$\exp\left[\frac{\pi}{n_2}\sqrt{Q^2P^2-(Q\cdot P)^2}\left(1+\mathcal{O}\left(\text{charge}^{-2}\right)\right)\right]$$

Dominant contribution comes from $n_2 = 1$.

The leading contribution to $d_{micro}(Q,P)$ agrees with the classical supergravity contribution to the quantum entropy function.

In principle the power corrections can be compared with the systematic quantum and higher derivative corrections to the quantum entropy function.

This has been done in various limits but a completely systematic comparison is still lacking.

What about the exponentially suppressed corrections?

Since they have a different exponential factor it would seem that they represent contribution from a different saddle point which has the same asymptotic field configuration as the Euclidean AdS_2 space-time.

Can we identify such a saddle point?

Proposal: These come from $\mathbb{Z}_N$ orbifolds of AdS_2 with $N = n_2$.

Classical contribution to $d(Q,P)$ from this saddle point is

$$\exp\left[\frac{\pi}{N}\sqrt{Q^2P^2-(Q\cdot P)^2}\right]$$

exactly as required.

(see Nabamita's talk for more details)

Summary

1. Proposal for relating the extremal black hole entropy to the microscopic degeneracy

$$d(\vec{q}) = \left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

– reduces to the relation between wald entropy and statistical entropy in the classical limit.

– in the spirit of AdS/CFT correspondence.

2. On the microscopic side we have a complete understanding of the degeneracies of a class of quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories.

A systematic analysis of the $d(q) = d_{micro}(q)$ postulate is still lacking, but the difficulties are mainly technical and may be overcome in the near future.