Yukawa Update with Magnetized branes

- Based on an ongoing work: Ignatios Antoniadis, AK and Binata Panda (also presented in Monsoon workshop, Mumbai)
- We discuss the computation of Yukawa coupling in magnetized brane constructions in the toroidal compactifications of type I string theories. magntized branes : D-branes where constant worldvolume fluxes have been turned on along compactified tori.
- Such computations have been performed explicitly for the examples when the fluxes along the branes respect the factorization of the internal T^6 into $T^2 \times T^2 \times T^2$. Then the fluxes are represented by the diagonal components of the three dimensional hermitian matrix with components $F_{z^i \bar{z}^j}$. Cremades, Ibanez, Marchesano, hep-th/0302105; hep-th/0404229
- However, the case when the fluxes do not repect such factorizations, i.e. when they are parameterized by the components of the general hermitian 3×3 matrices, the evaluation of the Yukawa term has been an open problem.

Motivation – contd.

- In this picture, the technical difficulties arise in dealing with the explicit form of the fermion wavefunctions on tori (in the presence of magnetic fluxes), in order to evaluate the overlap integrals of three such wavefunctions. [discussion of fermion wavefunctions : hep-th/0404229 and hep-th/0506259]
- Similar difficulties (in different forms) are present in other models such as intersecting brane constructions and possibly other (?) globally consistent models. (for some discussion: H. Omer, arXiv: 0811.3200).
- On ther hand, restricting to ‘magnetized brane’ constructions of ‘realistic’ particle physics models, one needs such ‘general’ (oblique) fluxes for phenomenological reasons as well as for the purpose of moduli stabilizations.

Plan

- Basic Introduction of magnetized branes and related GUT model
- Review of Yukawa computation in the factorized case
- Yukawa for non-factorized case: when ‘intesection’ matrices are positive (negative) definite
- Construction of wavefunctions when ‘interesection’ matrices have e.values of mixed signatures (application to Yukawa)
- Conclusions

Generalities

- Switching on constant internal magnetic fields has interesting consequences in type I string compactifications to four-dimensional (4d) (first proposed by Bachas '95, also used in Blumenhagen, Goerlich, Kors, Lust, hep-th/0007024 Angelantonj, Antoniadis, Dudas, Sagnotti, hep-th/0007090, Cascales, Uranga, hep-th/0303024 etc.. in open string constructions).
- Such magnetic fluxes are described by exact CFT and they give a spin dependent shift (for states which are charged under the corresponding gauge transformation) in the masses leading to a spectrum described by various Landau enery levels
- Moreover, in the present construction magnetic field is turned on along compact directions and is therefore quantized.
- One implication of the mass shift is the 'chirality generation'.
- Fluxes also generate noncommutativity on the internal tori.
- They in general break supersymmetry. However, in some special cases, a part of the supersymmetry can be preserved provided fluxes satisfy certain constraints.

generalities – contd.

- These constraints in turn can be used for stabilizing the close string moduli as in Antoniadis-Maillard '04.
(Also in I. Antoniadis, AK and T. Maillard, Nucl.Phys.B767 (2007) 139; AK, S. Mukhopadhyay and K. Ray, JHEP12 (2007) 032)
- Also, for particle physics model buidling, in order to obtain an odd number (3) of fermion generations, a NS-NS (Neveu-Schwarz) 2-form B-field background must be turned on – the quantization rule on the fluxes change. They are now half-integrally quantized.
- Based on these facts, it is possible to construct a 'minimal' example of a supersymmetric grand unified model in a toroidal compactification of type I string theory with magnetized $D9$-branes (i.e., replacing the ordinary $D9$'s by the magnetized ones) I. Antoniadis, AK and B. Panda, Nucl. Phys. B 795 (2008).

Application to GUT model

- The gauge symmetry is just $SU(5)$ (with no extra $U(1)$ factors)
- gauge non-singlet chiral spectrum contains only three families of quarks and leptons transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations
- It contains no chiral 'exotics'
- A susy SM embedding in a simple toroidal compactification (all previous constructions involve orbifolds which bring twisted moduli, or more complicated CFTs) with stabilized close string moduli.
- For consistency (supersymmetry + tadpole cancellation), certain charged scalar vev's need to be turned on along some of the (hidden) branes

Yukawa Coulings : factorized tori

- We review computations of Yukawa couplings for IIB orientifold constructions using magnetized branes
- Close form expressions for Yukawa interactions have been written down for string constructions involving branes at angles or those with magnetized branes: Cremades, Ibanez, Marchesano, hep-th/0302105; hep-th/0404229.
- These results have also been obtained in the magnetized brane picture, based on their gauge theoretic representation : valid for large volume and dilute flux approximation (but extendable to full string theory to all order in α' ?)
- In this case, the interactions are given by the overlap integral of three wavefunctions (contributing to interaction) along internal directions
- Such wavefunctions represent the open string states (ab), (bc), (ca) connecting branes (or stack of n_a banes): a, b, c etc.
- For torus compactification that we are discussing, the internal wavefunctions are factorized into those depending on three T^2 coordinates.

T^2 Wavefunctions

- Wavefunctions for the chiral wavefunctions on T^2's are expressed in terms of the basis wavefunctions $\psi^{j,N}$:

(1) $\psi^{j,N}(\tau,z) = \mathcal{N} \cdot e^{i\pi N z \mathrm{Im}\, z/\mathrm{Im}\, \tau} \cdot \vartheta \begin{bmatrix} \frac{j}{N} \\ 0 \end{bmatrix} (Nz, N\tau), \qquad j = 0, \ldots, N-1$

with N denoting the difference of the $U(1) \in U(n_a)$'th and $U(1) \in U(n_b)$'th magnetic gauge fluxes, representing stacks of n_a and n_b branes respectively and gives the degeneracy of the chiral fermions. We will also denote 'N' as I_{ab}, I_{bc} etc..

- Using such a basis, the chiral and anti-chiral (left and right handed fermions) fermion wavefunctions on T^2:

(2) $$\psi^j = \begin{pmatrix} \psi^j_+ \\ \psi^j_- \end{pmatrix},$$

wavefunctions

- satisfy the equations:

$$D\psi_+^j \equiv (\bar\partial + \frac{\pi N}{2Im\tau}(z+\zeta))\psi_+^j = 0,$$
$$D^\dagger(\psi_+^j)^* \equiv (\partial - \frac{\pi N}{2Im\tau}(\bar z+\bar\zeta))(\psi_+^j)^* = 0,$$
$$D^\dagger\psi_-^j \equiv (\partial - \frac{\pi N}{2Im\tau}(\bar z+\bar\zeta))\psi_-^j = 0,$$
$$D(\psi_-^j)^* \equiv (\bar\partial + \frac{\pi N}{2Im\tau}(z+\zeta))(\psi_-^j)^* = 0. \quad (3)$$

- Solutions:

$$\psi_+^j = \psi^{j,N}(\tau, z+\zeta),\ (\psi_+^j)^* = \psi^{-j,-N}(\bar\tau, \bar z+\bar\zeta),$$
$$\psi_-^j = \psi^{j,N}(\bar\tau, \bar z+\bar\zeta),\ \psi_+^j = \psi^{j,N}(\tau, z+\zeta). \quad (4)$$

- Furthermore, for expression of the chrial and anti-chiral solutions, as given in the basis defined earlier, are well defined, provided $N > 0$ for the wavefunctions ψ_+^j and $N < 0$ for the wavefunctions ψ_-^j.

Yukawa computation on factorized tori

- Can be seen from the convergence of Jacobi theta function series expansion.
- also, in these cases, for ψ_+^j and ψ_-^j to be properly normalized:

(5) $$\int_{T^2} dz d\bar{z} \psi_\pm^j (\psi_\pm^k)^* = \delta_{jk},$$

- an additional factor:

(6) $$\mathcal{N}_j = \left(\frac{2Im\tau|N|}{\mathcal{A}^2} \right)^{\frac{1}{4}}$$

needs to be introduced with $\mathcal{A}$ being the area of the T^2.

Yukawa computation on factorized tori

- We now first summarize the basic results of hep-th/0404229 regarding the computations of Yukawa interactions.
- Such four dimensional interaction terms were obtained through a dimensional reduction of the $D = 10$, $N = 1$ super-Yang-Mills theory to four dimensions in the presence of constant magnetic fluxes.
- The Yukawa interaction is of the form:

$$Y_{ijk} = \int_{\mathcal{M}} \psi_i^{a\dagger} \Gamma^m \psi_j^b \phi_{k,m}^c f_{abc} \tag{7}$$

- where $\mathcal{M}$ is the internal space on which the gauge theory has been compactified and ψ amd ϕ being the internal zero mode fluctuations of the gaugino and YM fields
- with f_{abc} being the structure constant of the higher dimensional gauge group.
- For torus compactification that we are discussing, the internal wavefunctions are factorized into those depending on three T^2 coordinates.

Yukawa computation on factorized tori

- The Yukawa interaction then reads (in terms of the basis function given earlier):

$$Y_{ijk} = \sigma_{abc} g \int_{T^2} dz d\bar{z} \psi^{i,I_{ab}}(\tau, z).\psi^{j,I_{ca}}(\tau, z).(\psi^{k,I_{cb}}(\tau, z))^* \tag{8}$$

  with $I_{bc} < 0$, since $I_{ab} + I_{ca} = I_{cb} = -I_{bc}$. A similar expression exists for $I_{bc} > 0$ as well.
- To evaluate this integral, however, one uses an identity, satisfied by the Theta functions appearing in the definitions of the basis functions given earlier.
- The aim of this identity is to establish a connection between the wavefunctions with fluxes N_1 and N_2 for bifundamental states in brane intersections ab and ca with the one in the intersection bc given by the flux $N_3 = N_1 + N_2$.
- The above integral is then evaluated using the orthogonality of the wavefunctions $\psi^{k,I_{cb}}$ and $(\psi^{l,I_{cb}})^*$

Jacobi Theta function identities

- Theta function identity (see Mumford: Tata lectures) used in hep-th/0404229 for computing Yukawa couplings mentioned above:

$$\vartheta\left[\begin{array}{c}\frac{r}{N_1}\\ 0\end{array}\right](z_1,\tau N_1)\cdot\vartheta\left[\begin{array}{c}\frac{s}{N_2}\\ 0\end{array}\right](z_2,\tau N_2)=$$
$$\sum_{m\in\mathbf{Z}_{N_1+N_2}}\vartheta\left[\begin{array}{c}\frac{r+s+N_1m}{N_1+N_2}\\ 0\end{array}\right](z_1+z_2,\tau(N_1+N_2))$$
$$\times\vartheta\left[\begin{array}{c}\frac{N_2r-N_1s+N_1N_2m}{N_1N_2(N_1+N_2)}\\ 0\end{array}\right](z_1N_2-z_2N_1,\tau N_1N_2(N_1+N_2)) \tag{9}$$

where ϑ is the Jacobi theta-function:

$$\vartheta\left[\begin{array}{c}a\\ b\end{array}\right](\nu,\tau)=\sum_{l\in\mathbf{Z}}e^{\pi i(a+l)^2\tau}\ e^{2\pi i(a+l)(\nu+b)}. \tag{10}$$

Application to Yukawa computation for factorized tori

- The identity can be proved explicitly by expanding the lhs. in series expansion, change over to the variables appearing in the rhs. and then matching it with the series expansion in the rhs.
- We now make use of the explicit form of the fermion and scalar wavefunctions, defined in terms of the basis function to write the expression for the Yukawa interaction term.
- More precisely, to evaluate the Yukawa interaction, one uses the Theta identity shown earlier and the basis function in $\psi^{i,I_{ab}}$ to write down:

$$\psi^{i,I_{ab}}(\tau,z).\psi^{j,I_{ca}}(\tau,z) = \left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{2}} (I_{ab}I_{ca})^{\frac{1}{4}} e^{i\pi(N_1+N_2)z\mathrm{Im}\, z/\mathrm{Im}\,\tau} \times$$
$$\times\vartheta\begin{bmatrix} \frac{i}{N_1} \\ 0 \end{bmatrix}(N_1 z, N_1\tau)\cdot\vartheta\begin{bmatrix} \frac{j}{N_2} \\ 0 \end{bmatrix}(N_2 z, N_2\tau),$$
$$i = 0,\ldots,N_1-1,\ j = 0,\ldots,N_2-1.$$

Jacobi Theta function identities

- where we have also made use of the normalization factor, $\mathcal{N}$ and also identified, for a T^2 compactification:

(11) $$N_1 = I_{ab}, \quad N_2 = I_{ca},$$

with

(12) $$I_{ab} = m_a - m_b, \quad etc.$$

giving

(13) $$N_3 = (N_1 + N_2) = I_{cb}.$$

- Now, using the Theta identity, product of wavefunctions for intersections ab and ca can be rewritten in the form:

$$\psi^{i,I_{ab}}(\tau,z).\psi^{j,I_{ca}}(\tau,z) = \left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}}\left(\frac{I_{ab}I_{ca}}{I_{cb}}\right)^{\frac{1}{4}} \cdot \sum_{m\in\mathbf{Z}_{I_{cb}}} \psi^{i+j+I_{ab}m,I_{cb}}(\tau,z)\times$$

$$\times\vartheta\begin{bmatrix} \frac{I_{ca}i - I_{ab}j + I_{ab}I_{ca}m}{I_{ab}I_{ca}I_{bc}} \\ 0 \end{bmatrix}(0, \tau I_{ab}I_{ca}I_{cb}).$$

Yukawa expression

- Yukawa interaction is then evaluated using the orthogonality property of the wavefunctions and reads:

$$Y_{ijk} = \sigma_{abc} g \left(\frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left(\frac{I_{ab} I_{ca}}{I_{cb}} \right)^{\frac{1}{4}}$$

$$\sum_{m \in \mathbf{Z}_{I_{cb}}} \delta_{k,i+j+I_{ab}m} \cdot \vartheta \begin{bmatrix} \frac{I_{ca}i - I_{ab}j + I_{ab}I_{ca}m}{I_{ab}I_{ca}I_{bc}} \\ 0 \end{bmatrix} (0, \tau I_{ab} I_{ca} I_{cb}).$$

- After imposing the Kronecker delta constraint, we obtain:

$$Y_{ijk} = \sigma_{abc} g \left(\frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left(\frac{I_{ab} I_{ca}}{I_{cb}} \right)^{\frac{1}{4}} \vartheta \begin{bmatrix} \left(\frac{j}{I_{ca}} + \frac{k}{I_{bc}} \right) / I_{ab} \\ 0 \end{bmatrix} (0, \tau I_{ab} I_{ca} I_{cb}). \tag{14}$$

Yukawa expression

- The final answer can be expressed as:

(15) $$Y_{ijk} = \sigma_{abc} g \left(\frac{2 Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left(\frac{I_{ab} I_{ca}}{I_{cb}} \right)^{\frac{1}{4}} \vartheta \left[\begin{array}{c} \delta_{ijk} \\ 0 \end{array} \right] (0, \tau I_{ab} I_{ca} I_{cb}).$$

with

(16) $$\delta_{ijk} = \frac{i}{I_{ab}} + \frac{j}{I_{ca}} + \frac{k}{I_{bc}}.$$

- The result can be easily extended to the case of factorized T^6 and the interaction is then written in in terms of products of Theta functions of the type appearing above.
- We now go on to the generalization when fluxes of both oblique and diagonal form are present.
- Such magenetic fluxes do not respect the factorization and hence involve the wavefunctions now expressed in terms of genralized Riemann Theta functions.

Generalization

- We now generalize Theta identities to the case of general Riemann Theta functions. The identity then takes the form:

$$\text{(17)}\quad \vartheta\begin{bmatrix}\vec{j_1}\\0\end{bmatrix}(\vec{z_1}|\mathbf{N_1}\Omega)\cdot\vartheta\begin{bmatrix}\vec{j_2}\\0\end{bmatrix}(\vec{z_2}|\mathbf{N_2}\Omega)=$$

$$\sum_{\vec{m}}\vartheta\begin{bmatrix}(\vec{j_1}\mathbf{N_1}+\vec{j_2}\mathbf{N_2}+\vec{m}.\mathbf{N_1})(\mathbf{N_1}+\mathbf{N_2})^{-1}\\0\end{bmatrix}(\vec{z_1}+\vec{z_2}|(\mathbf{N_1}+\mathbf{N_2})\Omega)$$

$$\cdot\quad\vartheta\begin{bmatrix}[(\vec{j_1}-\vec{j_2})+\vec{m}]\frac{\mathbf{N_1}(\mathbf{N_1}+\mathbf{N_2})^{-1}\mathbf{N_2}}{det\mathbf{N_1}det\mathbf{N_2}}\\0\end{bmatrix}$$

$$((det\mathbf{N_1}det\mathbf{N_2})(\mathbf{N_1}^{-1}\vec{z_1}-\mathbf{N_2}^{-1}\vec{z_2})|(det\mathbf{N_1}det\mathbf{N_2})^2(\mathbf{N_1}^{-1}(\mathbf{N_1}+\mathbf{N_2})\mathbf{N_2}^{-1}\Omega^T)),$$

- We leave the proof (though a bit involved) for the moment.

Yukawa expressions for oblique fluxes

- We now use the wavefunctions with general Riemann Theta functions (given already in hep-th/0404229), to obtain the expression of Yukawa interaction when oblique fluxes, specified by matrices ('intersection matrices)

(18)
$$\mathbf{N_1} = F_a - F_b,\ \mathbf{N_2} = F_b - F_c,\ \mathbf{N_3} = F_c - F_a.$$

are turned on along branes a, b and c.

- F_a, F_b and F_c are all real symmetric matrices due to the absence of components p_{xx}, p_{yy} along real 2-cycles. But in general they can be hermitian.
- In addition complex structure matrix is proportional to identity: τI_n (which is the case in the GUT model mentioned earlier).
- We then have:

$$\psi^{\vec{j},\mathbf{N_1}}(\vec{z}, \mathbf{\Omega} = \tau I_n) \cdot \psi^{\vec{j},\mathbf{N_2}}(\vec{z}, \mathbf{\Omega} = \tau I_n) =$$
$$\left(2^{\frac{n}{2}}\right) \left(Vol(T^{2n})\right)^{-1} \left(|det\mathbf{N_1}|.|det\mathbf{N_2}|\tau^2\right)^{\frac{1}{4}} \times$$
$$\times e^{i\pi \mathbf{N_3}.\vec{z}\mathrm{Im}\,\vec{z}/\mathrm{Im}\,\tau} \vartheta \begin{bmatrix} \vec{j_1} \\ 0 \end{bmatrix} (\vec{z_1}|\mathbf{N_1} \cdot \tau) \cdot \vartheta \begin{bmatrix} \vec{j_2} \\ 0 \end{bmatrix} (\vec{z_2}|\mathbf{N_2} \cdot \tau)$$

Yukawa expressions for oblique fluxes

- Then using Riemann Theta identity, above equation can be rewritten as:

$$\psi^{\vec{i},\mathbf{N_1}}(\vec{z}) \cdot \psi^{\vec{j},\mathbf{N_2}}(\vec{z}) = \sum_{\vec{m}} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|det\mathbf{N_1}|.|det\mathbf{N_2}|\tau)}{|det\mathbf{N_3}|}\right]^{\frac{1}{4}} \times$$

$$\psi^{(\mathbf{N_1}\vec{i}+\mathbf{N_2}\vec{j}+\vec{m}\mathbf{N_1})\mathbf{N_3}^{-1},\mathbf{N_3}}(\vec{z}) \cdot \vartheta \left[\begin{array}{c} [(\vec{i}-\vec{j})+\vec{m}]\frac{\mathbf{N_1}\mathbf{N_3}^{-1}\mathbf{N_2}}{det\mathbf{N_1}\,det\mathbf{N_2}} \\ 0 \end{array}\right]$$

$$(0|(det\mathbf{N_1}det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1})\tau)$$

Yukawa expressions for oblique fluxes

- Note that integrality condition is maintained by $\psi^{(\vec{i}+\vec{j}+\vec{m}\mathbf{N_1})\mathbf{N_3}^{-1},\mathbf{N_3}}(\vec{z})$ appearing in the RHS of the above equation, since the expression

$$\left[(\vec{i}+\vec{j}+\vec{m}\mathbf{N_1})\mathbf{N_3}^{-1}\right]\cdot\mathbf{N_3} \tag{19}$$

is always an integer.

- Then, as in the T^2 case, orthonormality of wavefunction, implies the Yukawa coupling of the form:

$$Y_{ijk}=\sigma_{abc}g\left(2^{\frac{n}{2}}\right)^{\frac{1}{2}}\left(Vol(T^{2n})\right)^{-\frac{1}{2}}\left[\frac{(|det\mathbf{N_1}|.|det\mathbf{N_2}|\tau)}{|det\mathbf{N_3}|}\right]^{\frac{1}{4}}\times$$

$$\sum_{m\in\mathbf{Z}_{I_{cb}}}\delta_{\vec{k},\mathbf{N_3}^{-1}(\mathbf{N_1}\vec{i}+\mathbf{N_2}\vec{j}+\mathbf{N_1}\vec{m})}\times$$

$$\vartheta\begin{bmatrix}[(\vec{i}-\vec{j})+\vec{m}]\frac{\mathbf{N_1}\mathbf{N_3}^{-1}\mathbf{N_2}}{det\mathbf{N_1}\,det\mathbf{N_2}}\\0\end{bmatrix}(0|(det\mathbf{N_1}det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1})\tau)$$

Yukawa expressions for oblique fluxes

- Imposing the constraint from the Kronecker delta, we obtain:

$$Y_{ijk} = \sigma_{abc} g \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|det\mathbf{N_1}|.|det\mathbf{N_2}|)}{|det\mathbf{N_3}|}\right]^{\frac{1}{4}} \times$$
$$\vartheta \begin{bmatrix} (\vec{j}+\vec{k})\frac{\mathbf{N_2}}{det\mathbf{N_1}\,det\mathbf{N_2}} \\ 0 \end{bmatrix} (0|(det\mathbf{N_1}det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1})\tau)$$

- The above expression reduces to the case of factorized tori.
- To compare the two expressions, note that indices i, j, k in the factorized case are scaled with respect to the one for the general tori, by factors $\frac{1}{N_1}$, $\frac{1}{N_2}$ and $\frac{1}{N_3}$.
- Then the Kronecker delta constraint also precisely matches with the one given earlier.

Yukawa expressions for oblique fluxes

- In the case of general tori, the constraint implies that the interaction terms involving only those states are nonzero which satisfy the equation

$$\mathbf{N_3}\vec{k} = (\mathbf{N_1}\vec{i} + \mathbf{N_2}\vec{j} + \mathbf{N_1}\vec{m}) \quad (20)$$

among vectors $\mathbf{N_1}\vec{i}$, $\mathbf{N_2}\vec{j}$, $\mathbf{N_3}\vec{k}$ having integer entries for $\vec{m}$ having integer entries also and is inside the unit cell mentioned earlier.

Fermion Wavefunctions

- However, just as in the T^2 case, the fermion wavefunctions $\psi^{\vec{j},\mathbf{N}}(\vec{z},\mathbf{\Omega} = \tau I_n)$ we have used in obtaining the Yukawa coupling, is well defined only for some restricted class of intersection matrices.
- For the T^2 case we had ψ_+^j well defined for $N > 0$ and ψ_-^j well defined for $N < 0$
- For general Tori, the constriant (setting $\tau = i$) on wavefunction: $\psi^{\vec{j},\mathbf{N}}(\vec{z},\mathbf{\Omega} = \tau I_n)$ to be well defined, matrix $\mathbf{N}$ needs to be positive definite.
- One can also have a well defined conjugate wavefunction with $\mathbf{N}$ negative definite.
- The situation where we can have mixed e. values for the intersection matrices $\mathbf{N}$ therefore remains to be addressed.

Dirac eq.

- To clarify the situation, we take the explicit example of the simplest nontrivial case, where oblique fluxes can appear : T^4 compactification .
- The flux as well as intersection matrices are now 2×2 hermitian matrices. Gauge potential A_μ has components: A_{z^i}, $A_{\bar{z}^i}$ with $i = 1, 2$.
- We write down the Dirac eq. in the presence of fluxes
- Equations satisfied by the various bi-fundamental Dirac components χ_{ab} corresponding to string states between branes-stacks a and b are (in complex T^4 coordinates $z^{1,2}$, $\bar{z}^{1,2}$):

Dirac eq.

- $$\begin{aligned}
\bar\partial_1 \chi_+^1 + \partial_2 \chi_+^2 + (A^a - A^b)_{\bar{z_1}} \chi_+^1 + (A^a - A^b)_{z_2} \chi_+^2 &= 0, \\
\bar\partial_2 \chi_+^1 - \partial_1 \chi_+^2 + (A^a - A^b)_{\bar{z_2}} \chi_+^1 - (A^a - A^b)_{z_1} \chi_+^2 &= 0, \\
\partial_1 \chi_-^2 + \partial_2 \chi_-^1 + (A^a - A^b)_{z_1} \chi_-^2 + (A^a - A^b)_{z_2} \chi_-^1 &= 0, \\
\bar\partial_2 \chi_-^2 - \bar\partial_1 \chi_-^1 + (A^a - A^b)_{\bar{z_2}} \chi_-^2 - (A^a - A^b)_{\bar{z_1}} \chi_-^1 &= 0,
\end{aligned} \tag{21}$$

  with subscript a, b being dropped from χ_{ab}
- where $\chi_+^{1,2}$ are the two positive chirality components and $\chi_-^{1,2}$ are the two negative chirality ones.

+ve chirality wavefunctions

- In particular, for the +ve chirality component, χ_+^1 equations reduce to:

$$\bar{D}_1\chi_+^1 \equiv \bar{\partial}_1\chi_+^1 + (A^a - A^b)_{\bar{z_1}}\chi_+^1 = 0,$$
$$\bar{D}_2\chi_+^1 \equiv \bar{\partial}_2\chi_+^1 + (A^a - A^b)_{\bar{z_2}}\chi_+^1 = 0. \tag{22}$$

- The generalization of the Yukawa interaction, using wavefunction: $\psi^{\vec{j},\mathbf{N}}(\vec{z},\mathbf{\Omega} = \tau I_n)$, that we presented earlier, then corresponds to the wavefunction of the type χ_+^1.
- The intersection matrix $\mathbf{N}$ (which needs to be positive definite) is coming from the linear form of the $A^a - A^b$ gauge potential appearing in the eq. above, so as to give constant fluxes.
- Therefore the solution is written in terms of the intersection matrices $\mathbf{N} = \mathbf{F^a} - \mathbf{F^b}$, as stated earlier.
- While writing down the solution one should also note that periodicity conditions need to be imposed on the wavefunction, which gives the infinite series with soln. $\psi^{\vec{j},\mathbf{N}}(\vec{z},\mathbf{\Omega} = \tau I_n)$.

-ve chirality

- The conjugate wavefunction χ_+^2 (again of +ve chirality), on the other hand is well defined for $\mathbf{N}$ negative definite and satisfies the equations: $D_1\chi_+^2 = D_2\chi_+^2 = 0$.
- So, in order to accommodate intersection matrices that are not necessarily positive (negative) definite, one needs to study the negative chirality solutions $\chi_-^{1,2}$ as well.
- Need for understanding the negative chirality spinor components is also confirmed from the fact that in the factorized case: on $T^2 \times T^2$, if the intersection matrix is diagonal with components : (N_d^1, N_d^2) such that $N_d^1 > 0$ and $N_d^2 < 0$
- Then the total wavefunction on T^4 is given by a product of a positive chirality wavefunction on the first T^2 and a negative chirality one on the second T^2.
- The T^4 wavefunction is therefore of negative chirality.

-ve chirality

- Now, going back to the positive chirality wavefunctions
- Note that the two equations for χ_+^2 can be simultaneously solved, since $[D_1, D_2] \sim F_{12}^{ab}$ which is zero, since all the $(2,0)$ components of the gauge fluxes are zero in order to maintain the supersymmetry.
- Superscript ab implies, we need to take the difference of fluxes in brane stacks a and b.
- Same is true for the two χ_+^1 equations: $\bar{D}_1\chi_+^1 = \bar{D}_2\chi_+^1 = 0$, since $(0,2)$ components of the fluxes are zero as well.
- On the other hand the relevant equations for the negative chirality equations are: $D_1\chi_-^2 + D_2\chi_-^1 = 0$ and $\bar{D}_2\chi_-^2 - \bar{D}_1\chi_-^1 = 0$
- When only one of the two components $\chi_-^{1,2}$ is excited: $\chi_-^{1,2}$ now satisfy: $\bar{D}_1\chi_-^1 = D_2\chi_-^1 = 0$ or $D_1\chi_-^2 = \bar{D}_2\chi_-^2 = 0$
- But now these equations can not be consistently solved when oblique fluxes are present, since $[D_1, \bar{D}_2] \sim F_{1\bar{2}} \neq 0$

-ve chirality

- The two negative chirality components therefore need to be mixed up when the oblique fluxes are present
- we need to simultaneously excite both $\chi_-^{1,2}$.
- Taking $\chi_-^1 = \alpha\psi$ and $\chi_-^2 = \beta\psi$, equations become:
- $(\alpha\bar{D}_1 + \beta\bar{D}_2)\psi = 0$ and $(\beta D_1\chi_-^2 - \alpha D_2)\psi = 0$
- These two equations to have simultaneous soln. one then obatains the condition: $\alpha\beta F^{ab}_{1\bar{1}} - \alpha^2 F^{ab}_{2\bar{1}} + \beta^2 F^{ab}_{1\bar{2}} - \alpha\beta F^{ab}_{2\bar{2}} = 0$
  ($F^{ab} \equiv \mathbf{N}$ is the difference of fluxes in brane stacks a and b)
- Fortunately, this equation has arbitrary solutions of the type:

$$F = \hat{N}_{1\bar{1}} \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} + \tilde{N}_{2\bar{2}} \begin{pmatrix} x^2 & -x \\ -x & 1 \end{pmatrix} \tag{23}$$

  with $x = \frac{\beta}{\alpha}$ and $\hat{N}_{1\bar{1}}$.
- Rhs of the above relation is a general parameterization of a 2×2 symmetric matrix.

Explicit Solution

- Quantization condition on fluxes also appear while obtaining the well-defined solution on T^4.
- To generate an explicit solution, we made use of the coordinate transformation (acting on the wavefunction for the factorized tori): which mixes z_1, z_2 and $\bar{z}_1, \bar{z}_2$
- Such a transformation mixes up the two negative chirality components on T^4, but leaves the positive chirality components unchanged
- Seen from the spinor basis parameterization of the T^4 coordinate: X^i as $\mathbf{X} \equiv X^i \Gamma_i$:

$$\mathbf{X} \equiv \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix} \tag{24}$$

- Explicit solution of the equations of motion, satisfying the periodicity restrictions are then given by orthogonal wavefunctions:

$$\psi = \mathcal{N} \cdot f(z, \bar{z}) \cdot \hat{\Theta}(z, \bar{z}) \tag{25}$$

Explicit Solution

- We have the pure z dependence piece:

(26)
$$f(z,\bar{z}) = e^{i\pi[(\hat{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}} z_i Im z_j) - (\tilde{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}} \bar{z}_i Im \bar{z}_j)]}$$

and the extra factor to restore the periodicity properties:

$$\hat{\Theta}(z,\bar{z}) = \sum_{m_1, m_2 \in \mathbf{Z}^n} e^{\pi i(i)[(m_i+j_i)(\hat{\mathbf{M}}_{i\bar{j}})(m_j+j_j)]} . e^{2\pi i[(m_i+j_i)\hat{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}} z_j]} . e^{2\pi i(m_i+j_i)\tilde{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}} \bar{z_j}}$$

(27)
with extra factor on the first exponential factor representing the complex structure: $\tau = i\mathbf{I}$. Also

(28)
$$\hat{\mathbf{M}}_{i\bar{j}} = \hat{\mathbf{N}}_{i\bar{j}} - \tilde{\mathbf{N}}_{i\bar{j}}$$

where both $\hat{\mathbf{N}}$,$\tilde{\mathbf{N}}$ are real, symmetric matrices and so is $\hat{\mathbf{M}}$.

- The explicit parameterization of both $\hat{\mathbf{N}}$, $\tilde{\mathbf{N}}$ was already given earlier.

Explicit Solution

- The wavefunction satisfies the Dirac equation for the following gauge potentials:

$$
\begin{aligned}
(A^1 - A^2)_{z_1} &= (\hat{\mathbf{N}}_{\mathbf{1\bar{1}}} + \tilde{\mathbf{N}}_{\mathbf{1\bar{1}}})\bar{z}_1 + (\hat{\mathbf{N}}_{\mathbf{1\bar{2}}} + \tilde{\mathbf{N}}_{\mathbf{1\bar{2}}})\bar{z}_2 \\
(A^1 - A^2)_{z_2} &= (\hat{\mathbf{N}}_{\mathbf{2\bar{1}}} + \tilde{\mathbf{N}}_{\mathbf{2\bar{1}}})\bar{z}_1 + (\hat{\mathbf{N}}_{\mathbf{2\bar{2}}} + \tilde{\mathbf{N}}_{\mathbf{2\bar{2}}})\bar{z}_2
\end{aligned} \tag{29}
$$

- The intersection matrix $\mathbf{N}$ is therefore given by $\hat{\mathbf{N}} + \tilde{\mathbf{N}}$
- Once again: $\hat{\mathbf{N}}$ and $\tilde{\mathbf{N}}$:

$$
\frac{\alpha}{\beta} = \frac{-\hat{\mathbf{N}}_{\mathbf{1\bar{1}}}}{\hat{\mathbf{N}}_{\mathbf{1\bar{2}}}} = \frac{-\hat{\mathbf{N}}_{\mathbf{1\bar{2}}}}{\hat{\mathbf{N}}_{\mathbf{2\bar{2}}}} = \frac{\tilde{\mathbf{N}}_{\mathbf{1\bar{2}}}}{\tilde{\mathbf{N}}_{\mathbf{1\bar{1}}}} = \frac{\tilde{\mathbf{N}}_{\mathbf{2\bar{2}}}}{\tilde{\mathbf{N}}_{\mathbf{1\bar{2}}}} = \frac{1}{x} \tag{30}
$$

- The above solution provides an orthogonal basis function for writing down the Dirac wavefunctions of negative chirality.
- provided: $(\hat{\mathbf{N}} + \tilde{\mathbf{N}})$ is integrally quantized and $\vec{j}$ satisfies $\vec{j} \cdot (\hat{\mathbf{N}} + \tilde{\mathbf{N}}) \in Z$:

-ve chirality

- They also solve our problem of defining the wavefunction, when intersection matrix ($\hat{\mathbf{N}}$ + $\tilde{\mathbf{N}}$) has e.values of opposite signatures.
- For this, we note: $det(\hat{\mathbf{N}} + \tilde{\mathbf{N}}) = -det(\hat{\mathbf{N}} - \tilde{\mathbf{N}})$.
- Therefore the $\hat{M}$ matrix which appears in the series expansion of the wavefunction (in z independent piece) has both e. values of the same sign
- This is the dominant term inside the exponent at large values of the integers in the series expansion.
- Both e. values signs can be made positive by interchanging $\hat{\mathbf{N}}$ and $\tilde{\mathbf{N}}$
- The wavefunction is therefore well defined.

-ve chirality

- We presented an explicit computation of the yukawa interaction in the presence of oblique fluxes when the intersection matrix is positive definite.
- We also gave the construction of the fermion wavefunction, when the intersection matrix is such that it has e.values of mixed signatures. Still remains to obtain the explicit form of the interaction terms in this case.
- Finally, about full string theory result:
- Mass-shift derived from Field theory:

$$\delta M^2 = (2k+1)|qH| + 2qH\Sigma \tag{31}$$

- String theory answer: same except qH replaced by $\theta_L + \theta_R$ and

$$\theta_{L,R} = arctan(q_{L,R} H\alpha') \tag{32}$$

- Similar modification for the couplings as well? One needs to examine the vertex operators which are of the form of twist fields in the non-abelian orbfifold models and the corresponding correlators.