Equivariant K-theory of group compactifications: further developments

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- $R(\widetilde{T})_I := \bigoplus_{v \in C^I} R(\widetilde{T})^W \cdot f_v$.

Regular Compactifications

X- **equivariant compactification of** G

X- smooth complete variety, $G \subset X$ dense open subvariety, $(G \times G) \times G \to G$ (left/right multn. $(g_1, g_2)\gamma = g_1 \gamma g_2^{-1}$) extends to X.

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Regular G**-variety**$\Rightarrow X \supset X_G^0$- dense G-orbit, $X \setminus X_G^0$- union of normal crossing boundary divisors, $\overline{Gx}$- the transversal intersection of boundary divisors containing it, $T_x X / T_x Gx$ contains dense orbit of G_x.

Smooth complete toric varieties are exactly the regular compactifications of the torus. For the adjoint group G_{ad}, the **wonderful compactification** $\overline{G_{ad}}$ constructed by De Concini and Procesi is the unique regular compactification of G_{ad} with a unique closed $G_{ad} \times G_{ad}$-orbit.

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Recall that there exists an exact sequence

$$1 \to \mathcal{Z} \to \widetilde{G} := \widetilde{C} \times G^{ss} \xrightarrow{\pi} G \to 1 \tag{1}$$

where $\mathcal{Z}$ is a finite central subgroup, $\widetilde{C}$ is a torus and G^{ss} is semisimple and simply-connected. In particular, $\widetilde{G}$ is *factorial* and $\widetilde{B} := \pi^{-1}(B)$ and $\widetilde{T} := \pi^{-1}(T)$ are respectively a Borel subgroup and a maximal torus of $\widetilde{G}$.

Equivariant K-theory

Theorem : X-nonsingular projective variety, T acts on X with finitely many fixed points $x_1, \ldots, x_m$ and finitely many invariant curves. Then the image of $\iota^* : K_T(X) \to K_T(X^T)$ is $(f_1, \ldots, f_m) \in R(T)^m$ such that $f_i \equiv f_j \pmod{(1 - e^{-\chi})}$ whenever x_i and x_j lie in an invariant irreducible curve C and T-acts on C through the character χ.

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For a smooth projective complex G variety X we have the following isomorphisms:

- $R(\widetilde{T}) \otimes_{R(\widetilde{G})} K_{\widetilde{G}}(X) \simeq K_{\widetilde{T}}(X)$.
- $K_{\widetilde{G}}(X) \simeq K_{\widetilde{T}}(X)^W$.
- $\mathbb{Z} \otimes_{R(\widetilde{G})} K_{\widetilde{G}}(X) \simeq K(X)$.

where $\widetilde{G}$ and $\widetilde{T}$ act on X through the canonical surjections to G and T respectively.

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The maximal cones $\mathcal{F}_+(l)$ parametrize the closed $G \times G$-orbits in X each of which is isomorphic to $G/B^- \times G/B$. $X^{T \times T}$ is parametrized by $\mathcal{F}_+(l) \times W \times W$.

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(2) γ is conjugate in $W \times W$ to a curve γ' joining z_σ and $(s_\alpha, s_\alpha)z_\sigma$ of the closed orbit Z_σ, where $\gamma' \not\subset Z_\sigma$. In this case, the cone $\sigma \in \mathcal{F}_+(I)$ has a facet orthogonal to α.

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(3) γ is conjugate in $W \times W$ to a projective line γ' joining z_σ and $z_{\sigma'}$ which are respectively the base points of distinct closed orbits Z_σ and $Z_{\sigma'}$. In this case, the cones σ and σ' in $\mathcal{F}_+(I)$ have a common facet.

Example of the wonderful compactification of $PGL(2,\mathbb{C})$

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Let $G = PGL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm Id$. Then the projective space $\mathbb{P}(M(2,\mathbb{C}))$ is the wonderful compactification of $PGL(2,\mathbb{C})$, on which the action of $PGL(2,\mathbb{C}) \times PGL(2,\mathbb{C})$ by multiplication on the left and on the right extends.

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Let E_{ij} denote the elementary matrix with 1 as $(i, j)th$ entry and 0 elsewhere for $1 \leq i, j \leq 2$. In this case the Weyl group is $W = \{1 = Id, s_\alpha = -E_{12} + E_{21}\}$, and $\overline{T} \simeq \mathbb{P}^1$ consists of the diagonal matrices in $\mathbb{P}(M(2, \mathbb{C}))$.

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Further, the unique closed $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$-orbit consists of the matrices of rank 1 in $\mathbb{P}(M(2, \mathbb{C}))$ and is isomorphic to $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})/(B^- \times B^+)$, choosing as base point the matrix E_{11}. Furthermore, $PGL(2, \mathbb{C})$ is the open orbit with base point Id.

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The four $T \times T$ fixed points of $\mathbb{P}(M(2,\mathbb{C}))$ are: E_{11}, $E_{12} = (1, s_\alpha)E_{11}$, $E_{21} = (s_\alpha, 1)E_{11}$ and $E_{22} = (s_\alpha, s_\alpha)E_{11}$. Further

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Thus we see that the curves of type (1) lie entirely in the unique closed orbit, whereas the curves of type (2) meet the open orbit. Moreover, $\overline{N} = \overline{T} \sqcup (s_\alpha, 1)\overline{T}$ is the union of diagonal and the antidiagonal matrices. Hence $\overline{N}$ contains only the curves of type (2) and does not contain the curves of type (1).

In this case we do not have curves of type (3) since there is a unique closed $G \times G$-orbit.

K-theory of regular embeddings

By restricting to the $\widetilde{T} \times \widetilde{T}$-fixed points $(w, w') \cdot z_\sigma$ for $w, w' \in W$, followed by taking $W \times W$-invariants and further using the exact sequence

$$1 \to diag(\widetilde{T}) \to \widetilde{T} \times \widetilde{T} \to \widetilde{T} \to 1, \tag{2}$$

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Theorem: $K_{\widetilde{G} \times \widetilde{G}}(X)$ consists in all families $(f_\sigma)(\sigma \in \mathcal{F}_+(l))$ of elements of $R(\widetilde{T} \times \{1\}) \otimes R(diag(\widetilde{T}))$ such that

- *(i)* $(1, s_\alpha) f_\sigma(u, v) \equiv f_\sigma(u, v) \pmod{(1 - e^{-\alpha(u)})}$ whenever $\alpha \in \Delta$ and the cone $\sigma \in \mathcal{F}_+(l)$ has a facet orthogonal to α.
- *(ii)* $f_\sigma \equiv f_{\sigma'} \pmod{(1 - e^{-\chi(u)})}$ whenever $\chi \in X^*(\widetilde{T})$ and the cones σ and $\sigma' \in \mathcal{F}_+(l)$ have a common facet orthogonal to χ.

Equivariant line bundles on X

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- $L_h := \mathcal{L}_h |_{\overline{T}^+}$ is a $\widetilde{T} \times \widetilde{T}$-linearized line bundle on the toric variety $\overline{T}^+$ corresponding to $h \in PL(\mathcal{F}_+)$. In particular, $\widetilde{T}^- \times \widetilde{T}$ acts on the fibre $L_h |_{z_\sigma}$ by the character h_σ.

Equivariant K-ring of X

Theorem:

$$K_{\widetilde{G}\times\widetilde{G}}(X) = \bigoplus_{I\subseteq\Delta}\prod_{\alpha\in I}(1-e^{\alpha(u)})\cdot K_{\widetilde{T}}(\overline{T}^{+})\otimes R(\widetilde{T})_{I}. \qquad (3)$$

The above direct sum is a free $K_{\widetilde{T}}(\overline{T}^{+})\otimes R(\widetilde{G})$-module of rank $|W|$ with basis

$$\{\prod_{\alpha\in I}(1-e^{\alpha(u)})\otimes f_v : \ v\in C^I \ \textit{and} \ I\subseteq\Delta\}.$$

Moreover, we can identify the component $K_{\widetilde{T}}\otimes 1\subseteq K_{\widetilde{T}}(Z)\otimes R(\widetilde{T})^{W}$ in the above direct sum with the subring of $K_{\widetilde{G}\times\widetilde{G}}(X)$ generated by $Pic_{\widetilde{G}\times\widetilde{G}}(X)$.

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- $c^w_{v,v'} := c_K \mid_{R(\widetilde{G})} (a^w_{v,v'}) \in \mathbb{Z}$
- $\mathcal{R}(\overline{T}^+) := \mathbb{Z} \otimes_{R(\widetilde{G})} K_{\widetilde{T}}(\overline{T}^+) = \mathbb{Z} \otimes_{R(\widetilde{G})} K_{\widetilde{G}}(\widetilde{G} \times^{\widetilde{T}} \overline{T}^+) = K(\widetilde{G} \times^{\widetilde{T}} \overline{T}^+) = K(G \times^B \overline{T}^+)$ where B acts on $\overline{T}^+$ via its quotient T. In other words, $\mathcal{R}(\overline{T}^+)$ is the Grothendieck ring of the toric bundle over G/B associated with $\overline{T}^+$.

Ordinary K-ring of X

Theorem:

$$K(X) \simeq \bigoplus_{v \in W} \mathcal{R}(\overline{T}^{+}) \cdot \gamma_v \tag{4}$$

where

$$\gamma_v := 1 \otimes [\overline{f}_v] \in \mathcal{R}(\overline{T}^{+}) \otimes K(G/B)_I \tag{5}$$

for $v \in C^I$ for every $I \subseteq \Delta$. Here $\mathcal{R}(\overline{T}^{+})$ can be identified with the subring of $K(X)$ generated by $Pic(X)$. Further, the above isomorphism is a ring isomorphism, where

$$\gamma_v \cdot \gamma_{v'} := \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} (\overline{\lambda}_{I \cap I'} \cdot \overline{\lambda}_{(I \cup I') \setminus J}) \cdot c^{w}_{v,v'} \cdot \gamma_w . \tag{6}$$

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Theorem: The ring $K_{\widetilde{G} \times \widetilde{G}}(X)$ has the following presentation as a $K_{\widetilde{G} \times \widetilde{G}}(\overline{G_{ad}})$-algebra

$$K_{\widetilde{G} \times \widetilde{G}}(X) = \frac{K_{\widetilde{G} \times \widetilde{G}}(\overline{G_{ad}})[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]}{\mathfrak{I}} \qquad (7)$$

where $\mathfrak{I}$ is the ideal in $K_{\widetilde{G} \times \widetilde{G}}(\overline{G_{ad}})[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]$ generated by the elements X_F for $F \notin \mathcal{F}_+$ and $(\prod_{\rho_j \in \mathcal{F}_+(1)} X_j^{<u, v_j>}) - [\mathcal{L}_u]_{\widetilde{G} \times \widetilde{G}}$ for $u \in X^*(T)$.

Comparison with the ordinary K-ring of the wonderful compactification

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Theorem: The ring $K(X)$ has the following presentation as a $K(\overline{G_{ad}})$-algebra:

$$K(X) = \frac{K(\overline{G_{ad}})[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]}{\mathfrak{I}} \tag{8}$$

where $\mathfrak{I}$ is the ideal in $K(\overline{G_{ad}})[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]$ generated by the elements X_F for $F \notin \mathcal{F}_+$ and $(\prod_{\rho_j \in \mathcal{F}_+(1)} X_j^{<u,v_j>}) - [\mathcal{L}_u]$ for $u \in X^*(T)$. Here $[\mathcal{L}_u]$ denotes the class of the line bundle $\mathcal{L}_u$ in $K(\overline{G_{ad}})$.

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Theorem: The ring $K_{G^{ss} \times G^{ss}}(X)$ as a $K_{G^{ss} \times G^{ss}}(\overline{G_{ad}})$-algebra and $K(X)$ as a $K(\overline{G_{ad}})$-algebra are respectively isomorphic to the $G^{ss} \times G^{ss}$-equivariant and ordinary Grothendieck ring of the **toric bundle** $P \times_{T_{ad}} Z$ over $\overline{G_{ad}}$.

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