The Central Definition of Control Theory

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Abstract: I explain the notion of a 'controllable (dynamical) system' due to R.E. Kalman, and its post-modern generalization due to J.C. Willems.

1. Introduction

I am grateful for the opportunity to speak here.

In order to put in perspective the central definition of Control Theory, namely that of a 'controllable system', I must start with a little history.

There are roughly speaking, two main streams of the subject, which merged in the second half of the last century, but whose origins are quite different. The older, called 'trajectory optimization', goes back to the brachistochrone problem, solved by Newton and the Bernoulli brothers. Its history includes Euler, Lagrange, Hamilton, Jacobi ... and reaches its apogee with the maximum priciple of Pontrjagin. My talk however is not about this stream, but the other called the 'regulation problem', which is of engineering origin. This problem, as an academic discipline was initiated by Maxwell's paper 'On Governers' which dealt with the hunting problem in steam engines. This was the dawn of the industrial revolution, and as engineering specialized in the early part of the last century, so did the regulation problem specialize to different disciplines. In electrical engineering the problem was amplifier design, in aeronautical engineering it was powered flight, in chemical engineering it was control of chemical processes, and so on. These specializations developed their different ways until 'the enormous experience gained in the second world war' led to a unification of views, culminating in a paper by R.E. Kalman in 1960 [5].

In this paper Kalman proposed an elementary model of an engineering system about which he made an important definition:

The system is described by its phase or state x, which is a point in $\mathbb{R}^\ell$, and its evolution in time is given by the non-autonomous linear differential equation

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \tag{1}$$

where A is a linear endomorphism of $\mathbb{R}^\ell$, $u : [0,T] \to \mathbb{R}^m$ is a smooth function called the input, and $B : \mathbb{R}^m \to \mathbb{R}^\ell$ is a linear map.

This 'state space' system is said to be controllable if for any x_0, x_1 in $\mathbb{R}^\ell$, there is an input u such that the solution $x(t)$ of (1) satisfies $x(0) = x_0,\ x(T) = x_1$.

It turned out that this was the correct definition to make, and solutions to the problems of feedback stability, trajectory tracking, disturbance attenuation, and many others, were reduced to this notion of a controllable system.

As an example, consider the problem of feedback stability of the system (1): given a subset S of $\mathbb{C}$, ℓ in number counting multiplicity and symmetric about $\mathbb{R}$, is there a linear map $F : \mathbb{R}^\ell \to \mathbb{R}^m$ called feedback such that the autonomous system

$$\frac{d}{dt}x = Ax + BFx = (A + BF)x$$

has poles at S, which is to say that the eigenvalues of $(A+BF)$ is S? The answer is yes there is, precisely when (1) is a controllable system.

And thus did 'post-war' control theory grow around Kalman's definition. One important reason for its success was the computability of the theory ([3]), for instance (1) was controllable if and only if the matrix $(\lambda I - A,\ -B)$ had full rank for every $\lambda \in \mathbb{C}$.

The elementary linear model (1) was soon generalized to systems described by non-involutive distributions on smooth manifolds, to infinite dimensional systems given by a 1-parameter semi-group of operators on a Hilbert space, and to other situations. In time however some foundational questions led to a 'crisis' in the sense of T. Kuhn. For instance the division of signals x and u into state and input rested on a causal structure. Such structures were not unique, and often unnecessary. Questions such as these eventually resulted in a 'paradigm shift', due to J.C. Willems, and it is about this new paradigm that my talk is about.

One way to obliterate the difference between state and input is to write (1) as

$$\begin{pmatrix} \frac{d}{dt}I_\ell - A, & -B \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} = 0 \ ;$$

then to consider $f := (x,u)$ a point in $\mathbb{R}^{\ell+m}$ and the solutions of (1) as the kernel of the operator

$$\begin{pmatrix} \frac{d}{dt}I_\ell - A, & -B \end{pmatrix}: \ (\mathcal{C}^\infty)^k \ \longrightarrow \ (\mathcal{C}^\infty)^\ell$$

where $k = \ell + m$. But then one could consider the more general situation of a dynamical system whose possible trajectories are given by the kernel of the operator

$$P(\frac{d}{dt}) : (\mathcal{C}^\infty)^k \longrightarrow (\mathcal{C}^\infty)^\ell$$

where the entries of the $\ell \times k$ matrix $P(\frac{d}{dt})$ are from the ring $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$, and even more general situations. To proceed further, one needs a more general definition of a controllable system - it cannot be the ability to move from one state to another for there is no notion of state anymore - and it was such a definition that Willems provided [18, 19].

This new notion of a controllable system has been studied for various classes of systems (for instance [1, 2, 4, 9, 11, 12, 13] and the many references therein). I will now describe it for 'distributed systems', which are systems described by partial differential operators.

2. Linear Distributed Systems

In the framework of [18], let the space of 'independent variables' be $\mathbb{R}^n$, and let the 'universe of signals' $\mathcal{U}$ be a space of distributions on $\mathbb{R}^n$ (and so contained in $\mathcal{D}'(\mathbb{R}^n)$).

Examples of $\mathcal{U}$ that have been studied are [8, 9, 11, 14, 16]:
$\mathcal{D}', \mathcal{C}^\infty$, the space $\mathcal{S}'$ of temperate distributions; their duals $\mathcal{D}, \mathcal{E}'$ and the Schwartz space $\mathcal{S}$.

The Sobolev limits: for $s \in \mathbb{R}$, let $\mathcal{H}^s(\mathbb{R}^n)$ be the Sobolev space of order s. For $t < s$ there is a continuous inclusion $\mathcal{H}^s \hookrightarrow \mathcal{H}^t$, and we can consider the direct and inverse systems defined by the family $\{\mathcal{H}^s\}_{s\in\mathbb{R}}$ to obtain the direct and inverse limits $\overrightarrow{\mathcal{H}} = \varinjlim \mathcal{H}^s$ and $\overleftarrow{\mathcal{H}} = \varprojlim \mathcal{H}^s$.

Spaces generated by periodic functions: let $\mathsf{T} := \mathbb{R}^n/2\pi\mathbb{Z}^n$ be the torus and $\mathcal{C}^\infty(\mathsf{T})$ the space of smooth functions on $\mathbb{R}^n$ periodic with respect to the lattice $2\pi\mathbb{Z}^n$. For positive integers N_1 dividing N_2, the natural map $\mathcal{C}^\infty(\mathbb{R}^n/2\pi N_1\mathbb{Z}^n) \to \mathcal{C}^\infty(\mathbb{R}^n/2\pi N_2\mathbb{Z}^n)$ identifies the first space with a closed subspace of the second. This is a directed system indexed by the positive integers and the direct limit $\varinjlim \mathcal{C}^\infty(\mathbb{R}^n/2\pi N\mathbb{Z}^n)$ is denoted $\mathcal{C}^\infty(\mathsf{PT})$. Let $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}$ be those elements in $\mathcal{C}^\infty(\mathsf{T})$ which have only finitely many Fourier coefficients nonzero, and similarly for $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}$. Finally let $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}[x_1,\dots,x_n]$ be the space obtained by adjoining the elements $x_1,\dots,x_n$, that is the coordinate functions, to $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}$, and similarly the space $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}[x_1,\dots,x_n]$ obtained from $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}$; these are all spaces of smooth functions on $\mathbb{R}^n$.

Given a signal space $\mathcal{U}$, let $\mathrm{O}(\mathcal{U})$ be the collection of all pairs of open subsets of $\mathbb{R}^n$ for which the following is satisfied: (O_1,O_2) is in $\mathrm{O}(\mathcal{U})$ if given any two signals f_1,f_2 in $\mathcal{U}$, there is an f in $\mathcal{U}$ such that the restrictions of f_1,f_2,f to $\mathcal{D}'(O_1)$ and $\mathcal{D}'(O_2)$ satisfy the 'patching condition'

$$f|_{O_1} = f_1|_{O_1}, \qquad f|_{O_2} = f_2|_{O_2} \tag{2}$$

Signals f_1,f_2,f in $\mathcal{U}^k$ satisfy the patching condition if each of their components satisfies (2).

For example, if $\mathcal{U}$ is $\mathcal{D}$ or $\mathcal{C}^\infty$, then $\mathrm{O}(\mathcal{U}) = \{(O_1,O_2)|\ \overline{O}_1 \cap \overline{O}_2 = \varnothing\}$. On the other hand, if $\mathcal{U} = \mathcal{S}(\mathbb{R}^2)$, then the pair (O_1,O_2) where $O_1 = \{(x,y)\in\mathbb{R}^2|y<0\}$, $O_2 = \{(x,y)\in\mathbb{R}^2|y>e^{-x^2}\}$ is not in $\mathrm{O}(\mathcal{U})$ (but is in $\mathrm{O}(\mathcal{D})$ and in $\mathrm{O}(\mathcal{C}^\infty)$).

Definition: *A* system $\mathcal{B}$ *is a subset of* $\mathcal{U}^k$ *('consisting of signals that can occur'). The system* $\mathcal{B}$ *is* controllable *if for every pair* (O_1,O_2) *in* $\mathrm{O}(\mathcal{U})$*, the above patching condition holds within* $\mathcal{B}$*, namely, for every* f_1,f_2 *in* $\mathcal{B}$ *there is an* f *in* $\mathcal{B}$ *satisfying* (2).

The apparatus that restricts a signal, a priori in the universe $\mathcal{U}$, to a signal in $\mathcal{B}$, are the laws the system obeys.

Example: A priori, any smooth $f:\mathbb{R}^3\to\mathbb{R}^3$ could occur as a magnetic field, but because of the law that there are no magnetic monopoles in the universe, it must be that the divergence of f equals 0. In other words f must be in the kernel of the operator

$$\mathsf{div} : (\mathcal{C}^\infty(\mathbb{R}^3))^3 \longrightarrow (\mathcal{C}^\infty(\mathbb{R}^3))$$

More generally, let $\mathcal{A} = \mathbb{C}[D_1,\dots,D_n]$, $D_j = \frac{1}{\imath}\frac{\partial}{\partial x_j}$, be the ring of constant coefficient partial differential operators on $\mathbb{R}^n$ (i.e. the polynomial ring in the indeterminates $\{D_j\}$). Let $\mathcal{U}$ be one of the spaces of distributions listed above, they are all $\mathcal{A}$-modules. Let $p(D) = (p_1(D),\dots,p_k(D))$ be in $\mathcal{A}^k$, then a signal in $\mathcal{U}^k$ satisfies the law given by $p(D)$ if it is in the kernel of the operator

$$\begin{array}{rccc} p(D): & \mathcal{U}^k & \longrightarrow & \mathcal{U} \\ & f & \mapsto & p(D)f = \sum p_j(D)f_j \end{array}$$

Denote this kernel by $\mathcal{B}_{\mathcal{U}}(p)$. Given a collection $\{p_\alpha(D)\}$, a signal in $\mathcal{U}^k$ satisfies all the laws in this collection if it belongs to $\bigcap_\alpha \mathcal{B}_{\mathcal{U}}(p_\alpha)$. If $\mathcal{P}$ is the submodule of $\mathcal{A}^k$ generated by the p_α, then this intersection is also $\bigcap_{p\in\mathcal{P}}\mathcal{B}_{\mathcal{U}}(p)$, denoted $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$. If $\mathcal{P}$ is generated by $p_{\alpha_1},\dots,p_{\alpha_\ell}$, and if $P(D)$ is the matrix whose j-th row is $p_{\alpha_j} = (p_{j1},\dots,p_{jk})$, then $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ is the kernel of the operator

$$P(D) : \mathcal{U}^k \longrightarrow \mathcal{U}^\ell \tag{3}$$

This is the class of *linear distributed systems*, namely those $\mathcal{B}\subset\mathcal{U}^k$ given by kernels of such differential operators.

3. Controllable Systems

I will now describe necessary and sufficient conditions for linear distributed systems to be controllable, and then I will describe results from PDE on which they rest. The kernel $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ (in the signal space $\mathcal{U}$) is isomorphic to $\mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P},\ \mathcal{U})$, and results of Hörmander, Malgrange, Palamodov and others describe the functor $\mathsf{Hom}_{\mathcal{A}}(-,\ \mathcal{U})$, when $\mathcal{U}$ is $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$ and $\mathcal{D}$. Questions in control theory about this class of systems, questions such as 'when is $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ controllable', are about the functor $\mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P},\ -)$ and its behavior as we vary $\mathcal{U}$.

Some results about controllable systems:

1. The notion of controllability due to Willems is a generalization of Kalman's definition: the state space system (1) is controllable in the sense of Willems if and only if it is controllable in the sense of Kalman [18, 17].

2. If the system $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$, as described in (3), is an image, i.e. if

$$\mathcal{U}^{k_1} \xrightarrow{P_1(D)} \mathcal{U}^k \xrightarrow{P(D)} \mathcal{U}^\ell$$

is exact for some $P_1(D)$, then clearly the system is controllable [18, 11]. In this situation, physicists say that the system $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ admits a '(vector) potential'.
Example: The system given by the set of all smooth magnetic fields is controllable because

$$(\mathcal{C}^\infty(\mathbb{R}^3))^3 \xrightarrow{\mathsf{curl}} (\mathcal{C}^\infty(\mathbb{R}^3))^3 \xrightarrow{\mathsf{div}} (\mathcal{C}^\infty(\mathbb{R}^3))$$

is exact. (The $\mathsf{curl}, \mathsf{div}$ sequence is not exact in the Sobolev limits $\overleftarrow{\mathcal{H}}$ and $\overrightarrow{\mathcal{H}}$, (for example [16]), and it is not known whether the set of magnetic fields in these signal spaces is controllable.)

3. Let $\mathcal{U}$ be $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}[x_1, \ldots, x_n]$ or $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}[x_1, \ldots, x_n]$. Then a linear distributed system in one of these spaces is controllable if and only if it is an image, i.e. if and only if it admits a potential, [11, 14, 15, 8].

4. Let $\mathcal{P}$ be an $\mathcal{A}$-submodule of $\mathcal{A}^k$, and $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ the system defined by it in a signal space $\mathcal{U}$.
   (a) When $\mathcal{U}$ is $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$, $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ is controllable for every $\mathcal{P}$, [15].
   (b) When $\mathcal{U}$ is $\mathcal{D}'$ or $\mathcal{C}^\infty$, $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ is controllable if and only if $\mathcal{A}^k/\mathcal{P}$ is torsion free, [11].
   (c) $\mathcal{B}_{\mathcal{S}'}(\mathcal{P})$ is controllable if and only if the affine varieties in $\mathbb{C}^n$ of the nonzero associated primes of $\mathcal{A}^k/\mathcal{P}$ do not intersect $\mathbb{R}^n$, [15].
   (d) When $\mathcal{U}$ is $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}[x_1, \ldots, x_n]$, $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ is controllable if and only if the varieties in $\mathbb{C}^n$ of the nonzero associated primes of $\mathcal{A}^k/\mathcal{P}$ do not intersect $\mathbb{Q}^n$, [8].
   (e) When $\mathcal{U}$ is $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}[x_1, \ldots, x_n]$, $\mathcal{B}_{\mathcal{U}}(\mathcal{P})$ is controllable if and only if the varieties of the nonzero associated primes of $\mathcal{A}^k/\mathcal{P}$ do not intersect $\mathbb{Z}^n$, [8].

These results rest on seminal results of Malgrange and Palamodov describing the module structure of the following spaces ([7, 10]):

As modules over the ring $\mathcal{A} = \mathbb{C}[D_1, \ldots, D_n]$, $\mathcal{D}'$ and $\mathcal{C}^\infty$ are injective cogenerators, $\mathcal{S}'$ is injective, $\mathcal{S}$ is flat, and $\mathcal{E}'$ and $\mathcal{D}$ are faithfully flat.

Further, $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}[x_1,\ldots,x_n]$ is an injective envelope of $\mathcal{C}^\infty(\mathsf{PT})_{\mathsf{fin}}$, and $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}[x_1,\ldots,x_n]$ is an injective envelope of $\mathcal{C}^\infty(\mathsf{T})_{\mathsf{fin}}$ [8].

These theorems answer the solvability question for PDE: given the operator $P(D)$ in (3), when is a $g \in \mathcal{U}^\ell$ in its image? When $\mathcal{U}$ is an injective $\mathcal{A}$-module, the image of every $P(D)$ is the kernel of a differential operator, i.e.

$$\mathcal{U}^k \xrightarrow{P(D)} \mathcal{U}^\ell \xrightarrow{P^1(D)} \mathcal{U}^{\ell_1}$$

is exact for some $P^1(D)$, and so g is in the image of $P(D)$ precisely when it is in the kernel of $P^1(D)$. For general $\mathcal{U}$, the solvability question asks when an image is also a kernel. Thus, the notion of a controllable system, for the spaces listed in result 3 above, is dual to the solvability question. (There are also other interpretations of a controllable system, for instance [6].)

In the spaces for which the controllability problem has been solved, it has turned out that a system is controllable if and only if it is an image, or in the language of Physics, if and only if it admits a potential. This leads to a fundamental question:

Is the notion of a controllable system, first defined by Kalman and then generalized by Willems, identical to the notion of a potential, a notion going back to Newton?

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References

[1] H. Bourles and B. Marinescu, *Linear Time-Varying Systems*, LNCIS 410, Springer, 2011.
[2] H. Glüsing-Lüerssen, *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach*, Lecture Notes in Mathematics 1770, Springer 2002.
[3] M.L.J. Hautus, *Controllability and observability conditions for linear autonomous systems*, Nederlandse Akademie van Wetenschappen, Proc. Ser. A, 72:443-448, 1969.
[4] A. Ilchmann and V. Mehrmann, *A behavioral approach to time-varying linear systems*, SIAM jl. Control and Optimization, 44:1725-1747, 2005.
[5] R.E. Kalman, *On the general theory of control systems*, Proceedings of the 1st World Congress of the International Federation of Automatic Control, 481-493, Moscow, 1960.
[6] V. Lomadze, *Controllability as minimality*, SIAM jl. Control and Optimization, 50:357-367, 2012.
[7] B. Malgrange, Division des distributions, *Séminaire L.Schwartz*, Exposés 21-25, 1960.
[8] D. Napp, M. van der Put and S. Shankar, *Periodic behaviors*, SIAM jl. Control and Optimization, 48:4652-4663, 2010.
[9] U. Oberst, *Multidimensional constant linear systems*, Acta Applicandae Mathematicae, 20:1-175, 1990.
[10] V.P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer Verlag, 1970.
[11] H.K. Pillai and S. Shankar, *A behavioural approach to control of distributed systems*, SIAM jl. Control and Optimization, 37:388-408, 1998.
[12] J-F. Pommaret and A. Quadrat, *Algebraic analysis of linear multidimensional control systems*, IMA jl. on Mathematical Control and Information, 16:275-297, 1999.
[13] P. Rocha and J.C. Willems, *Behavioral controllability of delay-differential systems*, SIAM jl. Control and Optimization, 35:254-264, 1997.
[14] S. Shankar, *The Nullstellensatz for systems of PDE*, Advances in Applied Mathematics, 23:360-374, 1999.

[15] S. Shankar, *Geometric completeness of distribution spaces*, Acta Applicandae Mathematicae, 77:163-180, 2003.

[16] S. Shankar, *On the dual of a flat module in TOP*, Linear Algebra and its Applications, 433:1077-1081, 2010.

[17] S. Shankar, *The Hautus test and genericity results for controllable and uncontrollable behaviors*, SIAM jl. Control and Optimization, 52:32-51, 2014.

[18] J.C. Willems, *Paradigms and puzzles in the theory of dynamical systems*, IEEE Trans. Automatic Control, 36:259-294, 1991.

[19] J.C. Willems, *Open dynamical systems and their control*, Proceedings of the International Congress of Mathematicians, Berlin, Vol III: Invited lectures, 697-706, 1998.