The embedding problem of infinitely divisible probability measures on Lie groups

Riddhi Shah

School of Physical Sciences
Jawaharlal Nehru University, New Delhi
rshah@jnu.ac.in

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Abstract

We discuss infinitely divisible measure on Lie groups. We discuss the conditions under which an infinitely divisible measure can be embedded in a continuous one-parameter convolution semigroup of probability measures. We survey known results and discuss the latest results obtained (jointly with Dani and Guivarc'h). We also mention some techniques developed by Dani and McCrudden which gives the embeddability in some new cases.

Plan of Talk

- Introduction to the Embedding Problem
  Infinitely Divisible Measures
  Embeddable Measures
- Known Results
- 'new' results on Lie Groups
- Techniques: Fourier Analysis, Concentration Functions of Measures, Structure Theory of Lie Groups and Algebraic Groups

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It is the smallest topology in which, $\mu_n \to \mu$, if and only if for every bounded continuous function f,

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Examples: Dirac measures on $\mathbb{R}$, $\mathbb{R}^n$, S^1 or $\mathbb{Q}_p$, exponential measures.

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Note that every embeddable measure is infinitely divisible, for if μ is embeddable in $\{\mu_t\}_{t\geq 0}$, then $\mu = \mu_1 = \mu_{1/n}^n$, for all n.

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Why the embedding problem?
On $\mathbb{R}^n$, any embeddable measure is a convolution of a Dirac measure, a Gausssian measure and a Poisson measure. In general the structure of a continuous one-parameter convolution semigroup is well understood. Also, embeddable measures can be approximated by Poisson (exponential) measures. Infinitely divisible/embeddable measures appear in the central limit theory as limits of commutative infinitesimal triangular systems of measures.

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On p-adic linear groups (R. Shah)

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In fact they later labelled this class of Lie groups as class $\mathcal{C}$. They also characterised this class as follows:

Lie Groups in Class $\mathcal{C}$

Let R be the solvable radical of a connected Lie group G. Then $\overline{[R, R]}$ is connected and nilpotent and it has a maximal torus which they call the $\mathcal{C}$-kernel of G which is characteristic. Since G is connected, the $\mathcal{C}$-kernel is central.

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In particular the above holds for a connected Lie group G without any compact central subgroup of positive dimension.

Embedding problem on class $\mathcal{C}$ Lie Groups

Moreover, for any infinitely divisible measure μ on a connected Lie group G, there exists $\{z_n\}$ in G such that $z_n T$ centralises $\pi(\mu)$ in G/T, T as above, such that $z_n \mu z_n^{-1} \to \lambda$ and λ is embeddable. (This is called the week embedding of μ.)

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Observe that $z_n(\mu * \omega_T)z_n^{-1} = \mu * \omega_T = \lambda * \omega_T$, and hence $\mu * \omega_T$ is embeddable, where ω_T is the Haar measure supported on T. (One can also replace T by the $\mathcal{C}$-kernel of G).

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McCrudden has two survey articles which capture the early developments until 2001: In the proceedings of "Positivity in Lie Theory" and the proceedings of the CIMPA-TIFR School.

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For e.g. the group of this type with the smallest dimension possible for which the embedding problem was not solved is the so-called Walnut Group.

Walnut Group

The Walnut group is defined as follows:

Take a 3-dimensional Heisenberg group H with the one-dimensional center Z. Now take a discrete subgroup D in the center and take $N = H/D$. Then the center of N is $T = Z/D$ which is a one-dimensional torus.

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(Neither G nor W belongs to class $\mathcal{C}$ since $[N, N] = T$ as you can see below.)

Note that the action of $SL(2,\mathbb{R})$ on H is given by the following:

$$\begin{aligned} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} (a,b,c) &= (x_1 a + x_2 b, x_3 a + x_4 b, c + c'), \\ \text{where } c' &= \frac{1}{2} a^2 x_1 x_3 + ab x_2 x_3 + \frac{1}{2} b^2 x_2 x_4, \end{aligned}$$

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In particular, the action of the rotation group on the Heisenberg group H is defined as follows:

$$T_\theta(a,b,c) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta, c + c'),$$
$$\text{where } c' = \frac{1}{2}a^2\cos\theta\sin\theta - ab\sin^2\theta - \frac{1}{2}b^2\sin\theta\cos\theta.$$

Embedding Theorems (with Dani and Guivarc'h)

Theorem 1: Let G be a semidirect product of $SL(2, \mathbb{R})$ and N as above. Let M be a closed connected subgroup of G. Then any infinitely divisible measure on M is embeddable in $P(M)$. In particular, every infinitely divisible measure on the Walnut group is embeddable.

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The above is proven with the help of the following general result which is somewhat technical and has some conditions on the group and on the measure.

Main Theorem

Main Theorem: Let G be a Lie group admitting a surjective continuous homomorphism $p : G \to \tilde{G}$ onto an almost algebraic group $\tilde{G}$, such that $\ker p$ is contained in the center of G and $(\ker p)^0$ is compact. Let $T = (\ker p)^0$ and let $q : G \to G/T$ be the natural quotient homomorphism. Let μ belong to $P(G)$ be such that $q(\mu)$ has no idempotent factor. If μ is infinitely divisible, then μ is embeddable.

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The first condition is satisfied if $G/\overline{[G,G]}$ is compact or more generally, if $G/\overline{[G,G]N}$ is compact, where N is the nilradical of G. The latter condition is equivalent to the condition that the solvable radical of G is a compact extension of the nilradical.

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Now for a particular subgroup L containing the support of μ, it follows that $\lambda_i y_i = z_i \lambda_i z_i^{-1}$ for some $z_i \in Z(\mu) \cap N$ in L; where N is a simply connected nilpotent group in L.

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Note that if λ_i is an n-th root of μ, then so is $z_i \lambda_i z_i^{-1}$ since $z_i \in Z(\mu)$; as $(z_i \lambda_i z_i^{-1})^n = z_i \lambda_i^n z_i^{-1} = z_i \mu z_i^{-1} = \mu$.

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Applying the above for $p(\mu)$ on $\tilde{G}$ with roots $p(\lambda_i)$, we get $\{x_i \lambda_i x_i^{-1}\}$ is relatively compact, where λ_i are roots of μ and x_i are such that $p(x_i) = z_i \in Z(p(\mu))$, i.e. $x_i \in p^{-1}(Z(p(\mu)))$ which is usually larger than $Z(\mu)$.

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Here $p(x_i \mu x_i^{-1}) = z_i p(\mu) z_i^{-1} = p(\mu)$ and since the kernel of p is T which is compact, $\{x_i \mu x_i^{-1}\}$ is relatively compact. Then we show using an earlier result on concentration functions of measures that all the limit points of $\{x_i \mu x_i^{-1}\}$ are of the form $\tau(\mu)$ where τ is an automorphism which fixes T pointwise.

We consider the group $I(\mu) = \{x \in G \mid x\mu x^{-1} = \mu\}$. Using the following theorem, we get that $\{x_i\}/I(\mu)$ is relatively compact, i.e. x_i's can be replaced by elements of $I(\mu)$ and in particular, we get that $x_i\lambda_i x_i^{-1}$'s are roots of μ.

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This way, we get a relatively compact set of roots and its fractional powers 'upto' μ and we can choose a rational embedding which extends to a real embedding.

Theorem 3: Let K be a compact connected subgroup of $GL(d, \mathbb{R})$, $d \geq 2$, such that the K-action on $\mathbb{R}^d$ has no nonzero fixed point. Let λ be a K-invariant probability measure on $\mathbb{R}^d$ such that $\lambda(V) = 0$ for any proper subspace V of $\mathbb{R}^d$. Then λ^d has a density in $\mathbb{R}^d$. In particular, $\hat{\lambda}$ vanishes at infinity.

Joint work with S.G. Dani and Yves Guivarc'h; Math. Zeit.

Some observations and some new cases

Suppose μ is supported on the center of the group G, then μ is embeddable as $x_i \mu x_i^{-1} \to \lambda = \mu$ which is embeddable as λ is so.

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From the result of weak embeddabality by Dani and McCrudden mentioned earlier, one can deduce that μ is infinitely divisible on $H = G(\mu)M$, where $M = N^0(\mu * \omega_T)$ is the connected component of the normaliser of $G(\mu * \omega_T) = G(\mu)T$.

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Observe that $G(\mu)T$ is normal in H and it is connected and nilpotent if $G(\mu)$ is connected and nilpotent. In particular, μ is supported on the nilradical of H. Hence we can get the following:

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If μ is infinitely divisible on a connected Lie group G such that $G(\mu)$ is connected and nilpotent, then μ is embeddable.

If $G(\mu)$ is a discrete group, then $M/T = Z^0(\pi(\mu))$, i.e. it centralises $G(\pi(\mu))$, where $\pi : G \to G/T$. Also, the commutator subgroup of $\pi^{-1}(Z(\pi(\mu)))$ is contained in $Z(\mu)$, using this we can choose x_i as above in $Z(\mu)$ and hence we get the following:

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Now there is a new survey by F. Leddrappier and R. Shah in Contemporary Mathematics, 631 (2015), a volume dedicated to Dani on his 65th Birthday.

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