Groups with expansive automorphisms

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Automorphism, contraction groups

$G$ - locally compact totally disconnected group, that is, $G$ has arbitrarily small compact open subgroups.

By an automorphisms $\alpha$ of $G$, we mean a continuous automorphism.

For an automorphism $\alpha$ of $G$ we consider the following two subgroups:

$U_\alpha = \{ x \in G | \lim_{n \to \infty} \alpha^n(x) = e \}$ - known as the contraction group of $\alpha$

$U_\alpha^{-1} = \{ x \in G | \lim_{n \to -\infty} \alpha^n(x) = e \}$
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Remarks about $U_\alpha$ and $U_\alpha^{-1}$

We recall the following facts about the contraction groups. In general, neither $U_\alpha$ nor $U_\alpha^{-1}$ is closed; $U_\alpha$ is closed if and only if $U_\alpha^{-1}$ is closed ([BaWi-04]). If $G$ is a $p$-adic Lie group, $U_\alpha$ is closed, in fact an unipotent algebraic group ([Wa-84]).
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Expansive automorphism

An expansive automorphism $\alpha$ of a totally disconnected locally compact group $G$ is called expansive if there is a compact open subgroup $K$ of $G$ such that $\bigcap_{n \in \mathbb{Z}} \alpha^n(K) = \{e\}$.

Aim: Study the structure of groups that admit expansive automorphisms.

The following are easy to observe:

- Automorphisms on discrete groups are expansive.
- If $\alpha$ restricted to an open subgroup is expansive, then $\alpha$ is expansive.
- Equicontinuous automorphisms on a non-discrete group is not expansive. For instance $\alpha \in \text{GL}_n(\mathbb{Z}_p)$ is not expansive on $\mathbb{Q}_p$. 
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Examples - Contraction

Scalar multiplication on $\mathbb{Q}_p$ by $p$ is expansive. In this case, $U_{\alpha} = \mathbb{Q}_p$.

Automorphisms for which $U_{\alpha} = G$, are called contractive and groups admitting contractive automorphisms are called contraction groups. Any contractive automorphism is expansive.
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Let 
\[ G = \{ (x, y, z + Z_p) \mid x, y, z \in \mathbb{Q}_p \} \]
with multiplication given by
\[ (x, y, z + Z_p)(x', y', z' + Z_p) = (x + x', y + y', z + z' + xy' + Z_p) \]
and
\[ \alpha: G \to G \text{ be given by} \]
\[ \alpha(x, y, z + Z_p) = \left( \frac{x}{p}, py, z + Z_p \right) \]
Then \( \alpha \) is an expansive automorphism but \( G \) does not admit any contractive automorphism as the commutator of \( G \) is discrete.
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Then $\alpha$ is an expansive automorphism but $G$ does not admit any contractive automorphism as the commutator of $G$ is discrete.
Let $\alpha$ be an automorphism of $G$.

For a compact open subgroup $V$, consider the following:

$V^+ = \bigcap_{n \geq 0} \alpha^n(V)$,

$V^- = \bigcap_{n \leq 0} \alpha^n(V)$,

$V_0 = V^+ \cap V^-$,

$V^{++} = \bigcup_{n \geq 0} \alpha^n(V^+)$,

$V^{--} = \bigcup_{n \leq 0} \alpha^n(V^-)$.

In general neither of $V^{++}$, $V^{--}$ is closed. However, Theorem [Wi-94] there is a compact open subgroup $V$ such that $V^{++}$ and $V^{--}$ are closed and $V = V^+ V^-$. Such a subgroup is called a tidy subgroup for $\alpha$. 

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**Theorem [Wi-94]**

There is a compact open subgroup $V$ such that $V_{++}$ and $V_{--}$ are closed and $V = V_+ V_-$. Such a subgroup is called a tidy subgroup for $\alpha$. 
Levi factor

We define the Levi factor $M_{\alpha} = \{ x \in G | \{ \alpha_n(x) \} \text{ is compact} \}$ and we have $M_{\alpha}$ is a $\alpha$-invariant closed subgroup ([Wi-94]).

We observe that Proposition (GlR) $\alpha$ is expansive if and only if $\alpha$ restricted to $M_{\alpha}$ is expansive.

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**Proposition (GIR)**

$\alpha$ is expansive if and only if $\alpha$ restricted to $M_\alpha$ is expansive.
Consequences of expansiveness

Assuming \( \alpha \) is expansive on \( G \), we observe the following:

- \( G \) is metrizable;
- \( V_0 \) is trivial for some compact open subgroup \( V \); and
- \( V = V + V^- \).

\( U_\alpha \) could be given a locally compact group topology \( \tau \) so that \( \alpha \) is contraction on \( (U_\alpha, \tau) \) and the canonical injection \( (U_\alpha, \tau) \to G \) is continuous (also proved in [Si-88]).

\( U_\alpha U_\alpha^{-1} \) is open and the converse holds if \( U_\alpha \) is closed.

In general, the converse need not be true.
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- $U_\alpha U_\alpha^{-1}$ is open and the converse holds if $U_\alpha$ is closed. In general, the converse need not be true.
Counter-example

Take $G = \mathbb{K}Z$ where $\mathbb{K}$ is any compact group and $\alpha$ to be the right shift. In this situation, $U^\alpha U^{\alpha^{-1}} = G$. It can be shown that $\alpha$ is expansive iff $\mathbb{K}$ is finite. Here $\alpha$ is never contractive since $U^\alpha$ as well as $U^{\alpha^{-1}}$ is a proper subgroup.

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Normal series

Assume $\alpha$ is expansive on $G$.

Theorem (GlR)

There exists $\alpha$-stable subnormal series of closed subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ of $G$ such that

1. every $\alpha$-stable closed normal subgroup of $G_{j-1}/G_j$ is discrete or open.
2. each of the quotient groups $G_{j-1}/G_j$ is discrete, abelian or topologically perfect.

Proof

We first find an upper bound for the number of $j$ in any subnormal series $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ for which $G_{j-1}/G_j$ is not discrete. We choose a series that has maximum such $j$, hence subfactors of such a series satisfy (1). For each such $j$ we introduce $(G_{j-1} \supseteq M_j \supseteq N_j \supseteq G_j)$ so that the conclusion are valid for the subfactors.

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2. each of the quotient groups $G_{j-1}/G_j$ is discrete, abelian or topologically perfect.

**Proof**

- We first find an upper bound for number of $j$ in any subnormal series $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ for which $G_{j-1}/G_j$ is not discrete.

- We choose a series that has maximum such $j$, hence subfactors of such a series satisfy (1).
Normal series

Assume $\alpha$ is expansive on $G$.

**Theorem (GIR)**

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- We choose a series that has maximum such $j$, hence subfactors of such a series satisfy (1).
- For each such $j$ we introduce $(G_{j-1} \supseteq) M_j \supseteq N_j (\supseteq G_j)$ so that the conclusion are valid for the subfactors.
A basic property

Theorem (GlR)

If $\alpha$ is expansive on $G$ and $H$ is a closed normal $\alpha$-stable subgroup of $G$, then the factor of $\alpha$ is expansive on $G/H$.

The result was known for compact groups (see [Sch-95], [Wi-15]).

Proof

We restrict to the Levi factor and prove the expansiveness of the factor automorphism.
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Proof

We restrict to the Levi factor and prove the expansiveness of the factor automorphism.
The following are abelian groups with expansive automorphisms:

1. $\mathbb{Q}_n$ for some $n \in \mathbb{N}$, together with a linear automorphism $\beta: \mathbb{Q}_n \rightarrow \mathbb{Q}_n$ such that $\beta$ or $\beta^{-1}$ is contracting.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $\mathbb{C}_p$ be the cyclic group of order $p$ and $\mathbb{C}_{\mathbb{N}_0}^p$ be the restricted direct product.

2. $\mathbb{C}_{\mathbb{N}_0}^p \times \mathbb{C}_p$ with the right-shift;

3. $\mathbb{C}_{\mathbb{N}_0}^p \times \mathbb{C}_p$ with the left-shift;

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Let $A$ be an abelian, totally disconnected, locally compact group and $\alpha: A \rightarrow A$ be an expansive automorphism. Assume that $A = \cup \alpha U \alpha^{-1}$ and every $\alpha$-stable proper closed subgroup of $A$ is discrete. Then there exists a prime number $p$ such that $(A, \alpha)$ is isomorphic to one of the above.
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Lie groups over local fields

Proposition (GlR)

An automorphism $\alpha$ of a Lie group over a local field is expansive if and only if the differential $d\alpha$ has no eigenvalue of absolute value one.

Proposition [Bourbaki]

If a Lie algebra has an automorphism that has no eigenvalue of absolute value one, then the Lie algebra is nilpotent.

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If a Lie group over a local field has an expansive automorphism, then its Lie algebra is nilpotent.

Raja Expansive
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Even for a $p$-adic Lie group, $U_\alpha U_\alpha^{-1}$ may not be a group. However, Theorem (GlR)

Let $G$ be a $p$-adic Lie group with an expansive automorphism $\alpha$.

If $G$ has a continuous injection into $\text{GL}_n(\mathbb{Q}_p)$, then $G$ has a $\alpha$-stable nilpotent open subgroup.

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Even for a $p$-adic linear group, $U_\alpha$ may not normalize $U_\alpha^{-1}$: recall that $U_\alpha$ as well as $U_\alpha^{-1}$ both are closed.
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$p$-adic Lie groups contd.,

\[ H = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{Q}_p \right\} \]

and \( G = H \times H \).

Define \( \beta : H \to H \) by

\[ \beta \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & px/\p & z/\p^2 \\ 0 & 1 & y/\p \\ 0 & 0 & 1 \end{pmatrix} \]

and take \( \alpha = \beta \times \beta^{-1} \) on \( G \).

Then \( U_\alpha = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| x \in \mathbb{Q}_p \right\} \times \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| y, z \in \mathbb{Q}_p \right\} \)

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In this case \( U_\alpha U_\alpha^{-1} = G \), but neither \( U_\alpha \) nor \( U_\alpha^{-1} \) normalize the other.
Take $H = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Q}_p \right\}$ and $G = H \times H$. Define $\beta : H \to H$ by 

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$U_\alpha = \{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p \} \times \{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{Q}_p \}$ and

$U_{\alpha^{-1}} = \{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{Q}_p \} \times \{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p \}$. In this case $U_\alpha U_{\alpha^{-1}} = G$.
Take $H = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Q}_p \right\}$ and $G = H \times H$.

Define $\beta: H \to H$ by $\beta \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & px & z/p \\ 0 & 1 & y/p^2 \\ 0 & 0 & 1 \end{pmatrix}$ and take $\alpha = \beta \times \beta^{-1}$ on $G$.

Then $U_\alpha = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p \right\} \times \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{Q}_p \right\}$ and $U_{\alpha^{-1}} = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{Q}_p \right\} \times \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p \right\}$.

In this case $U_\alpha U_{\alpha^{-1}} = G$ but neither $U_\alpha$ nor $U_{\alpha^{-1}}$ normalize the other.
Thanks for your attention!!!