An invariant for homogeneous spaces of compact quantum groups

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Abstract

The central notion in Connes' formulation of non commutative geometry is that of a spectral triple. Given a homogeneous space of a compact quantum group, restricting our attention to all spectral triples that are 'well behaved' with respect to the group action, we construct a certain dimensional invariant. In particular, taking the (quantum) group itself as the homogeneous space, this gives an invariant for a compact quantum group. Computations of this invariant in several cases, including all type A quantum groups, are given.

1 Introduction

Geometry can be broadly interpreted as the study of cycles and their intersection properties in some suitable homology theory. Noncommutative Geometry is no exception. In Alain Connes' interpretation, noncommutative geometry is the study of spectral triples or unbounded Kasparov-modules with finer properties ([6, 7]). Often these finer properties encode information about metric, dimension etc. In fact for an unbounded K-cycle to encode useful information, it is not always necessary that the cycle should be homologically nontrivial. A cycle may be homologically trivial but still it may contain metric or dimensional information. One prime example is the Laplacian on odd dimensional manifolds. Study of cycles not necessarily nontrivial is not new. Apart from Connes (see for example [5]), it also includes Voiculescu's work ([22], [23]) on norm ideal perturbations or Rieffel's work ([20]) on extending the notion of metric spaces. Voiculescu answers the question of existence of bounded K-cycles in a given representation of a C^*-algebra. He does not comment on the nontriviality of the K-cycle as a K-homology class. Rieffel uses spectral triples to produce compact quantum metric spaces but in his construction nontriviality do not play any major role. Here we take a similar approach. We utilise the notion of spectral triples to produce dimensional invariants for ergodic C^*-dynamical systems. However, we should emphasize a crucial distinction between the above cited works and the present one. In Rieffel's work, Dirac operators are used as a source to produce compact quantum metric spaces; but there are situations ([1]) where one produces compact quantum metric spaces even without using spectral triples. Whereas in our case, the concept of spectral triple is used in an essential manner, not as a source of examples.

Origin of the present paper lies in the search for non trivial spectral triples for the quantum $SU(2)$ ([2]) and quantum spheres ([4]). Instead of KK-theoretic machinery, our method tries to characterize equivariant spectral triples. In the process, one observes that even without the condition of nontriviality of the corresponding K-cycle, in certain cases there are canonical spectral triples encoding essential information about the space. That leads to the present invariant.

Let us recall the classical fact that if M is a d-dimensional compact Riemannian manifold and we have an elliptic operator D of order 1 then the n-th singular value of $|D|$ grows like $n^{1/d}$. Using this Connes has indicated how to define the dimension of a spectral triple. This observation can be utilized to obtain invariants for C^*-algebras provided one could associate natural spectral triples with them. But unlike the classical case it is difficult to define natural spectral triples for C^*-algebras. As one tries to answer this question one faces two main dfficulties: one, it is difficult to get hold of a canonical representation other than the GNS representation, which normally is too big; two: once a 'natural' representation has been identified, it is impossible to classify all spectral triples so as to be able to extract any meaningful quantity out of them. Another important point pertaining to the last problem above is that when considering a spectral triple, one should look at a spectral triple for some dense *-subalgebra of the C^*-algebra, but in general there is no canonical choice of a dense *-subalgebra; but properties of the behaviour of the spectral triple is sensitive to this choice.

If we restrict our attention to homogeneous spaces of compact quantum groups, or which is the same thing, to ergodic C^*-dynamical systems, then these problems can be resolved to a large extent. Ergodicity gives us a unique invariant state, so passing to its GNS representation gives us a way of fixing a canonical representation. Once that is done, we can again use the group action to identify an appropriate dense *-subalgebra. As we shall see, in many situations, it is then possible to investigate a subclass of spectral triples for this dense *-algebra that are 'well-behaved' with respect to the structure at hand; and one can study them in enough details in order to be able to come up with an invariant.

In the next section, we make all these notions precise and define the invariant. In the remaining sections, we compute it in several cases. In particular, in section 5, this is done for all type A quantum groups. When one introduces an invariant for a class of objects, one of the major issues one must look at is its computability. Thus section 5 can be looked upon as the heart of the present paper.

2 The Invariant

Let us start by recalling the notion of a spectral triple.

Definition 2.1 *Let $\mathcal{A}$ be an associative unital *-algebra. A* ***spectral triple*** *for $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, D)$ where*

1. *$\mathcal{H}$ is a (complex separable) Hilbert space,*
2. *$\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$ is a *-representation (usually assumed faithful),*
3. *D is a self-adjoint operator with compact resolvent such that $[D, \pi(a)] \in \mathcal{L}(\mathcal{H})$ for all $a \in \mathcal{A}$.*

One often writes $(\mathcal{H}, \mathcal{A}, D)$ in place of $(\mathcal{H}, \pi, D)$ if the representation π is clear from the context. We say that a spectral triple is p-summable if $|I + D^2|^{-p/2}$ belongs to the ideal $\mathcal{L}^1$ of Trace-class operators.

Remark 2.2 *Since D has compact resolvent, the above is equivalent to saying that $|D|^{-p}$ is Trace-class on the complement of its kernel. In some of our earlier papers ([2, 4]), the phrase 'p-summable' was used to mean that $|I + D^2|^{-p/2}$ is in the ideal $\mathcal{L}^{(1,\infty)}$ of Dixmier traceable operators, which is slightly different from the definition that we use here.*

It is a well known fact [10] that if M is a compact Riemannian manifold of dimension d and D is a elliptic differential operator of order r then $\mu_n(|D|)$, the n-th smallest singularvalue of D behaves like $n^{\frac{r}{d}}$. Therefore one has the equality

$$d = \inf\{\delta : \mathrm{Tr}|D|^{-\frac{\delta}{r}} < \infty\}.$$

Motivated by this Alain Connes defined the dimension of a spectral triple $\mathcal{E} = (\mathcal{A}, \mathcal{H}, D)$ as

$$\dim \mathcal{E} = \inf\{\delta : \mathrm{Tr}|D|^{-\delta} < \infty\}.$$

We want to utilize this concept to define a dimensional invariant for a C^*-algebra. There are two obstacles, firstly there is no canonical representation associated with a C^*-algebra and secondly even after fixing a representation there is no canonical Dirac operator. One simple strategy to overcome the first problem is to consider C^*-dynamical systems (A, G, τ) with an ergodic action of a compact quantum group G. Then by passing to the GNS representation of the invariant state we get a canonical representation. To tackle the second problem we exploit the notion of equivariance.

Let us now recall various terminologies used here

Definition 2.3 *([26]) A compact quantum group G consists of a unital C^*-algebra $C(G)$ along with a unital homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ such that*

1. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$,

2. *both* $\{(a \otimes I)\Delta(b) : a, b \in C(G)\}$ *and* $\{(I \otimes a)\Delta(b) : a, b \in C(G)\}$ *are total in* $C(G)$.

It is known that any compact quantum group admits a unique invariant state h, called the Haar state. The invariance property that the Haar state has is the following:

$$(h \otimes id)\Delta(a) = h(a)I = (id \otimes h)\Delta(a) \quad \text{for all } a \in C(G).$$

Definition 2.4 *([18]) We say that a compact quantum group* G *acts on a* C^**-algebra* A *if there is a homomorphism* $\tau : A \to A \otimes C(G)$ *such that*

1. $(\tau \otimes id)\tau = (id \otimes \Delta)\tau$,

2. $\{(I \otimes b)\Delta(a) : a \in A, b \in C(G)\}$ *are total in* $A \otimes C(G)$.

The C^**-algebra* A *is called a* ***homogeneous space*** *of* G *if the fixed point subalgebra* $\{a \in A : \tau(a) = a \otimes I\}$ *is* $\mathbb{C}I$. *In such a case, the action* τ *is said to be* ***ergodic*** *and we call* (A, G, τ) *an ergodic* C^**-dynamical system.*

Recall (Proposition 1.9, [18]) that if A happens to be a quotient space of G (i.e. A is isomorphic to $C(G\backslash H)$ for some closed quantum subgroup H of G and the G-action on A is equivalent to the natural G-action on $G\backslash H$), then A is a homogeneous space of G and the action is ergodic.

A covariant representation (π, u) of a C^*-dynamical system (A, G, τ) consists of a unital *-representation $\pi : A \to \mathcal{L}(\mathcal{H})$, a unitary representation u of G on $\mathcal{H}$, i.e. a unitary element of the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ such that they obey the condition $(\pi \otimes \mathrm{id})\tau(a) = u(\pi(a) \otimes I)u^*$ for all $a \in A$.

Definition 2.5 *Suppose* (A, G, τ) *is a* C^**-dynamical system. An operator* D *acting on a Hilbert space* $\mathcal{H}$ *is said to be* ***equivariant*** *with respect to a covariant representation* (π, u) *of the system if* $D \otimes I$ *commutes with* u. *If* (π, u) *is a covariant representation of* (A, G, τ) *on a Hilbert space* $\mathcal{H}$ *and* $(\mathcal{H}, \pi, D)$ *is a spectral triple for a dense *-subalgebra* $\mathcal{A}$ *of* A*, then we say that* $(\mathcal{H}, \pi, D)$ *is equivariant with respect to* (π, u) *if the operator* D *is equivariant with respect to* (π, u).

A homogeneous space A for G admits an invariant state ρ that satisfies

$$(\rho \otimes id)\tau(a) = \rho(a)I, \quad a \in A.$$

This invariant state ρ is unique and is related to the Haar state h on G through the equality

$$(id \otimes h)\tau(a) = \rho(a)I, \quad a \in A.$$

Given an ergodic C^*-dynamical system (A, G, τ) with unique invariant state ρ, denote by $(\mathcal{H}_\rho, \pi_\rho, \eta_\rho)$ the GNS representation associated with the state ρ, i.e. $\mathcal{H}_\rho$ is a Hilbert space,

$\eta_\rho : A \to \mathcal{H}_\rho$ is linear with $\eta_\rho(A)$ dense in $\mathcal{H}_\rho$ and $\langle \eta_\rho(a), \eta_\rho(b)\rangle = \rho(a^*b)$; and $\pi_\rho : A \to \mathcal{L}(\mathcal{H}_\rho)$ is the *-representation of A on $\mathcal{H}_\rho$ given by $\pi_\rho(a)\eta_\rho(b) = \eta_\rho(ab)$. The action τ induces a unitary representation u_τ of G on $\mathcal{H}_\rho$ that makes the pair (π_ρ, u_τ) a covariant representation of the system (A, G, τ). We fix this covariant representation. Let $A(G)$ denote the dense *-subalgebra of $C(G)$ generated by the matrix entries of irreducible unitary representations of G, and let

$$\mathcal{A} = \{a \in A : \tau(a) \in A \otimes_{alg} A(G)\}.$$

By the results in [18], $\mathcal{A}$ is a dense *-subalgebra of A. Now consider the class $\mathcal{E}$ of spectral triples for $\mathcal{A}$ equivariant with respect to the covariant representation (π_ρ, u_τ). We define the spectral dimension of the system (A, G, τ) to be the quantity

$$\inf\{p > 0 : \exists\, D \text{ such that } (\mathcal{H}_\rho, \pi_\rho, D) \in \mathcal{E} \text{ and } D \text{ is } p\text{-summable}\}.$$

We will denote this number by $\mathcal{S}dim(A, G, \tau)$. Here we have taken infimum because if we have a spectral triple $\mathcal{D} = (\mathcal{H}, \pi, D)$ that is p-summable, then $\mathcal{D}_\alpha = (\mathcal{H}, \pi, D/|D|^\alpha), 0 < \alpha < 1$ is a spectral triple that is $p/(1-\alpha)$-summable. (see page 459, [11]; [8]).

The first instance of computation of this invariant can be traced back to Connes in [5], where he proved that, in the terminology of the present paper, if Γ is a discrete group containing the free group on two generators, then the Pontryagin dual $\hat{\Gamma}$, which is a compact quantum group, has spectral dimension ∞. In the rest of the paper we will focus on computation of this invariant in several other cases, emphasis being on examples where the number is finite.

3 A Commutative Example

In this section we will calculate the spectral dimension of $SU(2)$ under its own natural action.

Recall from [2] that the representation of $C(SU(2))$ on $L_2(SU(2))$ is given by

$$\begin{aligned} \alpha : e^{(n)}_{ij} &\mapsto a_+(n,i,j) e^{(n+\frac{1}{2})}_{i-\frac{1}{2},j-\frac{1}{2}} + a_-(n,i,j) e^{(n-\frac{1}{2})}_{i-\frac{1}{2},j-\frac{1}{2}}, \\ \beta : e^{(n)}_{ij} &\mapsto b_+(n,i,j) e^{(n+\frac{1}{2})}_{i+\frac{1}{2},j-\frac{1}{2}} + b_-(n,i,j) e^{(n-\frac{1}{2})}_{i+\frac{1}{2},j-\frac{1}{2}}, \end{aligned}$$

where $n \in \frac{1}{2}\mathbb{N}$, $i, j \in \{-n, -n+1, \cdots, n\}$ and

$$a_+(n,i,j) = \Big(\frac{(n-j+1)(n-i+1)}{(2n+1)(2n+2)}\Big)^{\frac{1}{2}}, \tag{3.1}$$

$$a_-(n,i,j) = \Big(\frac{(n+j)(n+i)}{2n(2n+1)}\Big)^{\frac{1}{2}}, \tag{3.2}$$

$$b_+(n,i,j) = -\Big(\frac{(n-j+1)(n+i+1)}{(2n+1)(2n+2)}\Big)^{\frac{1}{2}}, \tag{3.3}$$

$$b_-(n,i,j) = \Big(\frac{(n+j)(n-i)}{2n(2n+1)}\Big)^{\frac{1}{2}}, \tag{3.4}$$

Observe that the representation of (the complexification of) $\mathfrak{su}(2)$ on $L_2(SU(2))$ is given by

$$\begin{aligned} he_{ij}^{(n)} &= (n-2j)e_{ij}^{(n)}, \\ ee_{ij}^{(n)} &= j(n-2j+1)e_{i,j-1}^{(n)}, \\ fe_{ij}^{(n)} &= e_{i,j+1}^{(n)}, \end{aligned}$$

where h, e and f obey

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h.$$

Therefore, any equivariant self-adjoint operator with discrete spectrum must be of the form

$$D : e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}. \tag{3.5}$$

Commutators of this operator with α and β are given by

$$\begin{aligned} [D,\alpha]e_{ij}^{(n)} &= a_+(n,i,j)(d(n+\frac{1}{2},i-\frac{1}{2})-d(n,i))e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} \\ &\quad + a_-(n,i,j)(d(n-\frac{1}{2},i-\frac{1}{2})-d(n,i))e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} [D,\beta]e_{ij}^{(n)} &= b_+(n,i,j)(d(n+\frac{1}{2},i+\frac{1}{2})-d(n,i))e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} \\ &\quad + b_-(n,i,j)(d(n-\frac{1}{2},i+\frac{1}{2})-d(n,i))e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}. \end{aligned} \tag{3.7}$$

where $a_\pm$ and $b_\pm$ are now given by equations (3.1)–(3.4).

Lemma 3.1 ([2]) *Suppose D is an operator on $L_2(SU(2))$ given by (3.5) and having bounded commutators with α and β. Then D can not be p-summable for $p \leq 3$.*

Proof: Conditions for boundedness of the commutators give us

$$|d(n+\frac{1}{2},i+\frac{1}{2})-d(n,i)| = O\Big(\Big(\frac{2n+2}{n+i+1}\Big)^{\frac{1}{2}}\Big), \tag{3.8}$$

$$|d(n+\frac{1}{2},i-\frac{1}{2})-d(n,i)| = O\Big(\Big(\frac{2n+2}{n-i+1}\Big)^{\frac{1}{2}}\Big). \tag{3.9}$$

Observe from (3.8) and (3.9) that if we restrict ourselves to the region $i \geq 0$, then

$$|d(n+\frac{1}{2},i+\frac{1}{2})-d(n,i)| = O(1), \tag{3.10}$$

and if we restrict to $i \leq 0$, then

$$|d(n+\frac{1}{2},i-\frac{1}{2})-d(n,i)| = O(1). \tag{3.11}$$

Also, it is not too difficult to see that

$$|d(n+1,0)-d(n,0)| = O(1). \tag{3.12}$$

Suppose $C > 0$ is a constant that works for (3.8)–(3.12). Then

$$|d(n,i)| < 2Cn \tag{3.13}$$

Therefore, for $\alpha \leq 3$

$$\text{Tr}|D|^{-\alpha} > \sum_{n \geq 0} (2n+1)^2 (2Cn)^{-\alpha} = \infty$$

□

Theorem 3.2 *Spectral dimension of $SU(2)$ is 3.*

Proof: The previous lemma shows that $\mathcal{S}dim(C(SU(2))) \geq 3$. Note that D given by $d(n,i) = n$ will give rise to a spectral triple that is p-summable for all $p > 3$. Hence the result. □

4 $\mathbb{T}^n$ action on noncommutative torus

Our next example deals with probably the most well known ergodic C^*-dynamical system, namely that of $\mathbb{T}^n$ acting ergodically on the noncommutative torus. By a result of Milnes and Walters ([14]), this describes all the primitive C^*-algebras with free and ergodic action of $\mathbb{T}^n$. Recall ([19]) that the noncommutative n-torus is the universal C^*-algebra generated by n-unitaries $U_1, \cdots, U_n$ satisfying the commutation relation $U_j U_k = exp(2\pi i \theta_{jk}) U_k U_j$, where $\Theta = (\!(\theta_{jk})\!)$ is a skew symmetric matrix with real entries. This C^*-algebra is referred as the noncommutative torus and denoted by A_Θ. If Θ has sufficient irrationality (i.e. $p^t \Theta q \in \mathbb{Z}$ for all $q \in \mathbb{Z}^n$ implies $p = 0$), then A_Θ is simple. Non-commutative torus admits an action of $\mathbb{T}^n$. To specify the action it is enough to prescribe them on the generators by $\alpha_z(U_j) = z_j U_j, 1 \leq j \leq n, z = (z_1, \cdots, z_n) \in \mathbb{T}^n$. This action is ergodic. Therefore the dynamical system $(A_\Theta, \mathbb{T}^n, \alpha)$ satisfies our hypothesis and we can ask what is the spectral dimension of this system.

Proposition 4.1 *The spectral dimension of $(A_\Theta, \mathbb{T}^n, \alpha)$ is n.*

Proof: The unique invariant state is specified by

$$\tau(a) = \int_{\mathbb{T}^n} \alpha_z(a) dz.$$

The algebra of smooth elements $A_\Theta^\infty = \{a \in A_\Theta : z \mapsto \alpha_z(a) \text{ is smooth}\}$ is dense in A_Θ and can be described as $\{a : a = \sum a_{k_1,k_2,\cdots,k_n} U_1^{k_1} \cdots U_n^{k_n}, a_{\underline{k}} \in S(\mathbb{Z}^n)\}$. On A_Θ^∞ the invariant state is specified by $\tau(\sum a_{k_1,k_2,\cdots,k_n} U_1^{k_1} \cdots U_n^{k_n}) = a_{0,\cdots,0}$. This state is tracial and the associated GNS space can be identified as $l_2(\mathbb{Z}^n)$. If we denote by $\{e_{k_1,\cdots,k_n} : k_1, \cdots, k_n \in \mathbb{Z}\}$ the canonical basis elements then the GNS representation is given by

$$U_j(e_{k_1,\cdots,k_n}) = \exp\left(2\pi i \sum_{r=1}^{j-1} \theta_{jr} k_r\right) e_{\underline{k}+\epsilon_j},$$

where $\epsilon_j = (0, \cdots, 1, \cdots, 0)$, the 1 is at the j-th position. The $e_{\underline{k}}$ are the spectral subspaces and hence an equivariant D must be of the form

$$D : e_{\underline{k}} \mapsto d(\underline{k}) e_{\underline{k}}.$$

The boundedness of the commutator condition is equivalent to

$$|d(\underline{k} + \epsilon_j) - d(\underline{k})| < C, \quad \forall\, \underline{k} \in \mathbb{Z}^n, \quad 1 \leq j \leq n \tag{4.14}$$

for some constant C. Therefore

$$|d(\underline{k})| < C|\underline{k}|,$$

where $|\underline{k}| = k_1 + \cdots + k_n$. Then

$$\begin{aligned}
\mathrm{Tr}|D|^{-\alpha} &= \sum_{k_1, \cdots, k_n \in \mathbb{Z}} C^{-\alpha} |k|^{-\alpha} \\
&> C'^{-\alpha} \sum_{k_1, \cdots, k_n = 2}^{\infty} \left(\sum_{j=1}^{n} k_j^2 \right)^{-\frac{\alpha}{2}} \\
&\geq \int_1^\infty \cdots \int_1^\infty \left(\sum_{j=1}^{n} x_j^2 \right)^{-\frac{\alpha}{2}} dx_1 \cdots dx_n \\
&> \int_{\sqrt{n}}^\infty \int_{S^{n-1}} r^{-\alpha} r^{n-1} dr d\sigma,
\end{aligned}$$

where $d\sigma$ is the normalized surface measure on the sphere. Therefore $\mathrm{Tr}|D|^{-\alpha}$ is finite only if $\int_1^\infty r^{-\alpha+n-1} dr < \infty$, that is $\alpha > n$.

On the other hand by taking $d(\underline{k}) = |\underline{k}|$ we obtain a D such that $\mathrm{Tr}|D|^{-\alpha} < \infty$ for all $\alpha > n$. Hence spectral dimension of A_Θ is n. □

5 $SU_q(\ell + 1)$ action on itself

In this section and the next, we will discuss two C^*-dynamical systems, both involving the quantum group $SU_q(\ell+1)$. In one case, the C^*-algebra under consideration will be the C^*-algebra $C(SU_q(\ell+1))$ of 'continuous functions' associated with the quantum group $SU_q(\ell+1)$ itself, and in the other case, it will be the C^*-algebra $C(S_q^{2\ell+1})$ of 'continuous functions' on the odd dimensional spheres $S_q^{2\ell+1}$. In both sections, q is assumed to lie in the open interval $(0, 1)$.

We will take G to be the quantum group $SU_q(\ell+1)$ and A to be the C^*-algebra $C(G)$. Action of G on A will just be the comultiplication of G. Haar state of G is the invariant state for this action and therefore the relevant covariant representation of this C^*-dynamical system is given by the triple $(L_2(G), \pi, u)$ where $L_2(G)$ is the GNS space of the Haar state on $A = C(G)$, π is the representation of A on $L_2(G)$ by left multiplication, and u is the right regular representation.

The C^*-algebra $C(SU_q(\ell+1))$ is the universal C^*-algebra generated by $\{u_{ij} : i, j = 1, \cdots, \ell+1\}$ obeying the relations (see [25]):

$$\sum_k u_{ki}^* u_{kj} = \delta_{ij} I, \quad \sum_k u_{ik} u_{jk}^* = \delta_{ij} I$$

$$\sum_{k_i\text{'s distinct}} (-q)^{I(k_1,k_2,\cdots,k_{\ell+1})} u_{j_1 k_1} \cdots u_{j_{\ell+1} k_{\ell+1}} = \begin{cases} (-q)^{I(j_1,j_2,\cdots,j_{\ell+1})} & j_i\text{'s distinct} \\ 0 & \text{otherwise} \end{cases}$$

where $I(k_1, k_2, \cdots, k_{\ell+1})$ is the number of inversions in $(k_1, k_2, \cdots, k_{\ell+1})$. The group laws are given by the folowing maps:

$$\begin{aligned} \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \qquad \text{(Comultiplication)} \\ S(u_{ij}) &= u_{ji}^* \qquad \text{(Antipode)} \\ \epsilon(u_{ij}) &= \delta_{ij} \qquad \text{(Counit)} \end{aligned}$$

For preliminaries on irreducible unitary representations of the group $SU_q(\ell+1)$, we refer the reader to section 7.3, [13] and section 4, [4]. For related computations that will be very crucial for us, we refer the reader to [4] and [16]. We will stick to the same notations as in those two papers. For convenience, let us summarize the main points. Let Λ denote the set of all Young tableaux

$$\{\lambda \equiv (\lambda_1, \ldots, \lambda_{\ell+1}\} : \lambda_i \in \mathbb{N}, \lambda_1 \geq \lambda_2 \geq \ldots \lambda_{\ell+1} = 0\}$$

and for $\lambda \in \Lambda$, let Γ_λ denote the set of all Gelfand-Tsetlin tableaux (will be referred to as GT tableaux) with top row equal to λ.

1. Irreducible unitary representations of G are indexed by Λ. We will denote them by u^λ.

2. dimension of the Hilbert space on which u^λ acts is $|\Gamma_\lambda|$,

3. Fixing an orthonormal basis $\{e_{\mathbf{r}} : \mathbf{r} \in \Gamma_\lambda\}$ of this Hilbert space, one gets the matrix entries $u^\lambda_{\mathbf{rs}}$, $\mathbf{r}, \mathbf{s} \in \Gamma_\lambda$.

4. $\{u^\lambda_{\mathbf{rs}} : \lambda \in \Lambda, \mathbf{r}, \mathbf{s} \in \Gamma_\lambda\}$ generate the C^*-algebra $A = C(G)$,

5. denote by $e^\lambda_{\mathbf{rs}}$ the normalized $u^\lambda_{\mathbf{rs}}$'s, i.e. $e^\lambda_{\mathbf{rs}} = \|u^\lambda_{\mathbf{rs}}\|^{-1} u^\lambda_{\mathbf{rs}}$. Then $\{e^\lambda_{\mathbf{rs}} : \lambda \in \Lambda, \mathbf{r}, \mathbf{s} \in \Gamma_\lambda\}$ is a complete orthonormal basis for $L_2(G)$.

6. For the Young tableaux $\mathbb{1} := (1, 0, \ldots, 0)$. We will omit the symbol $\mathbb{1}$ and just write u in order to denote $u^{\mathbb{1}}$. Notice that any GT tableaux $\mathbf{r}$ with first row $\mathbb{1}$ must be, for some $i \in \{1, 2, \ldots, \ell + 1\}$, of the form (r_{ab}), where

$$r_{ab} = \begin{cases} 1 & \text{if } 1 \leq a \leq i \text{ and } b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus such a GT tableaux is uniquely determined by the integer i. We will write just i for this GT tableaux $\mathbf{r}$. Thus a typical matrix entry of $u^{\mathbb{1}}$ will be written simply as u_{ij}.

7. Let $\mathbb{M}_k := \{M = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k : 1 \leq m_j \leq \ell + 2 - j, 1 \leq j \leq k\}$. For $M \in \mathbb{M}_k$ and a GT tableaux $\mathbf{r}$, denote by $M(\mathbf{r})$ the tableaux $\mathbf{s}$ defined by

$$s_{ij} = \begin{cases} r_{ij} + 1 & \text{if } j = m_i, 1 \leq i \leq k, \\ r_{ij} & \text{otherwise,} \end{cases} \tag{5.15}$$

and let $\text{sign}\,(M)$ denote the product $\prod_{a=1}^{k-1} \text{sign}\,(m_{a+1} - m_a)$ (here $\text{sign}\,(p) := 2\chi_{\mathbb{N}}(p) - 1$). Then one has

$$\pi(u_{ij})e_{\mathbf{rs}} = \sum_{\substack{M \in \mathbb{M}_i, N \in \mathbb{M}_j \\ m_1 = n_1}} \text{sign}\,(M)\text{sign}\,(N) q^{C(i,\mathbf{r},M)+C(j,\mathbf{s},N)+A(M)+K(M)+B(N)}(1+o(q))e_{M(\mathbf{r})N(\mathbf{s})}, \tag{5.16}$$

where

$$C(i, \mathbf{r}, M) = \sum_{a=1}^{i-1} \left(\sum_{b=m_a \wedge m_{a+1}}^{m_a \vee m_{a+1} - 1} H_{ab}(\mathbf{r}) + 2 \sum_{m_{a+1} < b < m_a} V_{ab}(\mathbf{r}) \right) + \sum_{m_i \leq b < \ell + 2 - i} H_{ib}(\mathbf{r}), \tag{5.17}$$

$$A(M) = \sum_{j=1}^{i-1} |m_j - m_{j+1}| - \#\{1 \leq j \leq i - 1 : m_j > m_{j+1}\}, \tag{5.18}$$

$$K(M) = \ell + 2 - i - m_i, \tag{5.19}$$

$$B(M) = A(M) + m_1 - m_i. \tag{5.20}$$

Let us next derive the general form of the operator D from the equivariance condition. The Hilbert space $\mathcal{H} = L_2(G)$ decomposes as a direct sum $\oplus_\lambda \mathcal{H}_\lambda$ where $\mathcal{H}_\lambda = \text{span}\{e^\lambda_{\mathbf{r},\mathbf{s}} : \mathbf{r}, \mathbf{s} \in \Gamma_\lambda\}$ and the restriction of u to each $\mathcal{H}_\lambda$ is equivalent to a direct sum of $\dim \lambda$ copies of u^λ and the operator D respects this decomposition. Looking at the restriction of D to each $\mathcal{H}_\lambda$ and using the fact that it commutes with u, it follows that the restriction of D to $\mathcal{H}_\lambda$ is of the form $\oplus_\mu d_{\lambda\mu} P_{\lambda\mu}$ where u commutes with each $P_{\lambda\mu}$ and the restriction of u to each $P_{\lambda\mu}$ is equivalent to u^λ. Write the restriction of D to $\mathcal{H}_\lambda$ in the form $\sum_\mu d_{\lambda\mu} P_{\lambda\mu}$ where the $d_{\lambda\mu}$'s are distinct for distinct μ's. Write $\mathcal{H}_{\lambda\mu} = P_{\lambda\mu}\mathcal{H}_\lambda$.

Proposition 5.1 *Choose and fix a μ. Then*

1. *If $\sum_{\mathbf{s},\mathbf{t}} c(\mathbf{s},\mathbf{t})e^{\lambda}_{\mathbf{s},\mathbf{t}} \in \mathcal{H}_{\lambda\mu}$, then $\sum_{\mathbf{s}} c(\mathbf{s},\mathbf{t})e^{\lambda}_{\mathbf{s},\mathbf{t}} \in \mathcal{H}_{\lambda\mu}$ for all $\mathbf{t} \in \Gamma_\lambda$.*

2. *If $\sum_{\mathbf{s}} c(\mathbf{s})e^{\lambda}_{\mathbf{s},\mathbf{t}_0} \in \mathcal{H}_{\lambda\mu}$ for some $\mathbf{t}_0 \in \Gamma_\lambda$, then $\sum_{\mathbf{s}} c(\mathbf{s})e^{\lambda}_{\mathbf{s},\mathbf{t}} \in \mathcal{H}_{\lambda\mu}$ for all $\mathbf{t} \in \Gamma_\lambda$.*

Proof: 1. Choose and fix an $\mathbf{t}_0 \in \Gamma_\lambda$. Take a linear functional on $\mathcal{H}_\lambda \subseteq C(G)$ that takes the value 1 at $e^{\lambda}_{\mathbf{t}_0\mathbf{t}_0}$ and vanishes at all other $e^{\lambda}_{\mathbf{st}}$'s. Extend it to a bounded linear functional ρ on $C(G)$. Write $u_\rho := (\mathrm{id} \otimes \rho)u$. Then

$$u_\rho\left(\sum_{\mathbf{s},\mathbf{t}} c(\mathbf{s},\mathbf{t})e^{\lambda}_{\mathbf{s},\mathbf{t}}\right) = \sum_{\mathbf{s}} c(\mathbf{s},\mathbf{t}_0)e^{\lambda}_{\mathbf{s},\mathbf{t}_0}.$$

Since D commutes with u_ρ and the $d_{\lambda\mu}$'s are distinct, it follows that $\sum_{\mathbf{s}} c(\mathbf{s},\mathbf{t}_0)e^{\lambda}_{\mathbf{s},\mathbf{t}_0} \in \mathcal{H}_{\lambda\mu}$.

2. Choose and fix $\mathbf{t}_1 \in \Gamma_\lambda$. In this case, take a linear functional ρ on $C(G)$ such that

$$\rho(e^{\lambda}_{\mathbf{t}_1\mathbf{t}_0}) = 1, \quad \rho(e^{\lambda}_{\mathbf{st}}) = 0 \quad \text{for all } (\mathbf{s},\mathbf{t}) \neq (\mathbf{t}_1,\mathbf{t}_0).$$

Then $u_\rho\left(\sum_{\mathbf{s}} c(\mathbf{s})e^{\lambda}_{\mathbf{s},\mathbf{t}_0}\right) = \sum_{\mathbf{s}} c(\mathbf{s})e^{\lambda}_{\mathbf{s},\mathbf{t}_1} \in \mathcal{H}_{\lambda\mu}$. □

Corollary 5.2 *There exists a unitary $((c^{\mathbf{r}}_{\mathbf{s}}))_{\mathbf{r},\mathbf{s}} \in GL(|\Gamma_\lambda|,\mathbb{C})$ and a partition $\{F_\mu : \mu\}$ of Γ_λ such that*

$$\mathcal{H}_{\lambda\mu} = \mathit{span}\left\{\sum_{\mathbf{s}} c^{\mathbf{r}}_{\mathbf{s}} e^{\lambda}_{\mathbf{st}} : \mathbf{r} \in F_\mu, \mathbf{t} \in \Gamma_\lambda\right\}. \tag{5.21}$$

Proof: It follows from the above proposition that there is a subset E_μ of $\mathbb{C}^{|\Gamma_\lambda|}$ such that

$$\mathcal{H}_{\lambda\mu} = \mathrm{span}\left\{\sum_{\mathbf{s}} c(\mathbf{s})e^{\lambda}_{\mathbf{st}} : (c(\mathbf{s}))_{\mathbf{s}} \in E_\mu, \mathbf{t} \in \Gamma_\lambda\right\}.$$

One can now further assume, without loss in generality, that E_μ is an independent set of unit vectors in $\mathbb{C}^{|\Gamma_\lambda|}$. Orthogonalizing the vectors $(c(\mathbf{s}))_{\mathbf{s}}$ next, we obtain a set of orthonormal vectors E_μ in $\mathbb{C}^{|\Gamma_\lambda|}$ such that

$$\mathcal{H}_{\lambda\mu} = \mathrm{span}\left\{\sum_{\mathbf{s}} c(\mathbf{s})e^{\lambda}_{\mathbf{st}} : (c(\mathbf{s}))_{\mathbf{s}} \in E_\mu, \mathbf{t} \in \Gamma_\lambda\right\}.$$

Since the spaces $\mathcal{H}_{\lambda\mu}$ are orthogonal for different μ, $\mathcal{H}_\lambda = \oplus_\mu \mathcal{H}_{\lambda\mu}$ and $\dim \mathcal{H}_{\lambda\mu} = |\Gamma_\lambda| \cdot |E\mu|$, it follows that the collection of vectors $(c(\mathbf{s}))_{\mathbf{s}} \in \cup_\mu E_\mu$ form a complete orthormal set of vectors in $\mathbb{C}^{|\Gamma_\lambda|}$. The matrix formed by taking the elements of $\cup_\mu E_\mu$ as rows now give us the required conclusion, with F_μ being the rows corresponding to $(c(\mathbf{s}))_{\mathbf{s}} \in E_\mu$. □

Proposition 5.3 *If we make a change of basis in the Hilbert space on which the irreducible u^λ acts using the matrix $(\!(c^{\mathbf{r}}_{\mathbf{s}})\!)$, then with respect to the new matrix entries $u^\lambda_{\mathbf{rs}}$ and $e^\lambda_{\mathbf{rs}} = \|u^\lambda_{\mathbf{rs}}\|^{-1}|u^\lambda_{\mathbf{rs}}$, one has*

$$\mathcal{H}_{\lambda\mu} = span\left\{e^\lambda_{\mathbf{rs}} : \mathbf{r} \in F_\mu, \mathbf{s} \in \Gamma_\lambda\right\}.$$

Proof: This follows from the observation that

$$\text{span}\left\{\sum_{\mathbf{s}} c^{\mathbf{r}}_{\mathbf{s}} e^\lambda_{\mathbf{st}} : \mathbf{t} \in \Gamma_\lambda\right\} = \text{span}\left\{\sum_{\mathbf{s},\mathbf{t}} c^{\mathbf{r}}_{\mathbf{s}} e^\lambda_{\mathbf{st}} \overline{c^{\mathbf{z}}_{\mathbf{t}}} : \mathbf{z} \in \Gamma_\lambda\right\},$$

and $\sum_{\mathbf{s},\mathbf{t}} c^{\mathbf{r}}_{\mathbf{s}} e^\lambda_{\mathbf{st}} \overline{c^{\mathbf{z}}_{\mathbf{t}}}$ are the matrix entries with respect to the new basis $f_{\mathbf{r}} := \sum_{\mathbf{s}} c^{\mathbf{r}}_{\mathbf{s}} e_{\mathbf{s}}$. □

Thus we can assume that D must be of the form

$$e^\lambda_{\mathbf{rs}} \mapsto d(\mathbf{r}) e^\lambda_{\mathbf{rs}}, \quad \lambda \in \Lambda, \mathbf{r}, \mathbf{s} \in \Gamma_\lambda. \tag{5.22}$$

From (5.16) we now have

$$\begin{aligned}&[D, \pi(u_{ij})]e^\lambda_{\mathbf{rs}} \\ &= \sum_{\substack{M\in\mathbb{M}_i, N\in\mathbb{M}_j\\ m_1=n_1}} \text{sign}\,(M)\text{sign}\,(N)(d(M(\mathbf{r})) - d(\mathbf{r}))q^{C(i,\mathbf{r},M)+C(j,\mathbf{s},N)+A(M)+K(M)+B(N)} e^\mu_{M(\mathbf{r})N(\mathbf{s})}.\end{aligned} \tag{5.23}$$

Therefore the condition for boundedness of commutators reads as follows:

$$|(d(M(\mathbf{r})) - d(\mathbf{r}))q^{C(i,\mathbf{r},M)+C(j,\mathbf{s},N)}| < c, \tag{5.24}$$

where c is independent of i, j, λ, μ, $\mathbf{r}$, $\mathbf{s}$, M and N. Choosing j, $\mathbf{s}$ and N suitably, one can ensure that (5.24) implies the following:

$$|(d(M(\mathbf{r})) - d(\mathbf{r}))| < cq^{-C(i,\mathbf{r},M)}. \tag{5.25}$$

It follows from (5.23) that this condition is also sufficient for the boundedness of the commutators $[D, u_{ij}]$.

Let us next form a graph $\mathcal{G}_c$ by taking the vertex set to be

$$\Gamma := \text{ the set of all GT tableaux } = \cup_\lambda \Gamma_\lambda,$$

and connecting two elements $\mathbf{r}$ and $\mathbf{r}'$ if $|d(\mathbf{r}) - d(\mathbf{r}')| < c$. We will assume the existence of a partition (Γ^+, Γ^-) that has the following important property: there does not exist infinite number of disjoint paths each going from a point in Γ^+ to a point in Γ^- (Here 'disjoint paths' mean paths for which the set of vertices of one does not intersect the set of vertices of the other). We will describe this by simply saying that the partition (Γ^+, Γ^-) does not admit any

infinite ladder. Existence of such a partition can be seen by looking at the sets $\{\mathbf{r} : d(\mathbf{r}) > 0\}$ and $\{\mathbf{r} : d(\mathbf{r}) < 0\}$ and exploiting the fact that D has compact resolvent. For any subset F of Γ, we will denote by $F^{\pm}$ the sets $F \cap \Gamma^{\pm}$. Our next job is to study this graph in more detail using the boundedness conditions above. Let us start with a few definitions and notations. By an **elementary move**, we will mean a map M from some subset of Γ to Γ such that γ and $M(\gamma)$ are connected by an edge. A **move** will mean a composition of a finite number of elementary moves. If M_1 and M_2 are two moves, M_1M_2 and M_2M_1 will in general be different. For a family of moves $M_1, M_2, \ldots, M_r$, we will denote by $\overrightarrow{\prod}_{j=1}^{r} M_j$ the move $M_1M_2 \ldots M_r$, and by $\overleftarrow{\prod}_{j=1}^{r} M_j$ the move $M_r \ldots M_2M_1$. For a non negative integer n and a move M, we will denote by M^n the move obtained by applying M successively n times. The following families of moves will be particularly useful to us:

$$M_{ik} = (i, i-1, \ldots, i-k+1) \in \mathbb{M}_k, \quad N_{ik} = (\underbrace{i+1, \ldots, i+1}_{k}, i, i, \ldots, i) \in \mathbb{M}_{\ell+2-i}.$$

For describing a path in our graph, we will often use phrases like 'apply the move $\overrightarrow{\prod}_{j=1}^{k} M_j$ to go from $\mathbf{r}$ to $\mathbf{s}$'. This will refer to the path given by

$$\Big(\mathbf{r},\, M_k(\mathbf{r}), M_{k-1}M_k(\mathbf{r}),\, \ldots,\, M_1M_2 \ldots M_k(\mathbf{r}) = \mathbf{s}\Big).$$

The following lemma will be very useful in the next two sections.

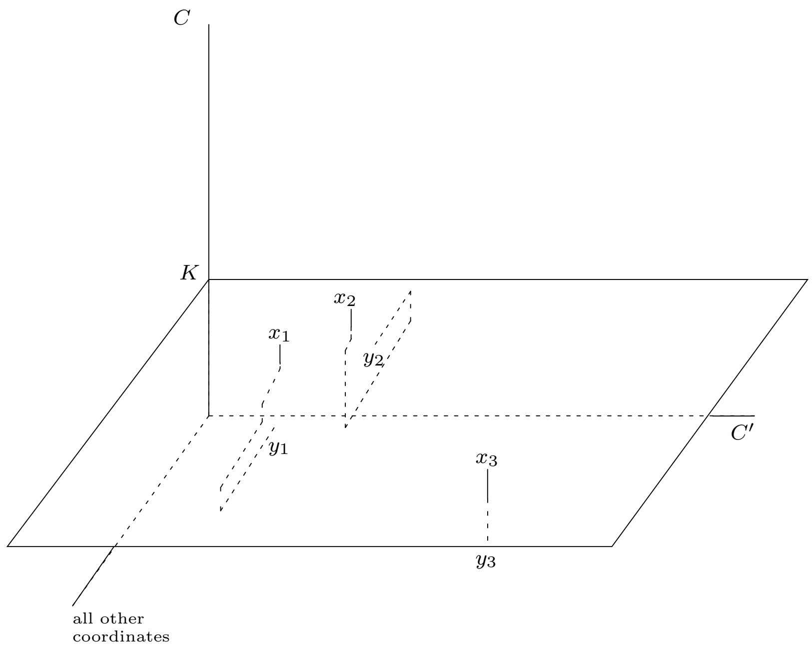
Lemma 5.4 *Let N_{jk} and M_{ik} be the moves defined above. Then*

1. $|d(\mathbf{r}) - d(N_{j0}(\mathbf{r}))| \leq c,$
2. $|d(\mathbf{r}) - d(M_{ik}(\mathbf{r}))| \leq cq^{-\sum_{a=1}^{k-1} H_{a,i+1-a} - \sum_{b=i}^{\ell} H_{k,b+k-1}}$. *In particular, if* $H_{a,i+1-a}(\mathbf{r}) = 0$ *for* $1 \leq a \leq k-1$ *and* $H_{k,b+k-1}(\mathbf{r}) = 0$ *for* $i \leq b \leq \ell$, *then* $|d(\mathbf{r}) - d(M_{ik}(\mathbf{r}))| \leq c$.

Proof: Direct consequence of (5.25). □

Next, we will derive a precise estimate of the singular values of D. The main ingredients in the proof are the finiteness of exactly one of the sets F^+ and F^- for appropriately chosen subsets F of Γ. General form of the argument for proving this will be as follows: for a carefully chosen coordinate C (in the present case, C would be one of the V_{a1}'s or H_{ab}'s), a sweepout argument will show that any γ can be connected by a path, throughout which $C(\cdot)$ remains constant, to another point γ' for which $C(\gamma') = C(\gamma)$ and all other coordinates of γ' are zero. This would help connect any two points γ and δ by a path such that $C(\cdot)$ would lie between $C(\gamma)$ and $C(\delta)$ on the path. This would finally result in the finiteness of at least one (and hence exactly one) of $C(F^+)$ and $C(F^-)$. Next, assuming one of these, say $C(F^-)$ is finite, one shows that for any other coordinate C', $C'(F^-)$ is also finite. This is done as follows. If $C'(F^-)$ is infinite, one chooses elements $y_n \in F^-$ with $C'(y_n) < C'(y_{n+1})$ for all n. Now starting at each

y_n, produce paths keeping the C'-coordinate constant and taking the C-coordinate above the plane $C(\cdot) = K$, where $C(F^-) \subseteq [-K, K]$. This will produce an infinite ladder. The argument is explained in the following diagram.



Our next job is to define an important class of subsets of Γ. Observe that lemma 5.4 tells us that for any $\mathbf{r}$ and any j, the points $\mathbf{r}$ and $N_{j0}(\mathbf{r})$ are connected by an edge, whenever $N_{j0}(\mathbf{r})$ is a GT tableaux. Let $\mathbf{r}$ be an element of Γ. Define the **free plane passing through r** to be the minimal subset of Γ that contains $\mathbf{r}$ and is closed under application of the moves N_{j0}. We will denote this set by $\mathscr{F}_{\mathbf{r}}$. The following is an easy consequence of this definition.

Lemma 5.5 *Let* $\mathbf{r}$ *and* $\mathbf{s}$ *be two GT tableaux. Then* $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$ *if and only if* $V_{a,1}(\mathbf{r}) = V_{a,1}(\mathbf{s})$ *for all* a *and for each* b*, the difference* $H_{a,b}(\mathbf{r}) - H_{a,b}(\mathbf{s})$ *is independent of* a*.*

Corollary 5.6 *Let* $\mathbf{r}, \mathbf{s} \in \Gamma$*. Then either* $\mathscr{F}_{\mathbf{r}} = \mathscr{F}_{\mathbf{s}}$ *or* $\mathscr{F}_{\mathbf{r}} \cap \mathscr{F}_{\mathbf{s}} = \phi$*.*

Let $\mathbf{r} \in \Gamma$. For $1 \leq j \leq \ell + 1$, define a_j to be an integer such that $H_{a_j,j}(\mathbf{r}) = \min_i H_{ij}(\mathbf{r})$. Note three things here:
1. definition of a_j depends on $\mathbf{r}$,
2. for a given j and given $\mathbf{r}$, a_j need not be unique, and
3. if $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$, then for each j, the set of k's for which $H_{kj}(\mathbf{s}) = \min_i H_{ij}(\mathbf{s})$ is same as the set of all k's for which $H_{kj}(\mathbf{r}) = \min_i H_{ij}(\mathbf{r})$. Therefore, the a_j's can be chosen in a manner such that they remain the same for all elements lying on a given free plane.

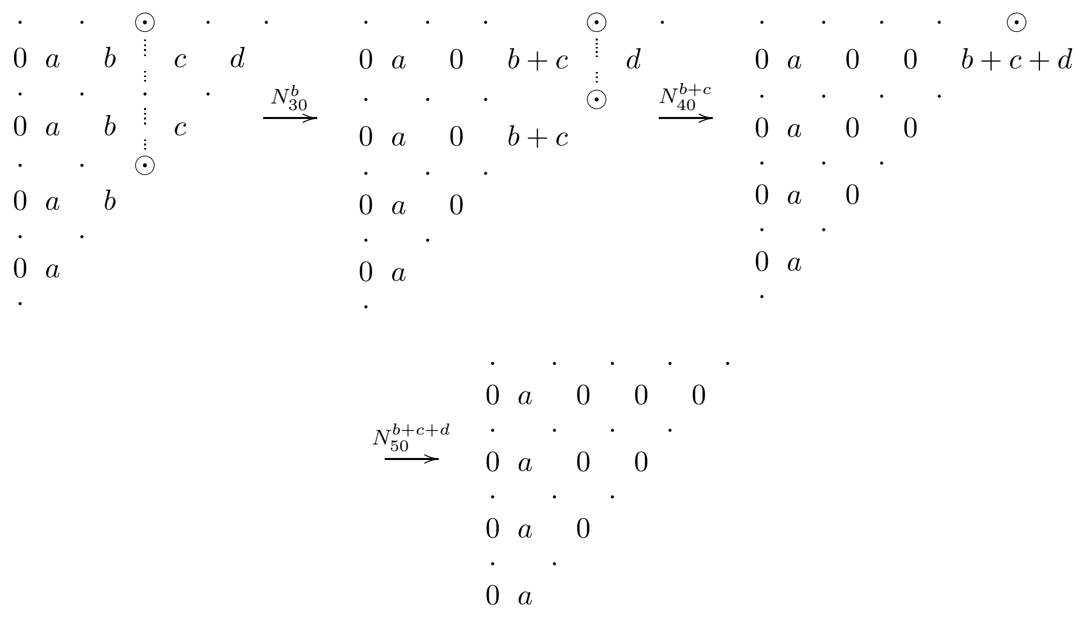
Lemma 5.7 *Let* $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$*. Let* $\mathbf{s}'$ *be another GT tableaux given by*

$$V_{a1}(\mathbf{s}') = V_{a1}(\mathbf{s}) \text{ and } H_{a1}(\mathbf{s}') = H_{a1}(\mathbf{s}) \text{ for all } a, \quad H_{a_b,b}(\mathbf{s}') = 0 \text{ for all } b > 1,$$

where the a_j*'s are as defined above. Then there is a path in* $\mathscr{F}_{\mathbf{r}}$ *from* $\mathbf{s}$ *to* $\mathbf{s}'$ *such that* $H_{11}(\cdot)$ *remains constant throughout this path.*

Proof: Let $c_b := \sum_{j=2}^{\ell+2-b} H_{a_j,j}(\mathbf{s})$. Apply the move $\overrightarrow{\prod}_{b=2}^{\ell} N_{\ell+3-b,0}^{c_b}$. □

The following diagram will help explain the steps involved in the above proof in the case where $\mathbf{r}$ is the constant tableaux.



A dotted line joining two circled dots signifies a move that increases the r_{ij}'s lying on the dotted line by one. Where there is one circled dot and no dotted line, it means one applies the move that raises the r_{ij} corresponding to the circled dot by one.

Proposition 5.8 *Let* $\mathbf{r}$ *be a GT tableaux. Then either* $\mathscr{F}_{\mathbf{r}}^{+}$ *is finite or* $\mathscr{F}_{\mathbf{r}}^{-}$ *is finite.*

Proof: Suppose, if possible, both $H_{11}(\mathscr{F}_{\mathbf{r}}^{+})$ and $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ are infinite. Then there exist two sequences of elements $\mathbf{r}_n$ and $\mathbf{s}_n$ with $\mathbf{r}_n \in \mathscr{F}_{\mathbf{r}}^{+}$ and $\mathbf{s}_n \in \mathscr{F}_{\mathbf{r}}^{-}$, such that

$$H_{11}(\mathbf{r}_1) < H_{11}(\mathbf{s}_1) < H_{11}(\mathbf{r}_2) < H_{11}(\mathbf{s}_2) < \cdots .$$

Now starting from $\mathbf{r}_n$, employ the forgoing lemma to reach a point $\mathbf{r}_n' \in \mathscr{F}_{\mathbf{r}}$ for which

$$V_{a1}(\mathbf{r}_n') = V_{a1}(\mathbf{r}_n) \text{ and } H_{a1}(\mathbf{r}_n') = H_{a1}(\mathbf{r}_n) \text{ for all } a, \quad H_{a_b,b}(\mathbf{r}_n') = 0 \text{ for all } b > 1.$$

Similarly, start at $\mathbf{s}_n$ and go to a point $\mathbf{s}'_n \in \mathscr{F}_\mathbf{r}$ for which

$$V_{a1}(\mathbf{s}'_n) = V_{a1}(\mathbf{s}_n) \text{ and } H_{a1}(\mathbf{s}'_n) = H_{a1}(\mathbf{s}_n) \text{ for all } a, \quad H_{a_b,b}(\mathbf{s}'_n) = 0 \text{ for all } b > 1.$$

Now use the move N_{10} to get to $\mathbf{s}'_n$ from $\mathbf{r}'_n$. The paths thus constructed are all disjoint, because for the path from $\mathbf{r}_n$ to $\mathbf{s}_n$, the H_{11} coordinate lies between $H_{11}(\mathbf{r}_n)$ and $H_{11}(\mathbf{s}_n)$. This means $(\mathscr{F}_\mathbf{r}^+, \mathscr{F}_\mathbf{r}^-)$ admits an infinite ladder, which can not happen. So one of the sets $H_{11}(\mathscr{F}_\mathbf{r}^+)$ and $H_{11}(\mathscr{F}_\mathbf{r}^-)$ must be finite. Let us assume that $H_{11}(\mathscr{F}_\mathbf{r}^-)$ is finite.

Let us next show that for any $b > 1$, $H_{ab}(\mathscr{F}_\mathbf{r}^-)$ is finite. Let K be an integer such that $H_{11}(\mathbf{s}) < K$ for all $\mathbf{s} \in \mathscr{F}_\mathbf{r}^-$. If $H_{ab}(\mathscr{F}_\mathbf{r}^-)$ was infinite, there would exist elements $\mathbf{r}_n \in \mathscr{F}_\mathbf{r}^-$ such that

$$H_{ab}(\mathbf{r}_1) < H_{ab}(\mathbf{r}_2) < \cdots .$$

Now start at $\mathbf{r}_n$ and employ the move N_{10} successively K times to reach a point in $\mathscr{F}_\mathbf{r}^+ = \mathscr{F}_\mathbf{r} \backslash \mathscr{F}_\mathbf{r}^-$. These paths will all be disjoint, as throughout the path, H_{ab} remains fixed.

Since the coordinates $(H_{11}, H_{12}, \ldots, H_{1,\ell})$ completely specify a point in $\mathscr{F}_\mathbf{r}$, it follows that $\mathscr{F}_\mathbf{r}^-$ is finite. □

Next we need a set that can be used for a proper indexing of the free planes. Such a set will be called a complementary axis.

Definition 5.9 A subset $\mathscr{C}$ of Γ is called a **complementary axis** if

1. $\cup_{\mathbf{r} \in \mathscr{C}} \mathscr{F}_\mathbf{r} = \Gamma$,

2. if $\mathbf{r}, \mathbf{s} \in \mathscr{C}$, and $\mathbf{r} \neq \mathbf{s}$, then $\mathscr{F}_\mathbf{r}$ and $\mathscr{F}_\mathbf{s}$ are disjoint.

Let us next give a choice of a complementary axis.

Theorem 5.10 *Define*

$$\mathscr{C} = \{\mathbf{r} \in \Gamma : \Pi_{a=1}^{\ell+1-b} H_{ab}(\mathbf{r}) = 0 \textit{ for } 1 \leq b \leq \ell\}.$$

The set $\mathscr{C}$ defined above is a complementary axis.

Proof: Let $\mathbf{s} \in \Gamma$. A sweepout argument almost identical to that used in lemma 5.7 (application of the move $\overrightarrow{\prod}_{b=1}^{\ell} N_{\ell+2-b,0}^{\sum_{j=1}^{\ell+1-b} H_{a_j,j}(\mathbf{s})}$) will connect $\mathbf{s}$ to another element $\mathbf{s}'$ for which $H_{a_b,b}(\mathbf{s}') = 0$ for $1 \leq b \leq \ell$ by a path that lies entirely on $\mathscr{F}_\mathbf{s}$. Clearly, $\mathbf{s}' \in \mathscr{C}$. Since $\mathbf{s}' \in \mathscr{F}_\mathbf{s}$, by corollary 5.6, $\mathbf{s} \in \mathscr{F}_{\mathbf{s}'}$.

It remains to show that if $\mathbf{r}$ and $\mathbf{s}$ are two distinct elements of $\mathscr{C}$, then $\mathbf{s} \notin \mathscr{F}_\mathbf{r}$. Since $\mathbf{r} \neq \mathbf{s}$, there exist two integers a and b, $1 \leq b \leq \ell$ and $1 \leq a \leq \ell + 2 - b$, such that $H_{ab}(\mathbf{r}) \neq H_{ab}(\mathbf{s})$. Observe that $H_{1\ell}(\cdot)$ must be zero for both, as they are members of $\mathscr{C}$. So b can not be ℓ here. Next we will produce two integers i and j such that the differences $H_{ib}(\mathbf{r}) - H_{ib}(\mathbf{s})$ and

$H_{jb}(\mathbf{r}) - H_{jb}(\mathbf{s})$ are distinct. If there is an integer k for which $H_{kb}(\mathbf{r}) = H_{kb}(\mathbf{s}) = 0$, then take $i = a$, $j = k$. If not, there would exist two integers i and j such that $H_{ib}(\mathbf{r}) = 0$, $H_{ib}(\mathbf{s}) > 0$ and $H_{jb}(\mathbf{r}) > 0$, $H_{jb}(\mathbf{s}) = 0$. Take these i and j. Since $H_{ib}(\mathbf{r}) - H_{ib}(\mathbf{s})$ and $H_{jb}(\mathbf{r}) - H_{jb}(\mathbf{s})$ are distinct, by lemma 5.5, $\mathbf{r}$ and $\mathbf{s}$ can not lie on the same free plane. □

Lemma 5.11 *Let* $\mathbf{r}$ *be a GT tableaux. Let* $\mathbf{s}$ *be the GT tableaux defined by the prescription*

$$V_{a1}(\mathbf{s}) = V_{a1}(\mathbf{r}) \text{ for all } a, \quad H_{ab}(\mathbf{s}) = H_{ab}(\mathbf{r}) \text{ for all } a \geq 2, \text{ for all } b, \quad H_{1,b}(\mathbf{s}) = 0 \text{ for all } b.$$

Then there is a path from $\mathbf{r}$ *to* $\mathbf{s}$ *such that* $V_{a1}(\cdot)$ *remains constant throughout the path.*

Proof: Apply the move $\overrightarrow{\prod}_{b=1}^{\ell} M_{b+1,1}^{H_{1,b}(\mathbf{r})}$. □

The above lemma is actually the first step in the following slightly more general sweepout algorithm.

Lemma 5.12 *Let* $\mathbf{r}$ *be a GT tableaux. Let* $\mathbf{s}$ *be the GT tableaux defined by the prescription*

$$V_{11}(\mathbf{s}) = V_{11}(\mathbf{r}), \quad V_{a1}(\mathbf{s}) = 0 \text{ for all } a > 1, \quad H_{ab}(\mathbf{s}) = 0 \text{ for all } a, b.$$

Then there is a path from $\mathbf{r}$ *to* $\mathbf{s}$ *such that* $V_{11}(\cdot)$ *remains constant throughout the path.*

Proof: Apply successively the moves

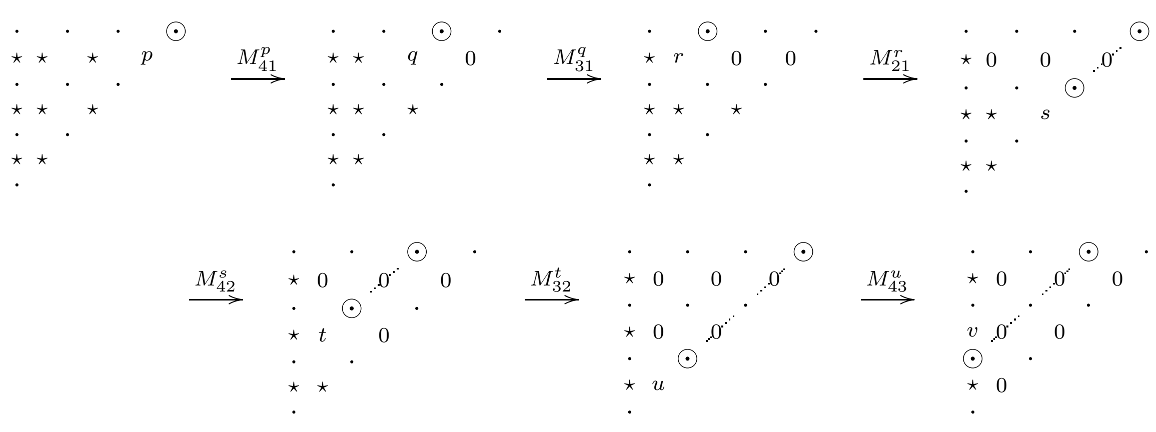
$$\overrightarrow{\prod}_{b=1}^{\ell} M_{b+1,1}^{H_{1,b}(\mathbf{r})}, \quad \overrightarrow{\prod}_{b=1}^{\ell-1} M_{b+2,2}^{H_{2,b}(\mathbf{r})}, \quad \ldots, \quad M_{\ell+1,\ell}^{H_{\ell,1}(\mathbf{r})},$$

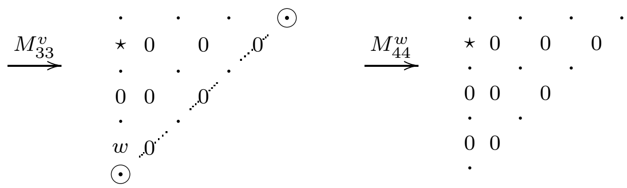
followed by

$$M_{33}^{V_{21}(\mathbf{r})}, \quad M_{44}^{V_{21}(\mathbf{r})+V_{31}(\mathbf{r})}, \quad \ldots, \quad M_{\ell+1,\ell+1}^{\sum_{a=2}^{\ell} V_{a1}(\mathbf{r})}. \tag{5.26}$$

□

The following diagram will help explain the procedure described above in a simple case.





Corollary 5.13 $|d(\mathbf{r})| = O(r_{11})$.

Proof: If one employs the sequence of moves

$$M_{22}^{V_{11}(\mathbf{r})}, \quad M_{33}^{V_{11}(\mathbf{r})+V_{21}(\mathbf{r})}, \quad \ldots, \quad M_{\ell+1,\ell+1}^{\sum_{a=1}^{\ell} V_{a1}(\mathbf{r})}$$

instead of the sequence given in (5.26), one would reach the constant (or zero) tableaux. Total length of this path from $\mathbf{r}$ to the zero tableaux is

$$\sum_{a=1}^{\ell}\sum_{b=1}^{\ell+1-a} H_{ab}(\mathbf{r}) + \sum_{b=1}^{\ell}\sum_{a=1}^{b} V_{a1}(\mathbf{r}),$$

which can easily be shown to be bounded by ℓr_{11}. □

Theorem 5.14 *Let $\widetilde{D}$ be the following operator:*

$$\widetilde{D} : e^{\lambda}_{\mathbf{r},\mathbf{s}} \mapsto r_{11} e^{\lambda}_{\mathbf{r},\mathbf{s}} \tag{5.27}$$

Then $(\mathcal{A}, \mathcal{H}, \widetilde{D})$ is an equivariant spectral triple.

Moreover, $\widetilde{D}$ is not p-summable if $p \le \ell(\ell+2)$, but is p-summable for all $p > \ell(\ell+2)$.

Proof: Boundedness of commutators with algebra elements follow from the observation that $|d(\mathbf{r}) - d(M(\mathbf{r})| \le 1$ and hence equation (5.25) is satisfied.

Let δ_n denote the dimension of the eigenspace of $\widetilde{D}$ corresponding to the eigenvalue n.

Observe that the number of Young tableux $\lambda = (\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1})$ with $n = \lambda_1 \ge \lambda_2 \ge \ldots \lambda_\ell \ge \lambda_{\ell+1} = 0$ is

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{\ell-1}=0}^{i_{\ell-2}} 1 = \text{polynomial in } n \text{ of degree } \ell-1.$$

Thus the number of such Young tableaux is $O(n^{\ell-1})$.

Next, let $\lambda : n = \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_\ell \ge 0$ be an Young tableaux, and let V_λ be the space carrying the irreducible representation parametrized by λ. Then by Weyl dimension formula,

$$\begin{aligned}
\dim V_\lambda &= \prod_{1\le i<j\le \ell+1} \frac{(\lambda_i - \lambda_{i+1}) + \ldots + (\lambda_{j-1} - \lambda_j) + j - i}{j-i} \\
&= \prod_{1\le i<j\le \ell+1} \frac{\lambda_i - \lambda_j + j - i}{j-i} \\
&\le (n+1)^{\frac{\ell(\ell+1)}{2}}.
\end{aligned}$$

Thus the dimension of an irreducible representation corresponding to a Young tableaux

$$n = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq \lambda_{\ell+1} = 0$$

is $O(n^{\frac{1}{2}\ell(\ell+1)})$.

Using the two observations above, it follows that

$$\delta_n \leq C n^{\ell-1} \left(n^{\frac{1}{2}\ell(\ell+1)} \right)^2 = C n^{\ell(\ell+2)-1},$$

where C denotes a generic constant.

This implies that for $p > \ell(\ell+2)$, one has

$$\text{Trace}\, |\widetilde{D}|^{-p} \leq \sum_n n^{-p} n^{\ell(\ell+2)-1} = \sum_n \frac{1}{n^{1+p-\ell(\ell+2)}} < \infty,$$

i.e. $\widetilde{D}$ is p-summable.

Next, let us take an $\epsilon \in (0, \frac{1}{4\ell})$. Then for large enough n, the number of Young tableux $\lambda = (\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1})$ with

$$\lambda_1 = n, \quad \left| \lambda_2 - \left(1 - \frac{1}{\ell}\right) n \right| < \epsilon n, \quad \left| \lambda_3 - \left(1 - \frac{2}{\ell}\right) n \right| < \epsilon n, \quad \ldots \quad \left| \lambda_\ell - \frac{1}{\ell} n \right| < \epsilon n, \quad \lambda_{\ell+1} = 0$$

is of the order $n^{\ell-1}$. For each such λ and for $1 \leq i < j \leq \ell+1$, one has $\lambda_i - \lambda_j > \left(\frac{j-i}{\ell} - 2\epsilon \right) n$, so that

$$\frac{\lambda_i - \lambda_j}{j-i} > \left(\frac{1}{\ell} - \frac{2\epsilon}{j-i} \right) n > \frac{n}{2\ell}.$$

Therefore

$$\dim V_\lambda = \prod_{1 \leq i < j \leq \ell+1} \frac{\lambda_i - \lambda_j + j - i}{j-i} > \left(\frac{n}{2\ell} + 1 \right)^{\frac{1}{2}\ell(\ell+1)} > C n^{\frac{1}{2}\ell(\ell+1)},$$

C being a generic constant. It now follows that

$$\delta_n \geq C n^{\ell-1} \left(n^{\frac{1}{2}\ell(\ell+1)} \right)^2 = C n^{\ell(\ell+2)-1}.$$

Hence for $p \leq \ell(\ell+2)$, we have

$$\text{Trace}\, |\widetilde{D}|^{-p} \geq \sum_n n^{-p} n^{\ell(\ell+2)-1} = \sum_n \frac{1}{n^{1+p-\ell(\ell+2)}} = \infty,$$

i.e. $\widetilde{D}$ can not be p-summable. □

As an important consequence of corollary 5.13 and the above theorem, we now derive the following.

Theorem 5.15 *The spectral dimension of the quantum group $SU_q(\ell+1)$ is $\ell(\ell+2)$.*

Proof: If $(G, L_2(G), D)$ is an equivariant spectral triple, then by corollary 5.13, the singular values of D grow slower than those of (a scalar multiple of) $\widetilde{D}$. Since $\widetilde{D}$ is not p-summable for $p \leq \ell(\ell+2)$, the operator D also can not be p-summable for $p \leq \ell(\ell+2)$. On the other hand, from theorem 5.14 we know that $(G, L_2(G), \widetilde{D})$ is an equivariant spectral triple that is p-summable for all $p > \ell(\ell+2)$. Therefore the result follows. □

6 $SU_q(\ell+1)$ action on $S_q^{2\ell+1}$

The C^*-algebra $A_\ell \equiv C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_{\ell+1}$ satisfying the following relations (see [12]):

$$\begin{aligned} z_i z_j &= q z_j z_i, & 1 \leq j < i \leq \ell+1, \\ z_i^* z_j &= q z_j z_i^*, & 1 \leq i \neq j \leq \ell+1, \\ z_i z_i^* - z_i^* z_i + (1-q^2)\sum_{k>i} z_k z_k^* &= 0, & 1 \leq i \leq \ell+1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

Let u_{ij} denote the generating elements of the C^*-algebra $C(SU_q(\ell+1))$ as in the previous section. The map

$$\tau(z_i) = \sum_k z_k \otimes u_{ki}^*$$

extends to a *-homomorphism τ from A_ℓ into $A_\ell \otimes C(SU_q(\ell+1))$ and obeys $(\mathrm{id}\otimes\Delta)\tau = (\tau\otimes\mathrm{id})\tau$. In other words this gives an action of $SU_q(\ell+1)$ on A_ℓ. Equivariant spectral triples for this dynamical system were studied in [4]. As we shall see shortly, that this dynamical system is ergodic as well as the computation of the spectral dimension of this dynamical system is a by product of the results there.

Let us recall from the description of the L_2 space of the sphere sitting inside $L_2(SU_q(\ell+1))$. Let $u^{1\!\!1}$ denote the fundamental unitary for $SU_q(\ell+1)$, i. e. the irreducible unitary representation corresponding to the Young tableaux $1\!\!1 = (1, 0, \ldots, 0)$. Similarly write $v^{1\!\!1}$ for the fundamental unitary for $SU_q(\ell)$. Fix some bases for the corresponding representation spaces. Then recall ([4]) that $C(SU_q(\ell+1))$ is the C^*-algebra generated by the matrix entries $\{u_{ij}^{1\!\!1}\}$ and $C(SU_q(\ell))$ is the C^*-algebra generated by the matrix entries $\{v_{ij}^{1\!\!1}\}$. Now define ϕ by

$$\phi(u_{ij}^{1\!\!1}) = \begin{cases} I & \text{if } i = j = 1, \\ v_{i-1,j-1}^{1\!\!1} & \text{if } 2 \leq i, j \leq \ell+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.28}$$

Then $C(SU_q(\ell+1)\backslash SU_q(\ell))$ is the C^*-subalgebra of $C(SU_q(\ell+1))$ generated by the entries $u_{1,j}$ for $1 \leq j \leq \ell+1$. Define $\psi : C(S_q^{2\ell+1}) \to C(SU_q(\ell+1)\backslash SU_q(\ell))$ by

$$\psi(z_i) = q^{-i+1} u_{1,i}^*.$$

This gives an isomorphism between $C(SU_q(\ell+1)\backslash SU_q(\ell))$ and $C(S_q^{2\ell+1})$, and the following diagram commutes:

$$\begin{array}{ccc} C(S_q^{2\ell+1}) & \xrightarrow{\quad\tau\quad} & C(S_q^{2\ell+1}) \otimes C(SU_q(\ell+1)) \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi\otimes\mathrm{id}} \\ C(SU_q(\ell+1)\backslash SU_q(\ell)) & \xrightarrow[\Delta]{} & C(SU_q(\ell+1)\backslash SU_q(\ell)) \otimes C(SU_q(\ell+1)) \end{array}$$

In other words, $(C(S_q^{2\ell+1}), SU_q(\ell+1), \tau)$ is the quotient space $SU_q(\ell+1)\backslash SU_q(\ell)$. Thus by proposition 1.9, [18], the action we are considering is ergodic. Also, note that $C(S_q^3) \cong C(SU_q(2))$ and the $SU_q(2)$-action on S_q^3 under this equivalence is same as the $SU_q(2)$-action on itself, which has been covered in the previous section. Therefore we will assume in the rest of this section that $\ell > 1$.

The choice of ψ makes $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ a span of certain rows of the $e_{\mathbf{r},\mathbf{s}}$'s. To be more precise, the right regular representation u of $SU_q(\ell+1)$ keeps the subspace $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ invariant, and the restriction of u to $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ decomposes as a direct sum of exactly one copy of each of the irreducibles given by the young tableaux $\lambda_{n,k} := (n+k, k, k, \ldots, k, 0)$, with $n, k \in \mathbb{N}$. Let Γ_0 be the set of all GT tableaux $\mathbf{r}^{nk}$ given by

$$r_{ij}^{nk} = \begin{cases} n+k & \text{if } i = j = 1, \\ 0 & \text{if } i = 1,\ j = \ell+1, \\ k & \text{otherwise}, \end{cases}$$

for some $n, k \in \mathbb{N}$. Let Γ_0^{nk} be the set of all GT tableaux with top row $\lambda_{n,k}$. Then the family of vectors

$$\{e_{\mathbf{r}^{nk},\mathbf{s}} : n, k \in \mathbb{N},\ \mathbf{s} \in \Gamma_0^{nk}\}$$

form a complete orthonormal basis for $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$. Thus the right regular representation u restricts to the subspace $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ and it also follows from the above discussion and equation (5.16) that the restriction of the left multiplication to $C(SU_q(\ell+1)\backslash SU_q(\ell))$ keeps $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ invariant. Let us denote the restriction of u to $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ by $\hat{u}$ and the restriction of π to $C(SU_q(\ell+1)\backslash SU_q(\ell))$ viewed as a map on $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ by $\hat{\pi}$. It is easy to check that $(\hat{\pi}, \hat{u})$ is a covariant representation for the system $(A_\ell, SU_q(\ell+1), \tau)$.

Theorem 6.1 *Spectral dimension of the odd dimensional quantum sphere* $S_q^{2\ell+1}$ *is* $2\ell+1$.

Proof: Let D_{eq} be the operator on $L_2(S_q^{2\ell+1})$ given by:

$$D_{eq} e_{\mathbf{r}^{nk},\mathbf{s}} = (n+k) e_{\mathbf{r}^{nk},\mathbf{s}}. \tag{6.29}$$

It follows from theorem 6.4, [4], $(\mathcal{A}(S_q^{2\ell+1}), L_2(S_q^{2\ell+1}), D_{eq})$ is an equivariant spectral triple.

Note that the eigenspace corresponding to the eigenvalue n is

$$\text{span}\,\{e_{\mathbf{r}^{n-k,k},\mathbf{s}} : 0 \le k \le n, \mathbf{s} \in \Gamma_0^{n-k,k}\}$$

. Let δ_n denote the dimension of this space.

Observe that for a given n and k, the set $\Gamma_0^{n-k,k}$ consists of all GT tableaux of the form

$$\mathbf{s} = \left(\begin{array}{cccccccc} c_1 = n & k & k & \cdots & k & k & d_1 = 0 \\ c_2 & k & k & \cdots & k & d_2 & \\ \cdots & & \cdots & & & & \\ c_{\ell-1} & k & d_{\ell-1} & & & & \\ c_\ell & d_\ell & & & & & \\ c_{\ell+1} = d_{\ell+1} & & & & & & \end{array}\right)$$

Therefore the number of GT tableaux $\mathbf{s} \in \Gamma_0^{n-k,k}$ is $O(n^{2\ell-1})$. Hence $\delta_n = O(n^{2\ell})$. This implies that for $p > 2\ell+1$, one has

$$\text{Trace}\,|D_{eq}|^{-p} \le \sum_n n^{-p} n^{2\ell} = \sum_n \frac{1}{n^{p-2\ell}} < \infty,$$

i.e. D_{eq} is p-summable.

Next, let us take an $\epsilon \in (0, \frac{1}{4\ell+2})$. Then for large enough n, the number of Young tableux $\lambda = \lambda_{n-k,k} \equiv (n, k, \ldots, k, 0)$ with

$$\left| k - \left(\frac{\ell+1}{2\ell+1}\right) n \right| < \epsilon n$$

is of the order n. For each such $\lambda_{n-k,k}$, the number of $\mathbf{s} \in \Gamma_0^{n-k,k}$ with

$$|d_2 - \frac{1}{2\ell+1} n| < \epsilon n, \quad |d_3 - \frac{2}{2\ell+1} n| < \epsilon n, \ldots, |d_{\ell+1} - \frac{\ell}{2\ell+1} n| < \epsilon n,$$
$$|c_\ell - \frac{\ell+2}{2\ell+1} n| < \epsilon n, \quad |c_{\ell-1} - \frac{\ell+3}{2\ell+1} n| < \epsilon n, \ldots, |c_2 - \frac{2\ell}{2\ell+1} n| < \epsilon n,$$

is $Cn^{2\ell-1}$, which implies that $\delta_n \ge Cn^{2\ell}$ (here C denotes a generic constant, independent of n). Hence for $p \le 2\ell+1$, we have

$$\text{Trace}\,|D_{eq}|^{-p} \ge \sum_n n^{-p} n^{2\ell} = \sum_n \frac{1}{n^{p-2\ell}} = \infty,$$

i.e. D_{eq} can not be p-summable.

Next, let D be a self-adjoint operator with compact resolvent on $L_2(S_q^{2\ell+1})$ that is equivariant with respect to the covariant representation $(\hat{\pi}, \hat{u})$. By theorem 6.4, [4], one then has $|D| \leq a + bD_{eq}$ for some constants a and b. Therefore D can not be p-summable for $p \leq 2\ell + 1$.

Thus the spectral dimension of the C^*-dynamical system under consideration is $2\ell + 1$. □

7 $SU_q(2)$ action on Podleś sphere S_{q0}^2

Quantum sphere was introduced by Podleś in [17]. The C^*-algebra $C(S_{q,0}^2)$ is the universal C^*-algebra generated by two elements ξ and η subject to the following relations:

$$\begin{aligned} \xi^* = \xi, \qquad \eta^*\eta = \xi - \xi^2, \\ \eta\xi = q^2\xi\eta, \qquad \eta\eta^* = q^2\xi - q^4\xi^2. \end{aligned}$$

Here the deformation parameters q satisfies $|q| < 1$. This space was studied in detail in [17]. Let us restate the relevant facts from that paper in our present notation, so as to be able to make use of the computations we have done in earlier sections.

Let u denote the fundamental unitary for $SU_q(2)$, which is the irreducible unitary representation corresponding to the Young tableaux $\mathbb{1} = (1, 0)$. Let $\mathbf{z}$ denote the function $t \mapsto t$ on the torus group $\mathbb{T}$. Then $C(\mathbb{T})$ is generated by the unitary $\mathbf{z}$. Define $\phi : C(SU_q(2)) \to C(\mathbb{T})$ by

$$\phi(u_{ij}) = \begin{cases} \mathbf{z} & \text{if } i = j = 1, \\ \mathbf{z}^* & \text{if } i = j = 2, \\ 0 & \text{if } i \neq j. \end{cases} \tag{7.30}$$

This is a quantum group homomorphism from $C(SU_q(2))$ onto $C(\mathbb{T})$. Then

$$C(SU_q(2)\backslash\mathbb{T}) := \{a \in C(SU_q(2) : (\phi \otimes \text{id})\Delta(a) = I \otimes a\}$$

is the C^*-subalgebra of $C(SU_q(2))$ generated by the elements $u_{11}u_{21}$ and $u_{12}u_{21}$. Define ψ : $C(S_{q0}^2) \to C(SU_q(2)\backslash\mathbb{T})$ by

$$\psi(\xi) = -q^{-1}u_{12}u_{21}, \quad \psi(\eta) = u_{11}u_{21}.$$

This gives an isomorphism between $C(SU_q(2)\backslash\mathbb{T})$ and $C(S_{q0}^2)$, and the action τ of $SU_q(2)$ on S_{q0}^2 is the action induced from the comultiplication map of $SU_q(2)$, i.e. the homomorphism that makes the following diagram commute:

$$\begin{array}{ccc} C(S_{q0}^2) & \xrightarrow{\ \tau\ } & C(S_{q0}^2) \otimes C(SU_q(2)) \\ \Big\downarrow \psi & & \Big\downarrow \psi\otimes\text{id} \\ C(SU_q(2)\backslash\mathbb{T}) & \xrightarrow{\ \Delta\ } & C(SU_q(2)\backslash\mathbb{T}) \otimes C(SU_q(2)) \end{array}$$

and the invariant state for this action is the restriction of the Haar state on $C(SU_q(2))$ to this C^*-subalgebra.

Let $A(S^2_{q0})$ denote the involutive algebra generated by ξ and η. Then one has

$$A(S^2_{q0}) = \text{span}\{e_{\mathbf{r}^{kk}\mathbf{s}} : k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{kk}\} = \text{span}\{e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}} : k, m \in \mathbb{N}, 0 \le m \le 2k\}.$$

Theorem 7.1 *The spectral dimension of S^2_{q0} is 0.*

Proof: It follows from the above discussion that the L_2 space $L_2(S^2_{q0})$ of the sphere is the closed subspace of $L_2(SU_q(2))$ spanned by $\{e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}} : k, m \in \mathbb{N}, 0 \le m \le 2k\}$ and the representation of $C(S^2_{q0})$ on this is the restriction of the left multiplication representation π of $C(SU_q(2))$ to $C(S^2_{q0})$. We will call this restriction $\hat{\pi}$. From equation (5.16), it follows that the action of the elements ξ and η on the basis elements are given by:

$$\begin{aligned}\hat{\pi}(\xi)&e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}}\\ &= -q^{k+m-1}e_{\mathbf{r}^{k-1,k-1}\mathbf{r}^{2k-m-1,m-1}} + (q^{2k}+q^{2m})e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}} - q^{k+m+1}e_{\mathbf{r}^{k+1,k+1}\mathbf{r}^{2k-m+1,m+1}}\end{aligned} \tag{7.31}$$

$$\begin{aligned}\hat{\pi}(\eta)&e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}}\\ &= -q^{k}e_{\mathbf{r}^{k-1,k-1}\mathbf{r}^{2k-m,m-2}} + q^{m}(1-q^{2k})e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m+1,m-1}} - q^{k+2m+2}e_{\mathbf{r}^{k+1,k+1}\mathbf{r}^{2k-m+2,m}}.\end{aligned} \tag{7.32}$$

Clearly the subspace $L_2(S^2_{q0})$ is an invariant subspace for the right regular representation of $SU_q(2)$. Let us call this restriction $\hat{u}$. Then $(\hat{\pi}, \hat{u})$ gives a covariant representation for the system $(S^2_{q0}, SU_q(2), \tau)$ on $L_2(S^2_{q0})$. Restriction of $\hat{u}$ to $\text{span}\{e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}} : 0 \le m \le 2k\}$ is equivalent to the irreducible $u^{(2k,0)}$. Therefore any equivariant Dirac operator D will be of the form

$$e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}} \mapsto d(k)e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}}, \quad k \in \mathbb{N}, 0 \le m \le 2k.$$

A necessary and sufficient condition for the boundedness of the commutators of D with the $\hat{\pi}(\xi)$ and $\hat{\pi}(\eta)$ (and hence with all $\hat{\pi}(a)$'s) is the following:

$$|d(k) - d(k+1)| = O(q^{-k}). \tag{7.33}$$

Thus an operator D given by

$$e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}} \mapsto q^{-k}e_{\mathbf{r}^{kk}\mathbf{r}^{2k-m,m}}, \quad k \in \mathbb{N}, 0 \le m \le 2k$$

makes $(L_2(S^2_{q0}), \hat{\pi}, D)$ an equivariant spectral triple. Dimension of the eigenspace corresponding to the eigenvalue q^{-k} is $2k+1$. Therefore it follows that for any $p > 0$, this spectral triple is p-summable. Thus the spectral dimension of the Podleś sphere S^2_{q0} is 0. □

8 $A_u(Q)$ action on Cuntz algebras

In this section we are going to compute the spectral dimension of the Cuntz algebras. For a brief account on Cuntz algebras, see [9]. Given a nonsingular $n \times n$ matrix $Q \in GL(n, \mathbb{C})$, the universal quantum groups $A_u(Q)$ were introduced by Van Daele and Wang in ([21]) as the universal compact quantum group $(A_u(Q), u)$ generated by $u_{ij}, (1 \leq i, j \leq n)$ with defining relations

$$u^*u = I_n = uu^*, u^tQ\overline{u}Q^{-1} = I_n = Q\overline{u}Q^{-1}u^t,$$

where $u = (u_{ij}), (\overline{u})_{ij} = u^*_{ij}$ and $(u^t)_{ij} = u_{ji}$. Wang showed ([24]) that $A_u(Q)$, with Q positive of trace 1 acts ergodically on the Cuntz algebra $\mathcal{O}_n$. Recall that $\mathcal{O}_n$ is the universal C^*-algebra generated by n-isometries $S_k (k = 1, \cdots, n)$ such that $\sum S_k S_k^* = 1$ and the action is specified by $\tau(S_j) = \sum_{i=1}^n S_i \otimes u_{ij}$. The dense $*$-subalgebra generated by the S_i's gives the span of spectral subspaces. Here is an explicit description of the spectral subspaces following [24]. For a multi-index $\alpha = (i_1, \cdots, i_r), 1 \leq i_1, \cdots, i_r \leq n$ of length $\ell(\alpha) = r$, let $S_\alpha = S_{i_1} \cdots S_{i_r}$. Then the spectral subspaces are given by $\mathcal{H}_{rs} = \text{Span}\{e_{\alpha,\beta} = S_\alpha S_\beta^* : \ell(\alpha) = r, \ell(\beta) = s\}$. An equivariant "Dirac" operator D must be constant on these subspaces, hence they are given by

$$D : e_{\alpha,\beta} \mapsto d(\ell(\alpha), \ell(\beta)) e_{\alpha,\beta},$$

where $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a real valued function. For a multi-index $\alpha = (i_1, \cdots, i_r)$ let $S_i\alpha$ and $S_i^{-1}\alpha$ denote the multi-indices given by $S_i\alpha = (i, i_1, \cdots, i_r), S_i^{-1}\alpha = (i_2, \cdots, i_r)$, if $i_1 = i$ and $e_{S_i^{-1}\alpha,\beta} = 0$, if $i \neq i_1$. In this notation we have $S_i e_{\alpha,\beta} = e_{S_i\alpha,\beta}$ and $S_i^* e_{\alpha,\beta} = e_{S_i^{-1}\alpha\beta}$. Let ρ_Q be the unique invariant state and M be such that $\|[D, S_i]\|, \|[D, S_i^*]\| < M$ for $1 \leq i \leq n$. Then

$$\begin{aligned}
[D, S_i]e_{\alpha,\beta} &= (d(\ell(\alpha) + 1, \ell(\beta)) - d(\ell(\alpha), \ell(\beta)))e_{S_i\alpha,\beta}, \\
\rho_Q(e^*_{\alpha,\beta} e_{\alpha,\beta}) &= \rho_Q(S_\beta S_\alpha^* S_\alpha S_\beta^*) \\
&= \rho_Q(S_\beta S_\beta^*) \\
&= \rho_Q(e_{\beta,\beta}) \\
&= \rho_Q(e^*_{S_i\alpha,\beta} e_{S_i\alpha,\beta)}
\end{aligned}$$

together imply

$$|d(\ell(\alpha) + 1, \ell(\beta)) - d(\ell(\alpha), \ell(\beta))| < M \quad \forall \alpha, \beta. \tag{8.34}$$

Similarly using $S_i^* e_{\phi,\beta} = e_{\phi,\beta'}$, with $\beta' = (j_1, \cdots, j_s, i)$, where $\beta = (j_1, \cdots, j_s)$ and

$$[D, S_i^*]e_{\phi,\beta} = (d(0, \ell(\beta)) - d(0, \ell(\beta)))e_{\phi,\beta}$$

we get

$$|d(0, \ell(\beta)) - d(0, \ell(\beta))| < M. \tag{8.35}$$

Thus combining (8.34) and (8.35) we get $|d(\ell(\alpha), \ell(\beta))| < M(\ell(\alpha) + \ell(\beta))$. Therefore D has eigenvalue k with multiplicity same as the cardinality of $\{e_{\alpha,\beta} : \ell(\alpha) + \ell(\beta) = k\}$, which is n^k. Clearly there is no positive number s such that $Tr|D|^{-s} = \sum_k n^k k^{-s} < \infty$. Hence $\mathcal{S}dim(\mathcal{O}_n, A_u(Q), \tau) = \infty$. Thus we have proved:

Theorem 8.1 *Spectral dimension of Cuntz algebra is infinite.*

9 Concluding remarks

Few remarks are in order.

1. In defining the invariant, we have not demanded nontriviality (of the K-homology class) of the spectral triples that we take into consideration. One reason behind that is the following. For many examples, if we just take the GNS space of the invariant state to be the Hilbert space where the spectral triples live and demand nontriviality, then there may not exist any such spectral triple. This, for example, is the case for the entire family $SU_q(\ell + 1)$, $\ell > 1$ (see [3]). On the other extreme, in the case of the classical $SU(2)$ for instance, as theorem (5.4) of ([2]) shows, the spectral dimension of $SU(2)$ will come out to be 4. On the other hand, if we want to allow Hilbert spaces other than just one copy of the GNS space, that takes us back to one of the problems that was mentioned in the introduction, namely, the choice of a 'natural' Hilbert space. One somewhat natural choice might be to look at the GNS space tensored with $\mathbb{C}^n$ where one allows n to be greater than one. But then another problem that we discussed in section 1 crops up, namely the collection of such spectral triples becomes big and somewhat intractable.

2. The fact that Podleś sphere has spectral dimension zero should not come as a surprise, because as shown in ([15]), on many counts its behaviour differs from the classical case. This particular example also illustrates that the invariant for a homogeneous space of the q-deformation of a classical Lie group is not merely the geometric dimension of its classical counterpart.

3. Noncommutative Geometry offers a new way of looking at classical situations. For example, we have seen in section 3 that for classical $SU(2)$ its spectral dimension is same as its dimension as a Lie group. It would then be tempting to conjecture that

   the spectral dimension of a homogeneous space of a (classical) compact Lie group is same as its dimension as a differentiable manifold.

It must be pointed out however that at the moment, except for the $SU(2)$ example, nothing much is known that one can cite as a strong evidence. The case of quantum $SU(\ell+1)$ and the quantum odd dimensional spheres do point towards the above statement, but one has to keep in mind that the behaviour of these spaces can be quite different when $q = 1$. So this should perhaps be looked upon as more of a pointer to future research. Also, if the above conjecture turns out to be false, then spectral dimension produces a new invariant; the question then is: is it some classically known quantity associated with the space?

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