ON THE COHOMOLOGY OF NORMALIZED BLOW-UPS

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ABSTRACT. These are edited notes from my talk at the Indo-French conference, IMSc, 12-Jan-2016, which is based on the pre-print "On the conjectures of Itoh and of Lipman on the cohomology of normalized blow-ups" jointly with Shreedevi Masuti, arXiv:1507.03343.

Let $(R, \mathfrak{m}, \Bbbk)$ be a three-dimensional noetherian Cohen-Macaulay excellent normal[1] domain and I an $\mathfrak{m}$-primary R-ideal. Write $X = \operatorname{Proj}\left(\oplus_{n \in \mathbb{N}} \overline{I^n} t^n\right)$ where $\overline{(.)}$ denotes integral closure. In other words, X is the normalization of the blow-up of $\operatorname{Spec} R$ along the closed subscheme $\operatorname{Spec} R/I$.

Write $\mathcal{R} = \oplus_{n \in \mathbb{N}} \overline{I^n} t^n$. We are interested in knowing when $\mathcal{R}$ is Cohen-Macaulay.

In this situation, a criterion due to Viêt [Viê93, Theorem 1.1] and to Goto and Nishida [GN94, Theorem (1.1)] can be translated as follows: Write X for $\operatorname{Proj} \mathcal{R}$. Then $\mathcal{R}$ is Cohen-Macaulay if and only if $\mathrm{H}^2(X, \mathscr{O}_X) = 0$ and the extended Rees ring $A = \oplus_{n \in \mathbb{Z}} \overline{I^n} t^n = \oplus_{n \in \mathbb{Z}} \mathrm{H}^0(X, I^n \mathscr{O}_X)$ is Cohen-Macaulay. One can also show that these conditions are equivalent to the following: $\mathrm{H}^2(X, \mathscr{O}_X) = 0$ and $\mathrm{H}^1(X, I^n \mathscr{O}_X) = 0$ for every $n \in \mathbb{Z}$. Write E for the effective Cartier divisor defined by $I\mathscr{O}_X$.

Here are the main results; see [KM15, Section 1] for the complete statements.

Theorem 0.1. *Suppose that* $\mathrm{H}^2(X, \mathscr{O}_X) = 0$. *Then:*

(a) $\mathrm{H}^3_{It+(t^{-1})}(A) = 0$, *and hence X is Cohen-Macaulay.*
(b) *Suppose that I has a reduction generated by three elements x, y, z. A is Cohen-Macaulay if and only if* $\mathrm{H}^1(X, \mathscr{O}_X(1)) = 0$. *If, additionally, R is equicharacteristic or $\overline{I} = \mathfrak{m}$ and A is not Cohen-Macaulay, then* $3 \operatorname{length}_R \mathrm{H}^1(X, I\mathscr{O}_X) - \operatorname{length}_R \mathrm{H}^1(X, I^2\mathscr{O}_X) \geq 3$.
(c) $\mathrm{H}^2_E(X, I^m \mathscr{O}_X) = 0$ *for every $m \in \mathbb{Z}$. In particular, if R is regular and I is such that X is pseudo-rational, $\widetilde{I^n} = I\widetilde{I^{n-1}}$ for every $n \geq \ell(I) = 3$, where $\widetilde{(-)}$ denotes the adjoint ideal in the sense of* [Lip94].

If $\mathrm{H}^2(X, \mathscr{O}_X) = 0$ and A is Cohen-Macaulay, we can conclude a Briançon-Skoda-type result: $\overline{I^n} = I\overline{I^{n-1}}$ for every $n \geq 3$. This, in part, forms the motivation for (a) and (b) of the above theorem. The last statement is a Briançon-Skoda-type theorem for adjoints, conjectured by Lipman in [Lip94], which is known to hold in characteristic zero.

The other motivation for proving the Cohen-Macaulayness of A is the following conjecture of Itoh: If R is Gorenstein and $\mathrm{H}^2(X, \mathscr{O}_X) = 0$ then A is Cohen-Macaulay. Itoh proved the conjecture when $\overline{I} = \mathfrak{m}$ [Ito92, Theorem 3]. Corso, Polini and Rossi [CPR14, Theorem 3.3] extended Itoh's result to more general Cohen-Macaulay rings $(R, \mathfrak{m})$ and ideals I satisfying $\overline{I} = \mathfrak{m}$ and $\operatorname{type}(R) \leq \operatorname{length}_R(\overline{I^2}/\mathfrak{m}I) + 1$. (Here $\operatorname{type}(R) = \dim_\Bbbk \operatorname{Ext}^3_R(\Bbbk, R) = \dim_\Bbbk \operatorname{soc}(\mathrm{H}^3_\mathfrak{m}(R))$.) A corollary of the above theorem generalizes this result of Corso-Polini-Rosso.

A word about the proofs. First we prove the vanishing $\mathrm{H}^3_\mathfrak{n}(A) = 0$. This is the connection between (a) and (b) of the theorem on the one hand, and (c) the other. The proofs involve analysis of various spectral sequences. The lower bound of 3 in (b) is obtained using Boij-Söderberg theory for coherent sheaves on projective spaces [ES10].

REFERENCES

[CPR14] A. Corso, C. Polini, and M. E. Rossi. Bounds on the normal Hilbert coefficients, 2014. arXiv:1410.4233v1 [math.AC]. 1

[ES10] D. Eisenbud and F.-O. Schreyer. Cohomology of coherent sheaves and series of supernatural bundles. *J. Eur. Math. Soc. (JEMS)*, 12(3):703–722, 2010. 2

[GN94] S. Goto and K. Nishida. *The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations*. American Mathematical Society, Providence, RI, 1994. Mem. Amer. Math. Soc. **1**10 (1994), no. 526. 1

[Ito92] S. Itoh. Coefficients of normal Hilbert polynomials. *J. Algebra*, 150(1):101–117, 1992. 1

[KM15] M. Kummini and S. K. Masuti. On conjectures of Itoh and of Lipman on the cohomology of normalized blow-ups, 2015. arXiv:1507.03343 [math.AC]. 1

[Lip94] J. Lipman. Adjoints of ideals in regular local rings. *Math. Res. Lett.*, 1(6):739–755, 1994. With an appendix by Steven Dale Cutkosky. 1

[Viê93] D. Q. Viêt. A note on the Cohen-Macaulayness of Rees algebras of filtrations. *Comm. Algebra*, 21(1):221–229, 1993. 1

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[1]One can formulate the results without assuming that R is normal, or for that matter, even a domain. Also, we use the excellence of R to ensure that the normalization map of finite type schemes over Spec R is finite, so it is not entirely necessary to assume that R is excellent, either.