ON THE COHOMOL OGY OF NORMALIZED BLOW-UPS
MANOJ KUMMINI

ABSTRACT. These are edited notes from my talk at the Indo-French conference, IMSc, 12-Jan-
2016, which is based on the pre-print “On the conjectures of Itoh and of Lipman on the coho-

Let \(( R, m, k)\) be a three-dimensional noetherian Cohen-Macaulay excellent normal domain
and \( I \) an \( m \)-primary \( R \)-ideal. Write \( X = \text{Proj} \left( \bigoplus_{n \in \mathbb{N}} \mathcal{O} \right) \) where \( \mathcal{O} \) denotes integral closure.
In other words, \( X \) is the normalization of the blow-up of \( \text{Spec} R \) along the closed subscheme \( \text{Spec} R/ I \).

Write \( \mathcal{R} = \bigoplus_{n \in \mathbb{N}} \mathcal{O} \). We are interested in knowing when \( \mathcal{R} \) is Cohen-Macaulay.

In this situation, a criterion due to Viêt [Viê93, Theorem 1.1] and to Goto and Nishida [GN94,
Theorem (1.1)] can be translated as follows: Write \( X \) for \( \text{Proj} \mathcal{R} \). Then \( \mathcal{R} \) is Cohen-Macaulay if
and only if \( H^2(X, \mathcal{O}_X) = 0 \) and the extended Rees ring \( A = \bigoplus_{n \in \mathbb{N}} \mathcal{O} \mathcal{H}^0(X, I^n \mathcal{O}_X) \)
is Cohen-Macaulay. One can also show that these conditions are equivalent to the following:
\( H^2(X, \mathcal{O}_X) = 0 \) and \( H^1(X, I^n \mathcal{O}_X) = 0 \) for every \( n \in \mathbb{Z} \). Write \( E \) for the effective Cartier divisor
defined by \( I \mathcal{O}_X \).

Here are the main results; see [KM15, Section 1] for the complete statements.

**Theorem 0.1.** Suppose that \( H^2(X, \mathcal{O}_X) = 0 \). Then:

(a) \( H^n_{H+1}\mathcal{O}_X(\mathcal{A}) = 0 \), and hence \( X \) is Cohen-Macaulay.

(b) Suppose that \( I \) has a reduction generated by three elements \( x, y, z \). A is Cohen-Macaulay if and
only if \( H^1(X, \mathcal{O}_X(1)) = 0 \). If, additionally, \( R \) is equicharacteristic or \( \mathcal{I} = m \) and \( A \) is not
Cohen-Macaulay, then \( 3 \) length \( \mathcal{R} \mathcal{H}_X(\mathcal{I}, \mathcal{O}_\mathcal{X}) \mathcal{H}^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) - \) length \( \mathcal{R} \mathcal{H}_X(\mathcal{I}, \mathcal{O}_\mathcal{X}) \mathcal{H}^2(\mathcal{X}, \mathcal{O}_\mathcal{X}) \geq 3 \).

(c) \( H^2(X, I^m \mathcal{O}_X) = 0 \) for every \( m \in \mathbb{Z} \). In particular, if \( R \) is regular and \( I \) is such that \( X \) is
defined by \( I \mathcal{O}_X \).

If \( H^2(X, \mathcal{O}_X) = 0 \) and \( A \) is Cohen-Macaulay, we can conclude a Briançon-Skoda-type result:
\( \mathcal{T}^n = \mathcal{T}^n_\mathcal{I} \) for every \( n \geq 3 \). This, in part, forms the motivation for (a) and (b) of the above
theorem. The last statement is a Briançon-Skoda-type theorem for adjoints, conjectured by
Lipman in [Lip94], which is known to hold in characteristic zero.

The other motivation for proving the Cohen-Macaulayness of \( A \) is the following conjecture
of Itoh: If \( R \) is Gorenstein and \( H^2(X, \mathcal{O}_X) = 0 \) then \( A \) is Cohen-Macaulay. Itoh proved the
conjecture when \( \mathcal{I} = m \) [Ito92, Theorem 3]. Corso, Polini and Rossi [CPR14, Theorem 3.3]
extended Itoh’s result to more general Cohen-Macaulay rings \( (R, m) \) and ideals \( I \) satisfying \( \mathcal{I} = m \) and
\( \text{type}(R) \leq \text{length}_R(\mathcal{I}/ \mathcal{I}^2) + 1 \). (Here \( \text{type}(R) = \text{dim}_k \text{Ext}_R^1(\mathcal{I}/ \mathcal{I}^2, R) = \text{dim}_k \text{soc}(H^3_{mR}(R)) \).)
A corollary of the above theorem generalizes this result of Corso-Polini-Rosso.

A word about the proofs. First we prove the vanishing \( H^1(A) = 0 \). This is the connection
between (a) and (b) of the theorem on the one hand, and (c) the other. The proofs involve
analysis of various spectral sequences. The lower bound of 3 in (b) is obtained using Boij-
Söderberg theory for coherent sheaves on projective spaces [ES10].

REFERENCES


