Singularities, Algebraic entropy and Integrability of discrete Systems

K.M. Tamizhmani

Pondicherry University, India.

Indo-French Program for Mathematics
The Institute of Mathematical Sciences, Chennai-2016

22.01.2016
Abstract

In this talk, we discuss about two integrability detectors, namely, singularity confinement approach and algebraic entropy method for discrete systems. The power of these methods are demonstrated by deriving discrete Painlevé equations and linearizable systems.
Painlevé developed $\alpha$-method to test the occurrence of branch point solution and to prove single-valuedness. He found the six Painlevé equations whose general solutions require the introduction of new transcendent called Painlevé transcendent.

This classification of second-order equations was completed by Gambier who presented the complete list of fifty equations including the six Painlevé equations that satisfied the requirement of absence of movable critical singularities.

Forty four of them can be solved by known functions. It is clear from the analysis of the above prominent personalities that the integrability of differential equation is related to the existence of the general solution which is single-valued and analytic.
A system of ODE is said to have the Painlevé property if its general solution has no movable critical singular point.

In other words, an ODE is said to possess the Painlevé property if all movable singularities of all solutions are poles.

For example, the solution of

\[ w' + w^2 = 0, \]

is \( w = (z - z_0)^{-1} \). Note that, if the initial condition is \( w(0) = 1 \), then \( z_0 = -1 \). On the other hand, if the initial condition is \( w(0) = 2 \), then \( z_0 \) moves to \( z_0 = -\frac{1}{2} \). That means that the location of the singularity at \( z_0 \) moves with initial condition. Such singularities are called movable singularities. For Painlevé property to hold the only forbidden singularities are the movable critical (multi-valued) singularities.
Consider the ODE
\[ w'' = w' \frac{2w - 1}{w^2 + 1}. \]
The singularities of this equation are \( w = \pm i, w = \infty \) and \( w' = \infty \). Series expansion can be developed for solutions exhibiting each of the above singular behaviors and the equations passes the Painlevé test. This equation, however, has the general solution
\[ w = \tan\{\log[k(z - z_0)]\}, \]
where \( k \) and \( z_0 \) are constants. For \( k \neq 0 \), \( w \) has poles at
\[ z = z_0 + k^{-1} \exp\{-\left(n + \frac{1}{2}\right)\pi\}, \]
for every integer \( n \). These poles accumulate at the movable point \( z_0 \), giving raise to a movable branched nonisolated essential singularity. This example clearly shows that passing the Painlevé test need not guarantee that the equation actually possess the Painlevé property.
Continuous Painlevé equations

\[ P_I \quad w'' = 6w^2 + z \]

\[ P_{\text{II}} \quad w'' = 2w^3 + zw + a \]

\[ P_{\text{III}} \quad w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(aw^2 + b) + cw^3 + \frac{d}{w} \]

\[ P_{\text{IV}} \quad w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - a)w - \frac{b^2}{2w} \]

\[ P_{\text{V}} \quad w'' = w'^2 \left( \frac{1}{2w} + \frac{1}{w-1} \right) - \frac{w'}{z} + \frac{(w-1)^2}{2z^2}(aw + \frac{b}{w}) + \frac{cw}{z} + \frac{dw(w+1)}{(w-1)} \]

\[ P_{\text{VI}} \quad w'' = \frac{w'^2}{2} \left( \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z} \right) - w' \left( \frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z} \right) + \frac{w(w-1)(w-z)}{2z^2(z-1)^2} \left( a - b \frac{z}{w^2} + c \frac{z-1}{(w-1)^2} - (d-1) \frac{z(z-1)}{(w-z)^2} \right) \]

where \( a, b, c, d \) are arbitrary constants.
All the solutions of the Painlevé equations are meromorphic functions and are called Painlevé transcendent. Details see Painlevé equations have many wonderful properties: By coalescence one can obtain all other Painlevé equations from $P_{VI}$ by taking special limits on the parameters; the special function solutions exist for specific values of parameters etc..

Recently, it has been found that Painlevé equations admit Lax pairs, bilinear forms as well. Theory of space of initial conditions was used extensively by to find the Hamiltonian structure of the Painlevé equations by the introduction of tau functions
“To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraical method of description of nature, that is to the elimination of continuous functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At the present time, however, such a program looks like an attempt to breathe in empty space".
Numerics
computer simulation of physical phenomena:
based on discretisation
Assumption: (close) parallel between continuous and discrete system
Valid only for small discretisation step(?)
Integrability of discrete systems
Example
Riccati equation

\[ x' = \alpha x^2 + \beta x + \gamma \]

Discretisation as a two-point mapping:
\( x(t) = x_n \) and \( x(t + \Delta t) = x_{n+1} \)
Discretisation

\[ x' \rightarrow (x_{n+1} - x_n)/\Delta t \]

But \( x^2 \)?

Possibilities \( x^2 \rightarrow x^2_n, x^2_{n+1}, x_nx_{n+1}, \)

\[(x_n + x_{n+1})x_n/2, (x_n + x_{n+1})^2/4, (x^2_n + x^2_{n+1})/2\]

Integrable discretisation \( x^2 \rightarrow x_nx_{n+1} \)

\[ \rightarrow \text{Homographic mapping:} \]

\[ x_{n+1} = \frac{bx_n + c}{1 + ax_n} \]

Linearizable through a Cole-Hopf transformation
Painlevé $P_1$ equation

\[ x'' = 3x^2 + t \]

Discretisation (explicit three-point mapping)

\[ x'' \rightarrow (x_{n+1} - 2x_n + x_{n-1})/(\Delta t)^2 \]

and \( 3x^2 \rightarrow x_n(x_{n+1} + x_n + x_{n+1}) \)

\rightarrow Integrable equation:

\[ x_{n+1} - 2x_n + x_{n+1} = \frac{3x_n^2 + z}{1 - x_n} \]

Integrable integrators
What are the discrete Painlevé equations?

Integrable, nonautonomous, discrete

Continuous limit: Painlevé equations (name of d-$\mathbb{P}$’s: second-order systems) Why are d-$\mathbb{P}$’s (and their study) interesting?

- Discrete systems: most fundamental objects
- Discrete $\rightarrow$ Continuous through simple limit (but continuous better known)
Study of discrete: perfect parallel between continuous and discrete integrable systems
- d-$\mathbb{P}$’s: integrable discretisations of c-$\mathbb{P}$’s
But few numerical studies to date
orthogonal polynomials
integrable discrete nonautonomous systems
Higher-order d-P’s
(no continuous limits computed)
They predate the continuous-P’s

Much (much) later
orthogonal polynomials

\[ x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + 1 \]

with \( z_n = \alpha n + \beta \)

(many years later was recognised as d-P₁)
contiguity relations of c-\(\mathbb{P}\)'s

From \(P_{II}\):

\[
 w'' = 2w^3 + tw + \alpha
\]

contiguity relation:

\[
 \frac{\alpha_n + 1/2}{x_{n+1} + x_n} + \frac{\alpha_n - 1/2}{x_n + x_{n-1}} = -(2x_n^2 + t)
\]

where \(x_n = w(t, \alpha_n)\) and \(\alpha_n = n + \alpha_0\).

No continuous limit was derived!

Same period, integrable difference recursion relations in field theoretical models

No relation to (possibly higher) d-\(\mathbb{P}\)’s established

Field-theoretical model of 2-D gravity

Recursion relation of Shohat

Computed the continuous limit!

Obtained \(w'' = 6w^2 + t\), i.e. Painlevé I

*The discrete \(\mathbb{P}\)'s were born!*
Obtained

\[ x_{n+1} + x_{n-1} = \frac{z_n x_n}{1 - x_n^2} \]

Continuous limit \( w'' = 2w^3 + tw \)

\( P_{II} \) equation for the zero value of parameter

Previously (with Capel) derived d-KdV and d-mKdV

Already (in the 70’s) derived by Hirota

Approach of

Similarity reduction of mKdV \( \rightarrow P_{II} \)

Similarity constraint of d-mKdV (nonlinear)

Similarity reduction \( \rightarrow d-P_{II} \)

same as
Beginning of 90’s
Examples of d-\(\mathbb{P}\)’s
Obtained with 3 of the 4 main methods:
• orthogonal polynomials (spectral methods)
• reductions
• contiguity relations.

*The time was ripe for the 4th method*

**Main idea:**
apply an integrability detector
to some postulated functional form
and select the integrable cases

**Starting point:**
The QRT, 2nd-order, integrable, autonomous mappings
Asymmetric mapping:

\[ x_{n+1} = \frac{f_1(y_n) - x_nf_2(y_n)}{f_2(y_n) - x_nf_3(y_n)} \]

\[ y_{n+1} = \frac{g_1(x_{n+1}) - y_ng_2(x_{n+1})}{g_2(x_{n+1}) - y_ng_3(x_{n+1})} \]

Symmetric case \((g_i = f_i)\):

\[ x_{m+1} = \frac{f_1(x_m) - x_{m-1}f_2(x_m)}{f_2(x_m) - x_{m-1}f_3(x_m)} \]

(identification \(x_n \rightarrow x_{2m}, y_n \rightarrow x_{2m+1}\))

Parameter counting: 8 for the asymmetric mapping and 5 for the symmetric

Conserved quantity \(K\) given by:

\[
(\alpha_0 + K\alpha_1)x_n^2y_n^2 + (\beta_0 + K\beta_1)x_n^2y_n + (\gamma_0 + K\gamma_1)x_n^2 \\
+ (\delta_0 + K\delta_1)x_ny_n^2 + (\epsilon_0 + K\epsilon_1)x_ny_n + (\zeta_0 + K\zeta_1)x_n \\
+ (\kappa_0 + K\kappa_1)y_n^2 + (\lambda_0 + K\lambda_1)y_n + (\mu_0 + K\mu_1) = 0
\]
For the symmetric case:

\[(\alpha_0 + K\alpha_1)x_{n+1}^2 x_n^2 + (\beta_0 + K\beta_1)x_{n+1}x_n(x_{n+1} + x_n)\]
\[+ (\gamma_0 + K\gamma_1)(x_{n+1}^2 + x_n^2) + (\epsilon_0 + K\epsilon_1)x_{n+1}x_n\]
\[+ (\zeta_0 + K\zeta_1)(x_{n+1} + x_n) + (\mu_0 + K\mu_1) = 0\]

Invariant relation between \(x_n\) and \(y_n\):

2-2 correspondence (similarly for \(x_n, x_{n+1}\))

Integration of symmetric correspondence:

elliptic functions (Euler)

Integration of the asymmetric case also (2001)

Solution of QRT mapping: sampling of an elliptic function.
Continuous \( \mathbb{P}'s \): non-autonomous extensions of elliptic functions

(\( \mathbb{P}'s \): same functional forms as the autonomous equations with coeffs depending on the independent variable)

**Strategy**: Start from QRT

allow coeffs to depend on indep. variable

select the integrable cases (through integrability detector)
The methodology is following

- Etienne Bezout (1779) theorem: The number of intersection of two plain curves is equal to the product of their degrees.
- Complexity- V.I. Arnold (1990): The growth of the degree sequence is measure of the complexity of the map.
- Slow growth - A.P. Veselov (1982)
Remark: The map is written in terms of projective coordinates.

By this process the map must be having homogeneous degrees in both the numerator and denominator, then the degree of the map is computed.

If there is no factorization of the polynomial the sequence then $d_n = d^m$ holds. On the other hand, if there is some factorization appear then it is obvious that the degree sequence should satisfies $d_n \leq d^m$. The drop may even be so important that the growth of $d_n$ becomes polynomial and not exponential.
Arnold:
Complexity number of intersection points of fixed curve with the iterates of a second curve

Exponential growth for generic mappings
Polynomial for integrable mappings
“integrability has an essential correlation with the weak growth of certain characteristics”

Viallet, Falqui, Hietarinta:
algebraic entropy
Mapping of degree $d$
→ $n$-th iterate: degree $d^n$, unless there exist simplifications
Integrable mappings: massive simplifications
→ polynomial degree growth
The measure of the complexity of the discrete dynamical systems is an interesting area of research. In recent years, this topic assumes importance in the study of the integrability of the dynamics of rational maps.

It has been observed that the dynamical complexity and degree of the composed map has close link.

One can define easily the algebraic entropy of a map from the growth of the degrees of its iterates.

Knowing the degree of the first few iterates, generally, it is possible to find a finite recurrence relation between the degrees which could be solved exactly and obtain a closed form expression.

In addition, it is also possible to find the generating function for the degree sequence. Moreover, from the generating function, one can easily find the exact value of the algebraic entropy.
First point to remember is that, we should consider the bi-rational map in the projective space.

This means that the map is written in terms of projective coordinates.

By this process the map must be having homogeneous degree in both the numerator and denominator.

After canceling the common factors between numerator and denominator, the degree of the map is computed.

If bi-rational map is denoted by $\phi$, then we can define the sequence $d_n$ of the degrees of the successive iterates $\phi^n$ of $\phi$. 
Here, $d_n$ denotes the number of intersection of the $n$’th image of $\phi$ generic line with a fixed hyper plane.

Then growth of the degree sequence $d_n$ is a measure of the complexity of $\phi$.

If there is no factorization of the polynomial the degree sequence $d_n = d^n$ holds. On the other hand, if there is some factorization appear then it induces a drop of the degree and thus the degree sequence should satisfy $d_n \leq d^n$.

The drop may even be so important that the growth of $d_n$ becomes polynomial and not exponential.
As an application of the method of algebraic entropy, we consider the map

\[ x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}. \]

Let us introduce the homogeneous coordinates \( x_0 = p, \ x_1 = q/r \), where we assume that \( \text{deg} \ p = 0, \ \text{deg} \ q = 1 \) and \( \text{deg} \ r = 1 \). Now, in term of homogeneous coordinates the first few iterates of the mapping can be written as

\[
\begin{align*}
  x_2 &= \frac{r^2 + aqr - pq^2}{q^2}, \\
  x_3 &= \frac{qP_4}{r(r^2 + aqr - pq^2)^2}, \\
  x_4 &= \frac{(r^2 + aqr - pq^2)P_6}{P_4^2}, \\
  x_5 &= \frac{P_4P_9}{rP_6^2}, \end{align*}
\]
where $p_k's$ are polynomials in $p, q$ of degree $k$. It is clear that the degree of each iterates can be computed as follows: 0, 1, 2, 5, 8, 13, 18, 25, 32, 41, . . . .

From the above sequence of the degree, we can arrive at the general formulae: $d_{2n} = 2n^2$ and $d_{2n+1} = 2n^2 + 2n + 1$. Since the degree growth obeys polynomial expression, by using

$$
\epsilon(\phi) = \lim_{n \to \infty} \frac{1}{n} \log(d_n). \quad (1)
$$

one can easily show that the algebraic entropy $\epsilon(\phi)$ is vanishing. This clearly indicates the integrability of the mapping.

Most important application of the algebraic entropy is that this method can be used in combination with singularity confinement to find the non-autonomous forms of the integrable mapping.

For this purpose, one of the basic assumption one can make is that the degree growth of the autonomous and non-autonomous mapping should be the same.
Since the generic mapping is of exponential growth, at some stage of the iteration the degree might be different from that of the autonomous case. At this stage, we need to impose certain restrictions on the parameters to bring down the degree. Solving the constraints of the unknown parameters, we get the non-autonomous integrable mapping.

- As an application, again we consider the same mapping mentioned earlier but now with \( a = a(n) = a_n \)

\[
x_{n+1} + x_{n-1} = \frac{a_n}{x_n} + \frac{1}{x_n^2}.
\]

Again by using the projective coordinates, we arrive at each iteration:

\[
x_2 = \frac{r^2 + a_1 qr - pq^2}{q^2},
\]

\[
x_3 = \frac{qQ_4}{r(r^2 + a_1 qr - pq^2)^2},
\]
\[
x_4 = \frac{(r^2 + a_1 qr - pq^2)Q_6}{Q_4^2},
\]
\[
x_5 = \frac{qQ_4Q_{12}}{r(r^2 + a_1 qr - pq^2)Q_7^2},
\]

where \(Q_k\)'s are polynomials in \(p, q\) of degree \(k\).

Therefore, we obtain the degrees as follows: 0, 1, 2, 5, 10, 21, 42, 85, .... Note that the degree growth is exponential

\[
d_{2m-1} = \frac{2^{2m} - 1}{3} \quad \text{and} \quad d_{2m} = 2d_{2m-1}.
\]

Observe that the degree \(d_5\) in the autonomous case is 8 and whereas 10 in the non-autonomous case. As per our assumption, the degree growth should be same for both autonomous and non-autonomous. So, at the iteration \(x_5\) we should impose restrictions on the parameter \(a_n\) to bring down the degree from 10 to 8.
The constraints we obtain

$$a_{n+1} - 2a_n + a_{n-1} = 0. \quad (2)$$

Solving (1), we get $a_n = \alpha_n + \beta$. Hence, we obtain the non-autonomous mapping

$$x_{n+1} + x_{n-1} = \frac{\alpha_n + \beta}{x_n} + \frac{1}{x_n^2}.$$  

The degree sequence is exactly same as the autonomous mapping and hence we have vanishing of the algebraic entropy. Thus, we conclude that the non-autonomous mapping as well integrable.
Consider the discrete Riccati

\[ x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta}. \]

By applying the algebraic entropy analysis one can find

\[ \{d_n\} = \{1, 1, \ldots\}. \]

Obviously, the degree growth is linear and hence the given map is linearizable.
Our second example is

\[ x_{n+1} = ax_{n-1} \left( \frac{x_n - a}{x_n - 1} \right), \]

The degrees of this map is 0, 1, 2, 3, 4, \ldots. Again the degree growth is linear and hence the mapping is linearizable.

Our final example is a non-integrable map

\[ x_{n+1} = ax_{n-1} \left( x_n + \frac{1}{x_n} \right). \]

Here, we can find that the degree sequence is 0, 1, 2, 4, 8, 14, 24, 40, 66, 108, \ldots. Notice that the degree growth is exponential. Therefore, the above map is non-integrable.
In this section, we extend the algebraic entropy method to differential-difference equations. We consider semi-discrete KdV equation:

\[ u_{n+1} = u_{n-1} + \frac{u'_n}{u_n}, \]  

As before, we start with \( u_0 = p, \ u_1 = \frac{q}{r} \), where \( \deg p = 0, \deg q = 1, \deg r = 1 \) and \( \deg t = 1 \) and compute the first few iterates of (2). We thus obtain

\[ u_2 = \frac{pqr - q'r - qr'}{qr}, \] 

\[ u_3 = \frac{q^2(pqr - q'r - qr' + r'^2 - rr'') + r^2(p'q^2 + qq'' - q^2)}{(pqr - q'r - qr')qr}, \] 

and so on.
Computing the degree of the successive iterates, we find
\[ d_n = 0, 1, 2, 4, 7, 11, 16, 22, \ldots \] i.e. given by
\[ d_n = \frac{n^2 - n + 2}{2} \] for \( n > 0 \). The fact that the degree growth is polynomial and hence the system (2) is integrable.
Singularity confinement

Painlevé test: great heuristic value

How to transpose singularity analysis to the study of discrete systems?

Rational discrete systems have singularities

*Do singularities play a role in integrability of discrete systems?*
Singularity confinement

*characteristic of systems integrable through spectral methods*

SC: integrability criterion? necessary? sufficient?

Difficulties

- Mappings not uniquely defined in both directions

Criterion of pre-image non-proliferation Grammaticos, Ramani and Tamizhmani (1994)

(Eliminates all polynomial nonlinear mappings)

- What do we mean by ‘singularity’?
  → “loss of a degree of freedom”

For a mapping $x_{n+1} = f(x_n, x_{n-1})$ this means

$$\frac{\partial x_{n+1}}{\partial x_{n-1}} = 0$$

“Confinement”

mapping recovers the lost degree of freedom

The only way to do this is by the appearance of an indeterminate form $0/0$, $\infty - \infty$, etc., in the subsequent iterations.
Kruskal:
The real problem is the indeterminate form not the simple infinity
Solution
Use continuity with respect to the initial conditions
Introduce a small parameter $\epsilon$
Consider the McMillan mapping:

\[ x_{n+1} + x_{n-1} = \frac{2\mu x_n}{1 - x_n^2} \]

Singularity: whenever \( x \) passes through \( \pm 1 \)

Assume, \( x_0 \) is finite and \( x_1 = 1 + \epsilon \)

We find:

\[ x_2 = -\mu/\epsilon - (x_0 + \mu/2) + O(\epsilon), \]
\[ x_3 = -1 + \epsilon + O(\epsilon^2) \]
\[ x_4 = x_0 + O(\epsilon) \]

Singularity confined and mapping recovered memory of the initial conditions through \( x_0 \).
Deautonomise the McMillan mapping

\[ x_{n+1} + x_{n-1} = \frac{a(n) + b(n)x_n}{1 - x_n^2} \]

Assume: regular \( x_n \) and \( x_{n+1} = \sigma + \epsilon \) where \( \sigma = \pm 1 \) We find:

\[ x_{n+2} = -\frac{b_{n+1} + \sigma a_{n+1}}{2\epsilon} + \frac{a_{n+1} - \sigma b_{n+1}}{4} - x_n \]

\[ x_{n+3} = -\sigma + \frac{2b_{n+2} - b_{n+1} - \sigma a_{n+1}}{b_{n+1} + \sigma a_{n+1}} \epsilon + \mathcal{O}(\epsilon^2) \] \hspace{1cm} (1.12)

Condition for \( x_{n+4} \) to be finite:

\[ b_{n+1} - 2b_{n+2} + b_{n+3} + \sigma(a_{n+1} - a_{n+3}) = 0 \]

Solution:

\[ b_n (\equiv z_n) = \alpha n + \beta \text{ and } a_n = \delta + \gamma(-1)^n \]

Ignore even-odd dependence (\( a = \text{constant} \))

\[ x_{n+1} + x_{n-1} = \frac{a + z_n x_n}{1 - x_n^2} \]

Discrete form of \( P_{\Pi} \)!
Similarly:

\[ x_{n+1} + x_n + x_{n-1} = a(n) + \frac{b(n)}{x_n} \]

Assume: \( x_n \) is regular while \( x_{n+1} \) vanishes. We set \( x_{n+1} = \epsilon \) and obtain:

\[ x_{n+2} = \frac{b_{n+1}}{\epsilon} + a_{n+1} - x_n + O(\epsilon) \]

\[ x_{n+3} = -\frac{b_{n+1}}{\epsilon} + a_{n+2} - a_{n+1} + x_n + O(\epsilon) \]

\( x_{n+4} \) diverges unless \( a_{n+3} - a_{n+2} = 0 \)
For confinement $a=\text{constant}$

$$x_{n+4} = \frac{b_{n+1} - b_{n+2} - b_{n+3}}{b_{n+1}} \epsilon + \mathcal{O}(\epsilon^2)$$

For $x_{n+5}$ finite, second condition:

$$b_{n+1} - b_{n+2} - b_{n+3} + b_{n+4} = 0$$

Solution $b_n = \alpha n + \beta + \gamma (-1)^n$

Ignore even-odd dependence, put $b_n \equiv z_n = \alpha n + \beta$:

$$x_{n+1} + x_n + x_{n-1} = a + \frac{z_n}{x_n}$$

Discrete form of $P_1$. 
Even more interesting result
Deautonomisation of the $f_2 = 0$ QRT
Integrable mapping:

$$x_{n+1}x_{n-1} = \frac{ab(x_n - cq_n)(x_n - dq_n)}{(x_n - a)(x_n - b)}$$

where $a, b, c, \text{ and } d$ are constants and

$$q_n = q_0\lambda^n$$

The continuous limit is $P_{III}$
Not a difference equation,
but a $q$- (multiplicative) mapping
Today $q$-discrete forms exist for all $d$-$P$’s From geometrical description gave a unified approach to derive discrete Painlevé equations derived certain asymmetric discrete Painlevé equations
In this short review, we briefly pointed out the historical background (certainly not a complete one) of considering the differential equations over complex domain. The main theme is to the search for differential equations whose general solution is single-valued and analytic. This lead to the discovery of six Painlevé transcendents.

The discovery of singularity confinement and introduction of algebraic entropy have been discussed and demonstrated their effectiveness with examples from maps and differential-difference equations.