UNIVERSITÉ Grenoble Alpes

INSTITUT DE FRANCE
Académie des sciences

Recent progress towards the Green-Griffiths-Lang and Kobayashi conjectures

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble Alpes & Académie des Sciences de Paris

in the memory of Professor Raghavan Narasimhan
January 11, 2016, Indo-French Conference
Institute of Mathematical Sciences, Chennai

Let X be a complex space. Given two points $p, q \in X$, consider a *chain of analytic disks* from p to q, i.e. holomorphic maps

$$f_j : \Delta := D(0,1) \to X \ \text{ and points } \ a_j, b_j \in \Delta, \quad 0 \leq j \leq k \ \text{ with}$$

$$p = f_0(a_0),\ q = f_k(b_k), \quad f_j(b_j) = f_{j+1}(a_{j+1}), \quad 0 \leq j \leq k-1.$$

Kobayashi pseudodistance and infinitesimal metric

Let X be a complex space. Given two points $p, q \in X$, consider a *chain of analytic disks* from p to q, i.e. holomorphic maps

$$f_j : \Delta := D(0,1) \to X \quad \text{and points} \quad a_j, b_j \in \Delta, \quad 0 \le j \le k \quad \text{with}$$

$$p = f_0(a_0),\ q = f_k(b_k), \quad f_j(b_j) = f_{j+1}(a_{j+1}), \quad 0 \le j \le k-1.$$

One defines the *Kobayashi pseudodistance* d_{Kob} on X to be

$$d_{\mathrm{Kob}}(p,q) = \inf_{\{f_j, a_j, b_j\}} d_{\mathrm{Poincaré}}(a_1, b_1) + \cdots + d_{\mathrm{Poincaré}}(a_k, b_k).$$

Kobayashi pseudodistance and infinitesimal metric

Let X be a complex space. Given two points $p, q \in X$, consider a *chain of analytic disks* from p to q, i.e. holomorphic maps

$$f_j : \Delta := D(0,1) \to X \quad \text{and points} \quad a_j, b_j \in \Delta, \quad 0 \le j \le k \quad \text{with}$$

$$p = f_0(a_0),\; q = f_k(b_k), \quad f_j(b_j) = f_{j+1}(a_{j+1}), \quad 0 \le j \le k-1.$$

One defines the *Kobayashi pseudodistance* d_{Kob} on X to be

$$d_{\mathrm{Kob}}(p,q) = \inf_{\{f_j, a_j, b_j\}} d_{\text{Poincaré}}(a_1, b_1) + \cdots + d_{\text{Poincaré}}(a_k, b_k).$$

The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric

$$\mathbf{k}_x(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\, f(0) = x,\, \lambda f'(0) = \xi\big\},\ \xi \in T_{X,x}.$$

The integrated pseudometric is precisely d_{Kob}.

Kobayashi hyperbolicity and entire curves

Definition

A complex space X is said to be Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{\mathrm{Kob}} : X \times X \to \mathbb{R}_+$ is a distance (i.e. everywhere non degenerate).

Kobayashi hyperbolicity and entire curves

Definition

A complex space X is said to be Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{\mathrm{Kob}} : X \times X \to \mathbb{R}_+$ is a distance (i.e. everywhere non degenerate).

By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.

Kobayashi hyperbolicity and entire curves

Definition

A complex space X is said to be Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{\mathrm{Kob}} : X \times X \to \mathbb{R}_+$ is a distance (i.e. everywhere non degenerate).

By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.

Theorem (Brody, 1978)

For a compact complex manifold X, $dim_{\mathbb{C}} X = n$, TFAE:

(i) X is Kobayashi hyperbolic

(ii) X is Brody hyperbolic, i.e. $\nexists$ entire curves $f : \mathbb{C} \to X$

(iii) The Kobayashi infinitesimal pseudometric $\mathbf{k}_x$ is everywhere non degenerate

Kobayashi hyperbolicity and entire curves

Definition

A complex space X is said to be Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{\mathrm{Kob}} : X \times X \to \mathbb{R}_+$ is a distance (i.e. everywhere non degenerate).

By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.

Theorem (Brody, 1978)

For a compact complex manifold X, $dim_{\mathbb{C}} X = n$, TFAE:

(i) X is Kobayashi hyperbolic

(ii) X is Brody hyperbolic, i.e. $\nexists$ entire curves $f : \mathbb{C} \to X$

(iii) The Kobayashi infinitesimal pseudometric $\mathbf{k}_x$ is everywhere non degenerate

Our interest is the study of hyperbolicity for projective varieties.
In dim 1, X is hyperbolic iff genus $g \geq 2$.

Kobayashi-Eisenman measures

In a similar way, one can introduce the *p-dimensional Kobayashi-Eisenman* infinitesimal metric on decomposable tensors $\xi = \xi_1 \wedge \ldots \wedge \xi_p$ of $\Lambda^p T_{X,x}$ (i.e. on the tautological line bundle over the Grassmann bundle $\mathrm{Gr}(T_X, p)$) by

$$\mathbf{e}_x^p(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \mathbb{B}^p \to X,\ f(0) = x,\ \lambda f'(0)\cdot\tau = \xi\big\},$$

where $\mathbb{B}^p \subset \mathbb{C}^p$ is the unit ball and $\tau = \frac{\partial}{\partial t_1} \wedge \ldots \wedge \frac{\partial}{\partial t_p}$.

Kobayashi-Eisenman measures

In a similar way, one can introduce the *p-dimensional Kobayashi-Eisenman* infinitesimal metric on decomposable tensors $\xi = \xi_1 \wedge \ldots \wedge \xi_p$ of $\Lambda^p T_{X,x}$ (i.e. on the tautological line bundle over the Grassmann bundle $\mathrm{Gr}(T_X, p)$) by

$$\mathbf{e}^p_x(\xi) = \inf\left\{\lambda > 0\,;\ \exists f : \mathbb{B}^p \to X,\ f(0) = x,\ \lambda f'(0)\cdot\tau = \xi\right\},$$

where $\mathbb{B}^p \subset \mathbb{C}^p$ is the unit ball and $\tau = \frac{\partial}{\partial t_1} \wedge \ldots \wedge \frac{\partial}{\partial t_p}$.

Definition

A complex space X is said to be *p-measure hyperbolic* in the sense of Kobayashi-Eisenman if $\mathbf{e}^p$ is non degenerate on a dense Zariski open set.

Volume hyperbolicity refers to the case $p = n = \dim X$.

Main conjectures

Conjecture of General Type (CGT)

- A compact complex variety X is volume hyperbolic $\Longleftrightarrow$ X is of general type, i.e. K_X is big [implication $\Longleftarrow$ is well known].

Main conjectures

Conjecture of General Type (CGT)

- A compact complex variety X is volume hyperbolic $\Longleftrightarrow$ X is of general type, i.e. K_X is big [implication $\Longleftarrow$ is well known].
- X Kobayashi (or Brody) hyperbolic should imply K_X ample.

Main conjectures

Conjecture of General Type (CGT)

- A compact complex variety X is volume hyperbolic $\iff$ X is of general type, i.e. K_X is big [implication $\Longleftarrow$ is well known].
- X Kobayashi (or Brody) hyperbolic should imply K_X ample.

Green-Griffiths-Lang Conjecture (GGL)

Let X be a projective variety$/\mathbb{C}$ of general type. Then $\exists Y \subsetneq X$ algebraic such that all entire curves $f : \mathbb{C} \to X$ satisfy $f(\mathbb{C}) \subset Y$.

Main conjectures

Conjecture of General Type (CGT)

- A compact complex variety X is volume hyperbolic $\iff X$ is of general type, i.e. K_X is big [implication $\Longleftarrow$ is well known].
- X Kobayashi (or Brody) hyperbolic should imply K_X ample.

Green-Griffiths-Lang Conjecture (GGL)

Let X be a projective variety/$\mathbb{C}$ of general type. Then $\exists Y \subsetneq X$ algebraic such that all entire curves $f : \mathbb{C} \to X$ satisfy $f(\mathbb{C}) \subset Y$.

Arithmetic counterpart (Lang 1987) – very optimistic !

If X is projective and defined over a number field $\mathbb{K}_0$, the smallest locus $Y = \mathrm{GGL}(X)$ in GGL's conjecture is also the smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

Main conjectures

Conjecture of General Type (CGT)

- A compact complex variety X is volume hyperbolic $\Longleftrightarrow$ X is of general type, i.e. K_X is big [implication $\Longleftarrow$ is well known].
- X Kobayashi (or Brody) hyperbolic should imply K_X ample.

Green-Griffiths-Lang Conjecture (GGL)

Let X be a projective variety$/\mathbb{C}$ of general type. Then $\exists Y \subsetneq X$ algebraic such that all entire curves $f : \mathbb{C} \to X$ satisfy $f(\mathbb{C}) \subset Y$.

Arithmetic counterpart (Lang 1987) – very optimistic !

If X is projective and defined over a number field $\mathbb{K}_0$, the smallest locus $Y = \mathrm{GGL}(X)$ in GGL's conjecture is also the smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

Consequence of CGT + GGL

A compact complex manifold X should be Kobayashi hyperbolic iff it is projective and every subvariety Y of X is of general type.

Solution of the Bloch conjecture

The following has been proved by Ochiai 77, Noguchi 77, 81, 84, Kawamata 80 in the algebraic situation.

Theorem (Ochiai 77, Noguchi 77,81,84, Kawamata 80)

Let $Z = \mathbb{C}^n/\Lambda$ be an abelian variety (resp. a complex torus). Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ of the image of every entire curve $f : \mathbb{C} \to Z$ is the translate of a subtorus.

Solution of the Bloch conjecture

The following has been proved by Ochiai 77, Noguchi 77, 81, 84, Kawamata 80 in the algebraic situation.

Theorem (Ochiai 77, Noguchi 77,81,84, Kawamata 80)

Let $Z = \mathbb{C}^n/\Lambda$ be an abelian variety (resp. a complex torus). Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ of the image of every entire curve $f : \mathbb{C} \to Z$ is the translate of a subtorus.

Corollary 1

Let X be a complex analytic subvariety of a complex torus Z . Assume that X is of general type. Then every entire curve drawn in X is analytically degenerate.

Solution of the Bloch conjecture

The following has been proved by Ochiai 77, Noguchi 77, 81, 84, Kawamata 80 in the algebraic situation.

Theorem (Ochiai 77, Noguchi 77,81,84, Kawamata 80)

Let $Z = \mathbb{C}^n/\Lambda$ be an abelian variety (resp. a complex torus). Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ of the image of every entire curve $f : \mathbb{C} \to Z$ is the translate of a subtorus.

Corollary 1

Let X be a complex analytic subvariety of a complex torus Z . Assume that X is of general type. Then every entire curve drawn in X is analytically degenerate.

Corollary 2

Let X be a complex analytic subvariety of a complex torus Z . Assume that X does not contain any translate of a positive dimensional subtorus. Then X is Kobayashi hyperbolic.

Results on the Kobayashi conjecture

Kobayashi conjecture (1970)

- Let $X \subset \mathbb{P}^{n+1}$ be a (very) generic hypersurface of degree $d \geq d_n$ large enough. Then X is Kobayashi hyperbolic.

Results on the Kobayashi conjecture

Kobayashi conjecture (1970)

- Let $X \subset \mathbb{P}^{n+1}$ be a (very) generic hypersurface of degree $d \geq d_n$ large enough. Then X is Kobayashi hyperbolic.
- By a result of M. Zaidenberg (1987), the optimal bound must satisfy $d_n \geq 2n+1$, and one expects $d_n = 2n+1$.

Results on the Kobayashi conjecture

Kobayashi conjecture (1970)

- Let $X \subset \mathbb{P}^{n+1}$ be a (very) generic hypersurface of degree $d \geq d_n$ large enough. Then X is Kobayashi hyperbolic.
- By a result of M. Zaidenberg (1987), the optimal bound must satisfy $d_n \geq 2n+1$, and one expects $d_n = 2n+1$.

Using "jet technology" and deep results of McQuillan for curve foliations on surfaces, the following has been proved:

Theorem (D., El Goul, 1998)

A very generic surface $X \subset \mathbb{P}^3$ of degree $d \geq 21$ is hyperbolic. Independently McQuillan got $d \geq 35$.

This was more recently improved to $d \geq 18$ (Păun, 2008).

Results on the Kobayashi conjecture

Kobayashi conjecture (1970)

- Let $X \subset \mathbb{P}^{n+1}$ be a (very) generic hypersurface of degree $d \geq d_n$ large enough. Then X is Kobayashi hyperbolic.
- By a result of M. Zaidenberg (1987), the optimal bound must satisfy $d_n \geq 2n+1$, and one expects $d_n = 2n+1$.

Using "jet technology" and deep results of McQuillan for curve foliations on surfaces, the following has been proved:

Theorem (D., El Goul, 1998)

A very generic surface $X \subset \mathbb{P}^3$ of degree $d \geq 21$ is hyperbolic. Independently McQuillan got $d \geq 35$.

This was more recently improved to $d \geq 18$ (Păun, 2008).
In 2012, Yum-Tong Siu announced a proof of the case of arbitrary dimension n, with a very large d_n (and a rather involved proof).

Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

Theorem (S. Diverio, J. Merker, E. Rousseau, 2009)

A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n := 2^{n^5}$ satisfies the GGL conjecture.

Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

Theorem (S. Diverio, J. Merker, E. Rousseau, 2009)

A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n := 2^{n^5}$ satisfies the GGL conjecture.
The bound was improved by (D-, 2012) to

$$d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor = O(\exp(n^{1+\varepsilon})), \quad \forall \varepsilon > 0.$$

Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

Theorem (S. Diverio, J. Merker, E. Rousseau, 2009)

A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n := 2^{n^5}$ satisfies the GGL conjecture.
The bound was improved by (D-, 2012) to

$$d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor \quad = O(\exp(n^{1+\varepsilon})), \quad \forall \varepsilon > 0.$$

Theorem (S. Diverio, S. Trapani, 2009)

Additionally, a generic hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 593$ is hyperbolic.

Category of directed varieties

- **Goal.** We are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X or, more generally, a (possibly singular) linear subspace, i.e. a closed irreducible analytic subspace of the total space T_X such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.

Category of directed varieties

- **Goal.** We are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X or, more generally, a (possibly singular) linear subspace, i.e. a closed irreducible analytic subspace of the total space T_X such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.
- **Definition.** *Category of directed varieties :*
  – Objects : pairs (X, V), X variety/$\mathbb{C}$ and $V \subset T_X$
  – Arrows $\psi : (X, V) \to (Y, W)$ holomorphic s.t. $\psi_* V \subset W$

Category of directed varieties

- **Goal.** We are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X or, more generally, a (possibly singular) linear subspace, i.e. a closed irreducible analytic subspace of the total space T_X such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.
- **Definition.** *Category of directed varieties :*
  - Objects : pairs (X, V), X variety$/\mathbb{C}$ and $V \subset T_X$
  - Arrows $\psi : (X, V) \to (Y, W)$ holomorphic s.t. $\psi_* V \subset W$
  - "Absolute case" (X, T_X), i.e. $V = T_X$
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)

Category of directed varieties

- **Goal.** We are interested in curves $f:\mathbb{C}\to X$ such that $f'(\mathbb{C})\subset V$ where V is a subbundle of T_X or, more generally, a (possibly singular) linear subspace, i.e. a closed irreducible analytic subspace of the total space T_X such that $\forall x\in X$, $V_x:=V\cap T_{X,x}$ is linear.
- **Definition.** *Category of directed varieties :*
  - Objects : pairs (X,V), X variety$/\mathbb{C}$ and $V\subset T_X$
  - Arrows $\psi:(X,V)\to(Y,W)$ holomorphic s.t. $\psi_*V\subset W$
  - "Absolute case" (X,T_X), i.e. $V=T_X$
  - "Relative case" $(X,T_{X/S})$ where $X\to S$
  - "Integrable case" when $[V,V]\subset V$ (foliations)
- **Fonctor "1-jet" :** $(X,V)\mapsto(\tilde X,\tilde V)$ where :

$$\tilde X = P(V) = \text{bundle of projective spaces of lines in } V$$

$$\pi:\tilde X=P(V)\to X,\quad (x,[v])\mapsto x,\quad v\in V_x$$

$$\tilde V_{(x,[v])}=\{\xi\in T_{\tilde X,(x,[v])}\,;\ \pi_*\xi\in\mathbb{C}v\subset T_{X,x}\}$$

Semple jet bundles (non singular case)

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

Semple jet bundles (non singular case)

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*

  – $(X_k, V_k) =$ k-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.

Semple jet bundles (non singular case)

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*

  – $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.

- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk } \tilde{V} = r = \text{rk } V$$

$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$

Semple jet bundles (non singular case)

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*

  – $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.

- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\, \tilde{V} = r = \text{rk}\, V$$

$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$

$$0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{X_k}(-1) \to 0 \qquad \Rightarrow \text{rk}\, V_k = r$$

$$0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0 \quad \text{(Euler)}$$

k-jets of curves

For $n = \dim X$ and $r = \mathrm{rk}\, V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

k-jets of curves

For $n = \dim X$ and $r = \operatorname{rk} V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rk} V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

We define the bundle $J^k V$ of k-jets of curves tangent to V by taking $J^k V_x$ to be the set of equivalence classes of germs $f : (\mathbb{C}, 0) \to (X, V)$ such that in some coordinates $f(t) = (f_1(t), \ldots, f_n(t))$ has a Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}).$$

k-jets of curves

For $n = \dim X$ and $r = \mathrm{rk}\, V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \overset{\pi_k}{\to} X_{k-1} \to \cdots \to X_1 \overset{\pi_1}{\to} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

We define the bundle $J^k V$ of k-jets of curves tangent to V by taking $J^k V_x$ to be the set of equivalence classes of germs $f : (\mathbb{C}, 0) \to (X, V)$ such that in some coordinates $f(t) = (f_1(t), \ldots, f_n(t))$ has a Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}).$$

Here we take $\xi_s = \frac{1}{s!}\nabla^s f(0)$ with respect to some local holomorphic connection on V (obtained e.g. from a trivialization). Thus $\xi_s \in V_x$ and

$$J^k V_x \simeq V_x^{\oplus k} \simeq \mathbb{C}^{kr} \quad \text{(non intrinsically).}$$

Semple bundles and reparametrization of curves

Consider the group $\mathbb{G}_k$ of k-jets of germs of biholomorphisms $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, i.e.

$$\varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k + O(t^{k+1})$$

and the natural $\mathbb{G}_k$ action on the right:

$$J^k V \times \mathbb{G}_k \to J^k V, \quad (f, \varphi) \mapsto f \circ \varphi.$$

The action is free on germs $J^k V^{\mathrm{reg}}$ of *regular curves* with $\xi_1 = f'(0) \neq 0$.

Semple bundles and reparametrization of curves

Consider the group $\mathbb{G}_k$ of k-jets of germs of biholomorphisms $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, i.e.

$$\varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k + O(t^{k+1})$$

and the natural $\mathbb{G}_k$ action on the right:

$$J^k V \times \mathbb{G}_k \to J^k V, \quad (f, \varphi) \mapsto f \circ \varphi.$$

The action is free on germs $J^k V^{\mathrm{reg}}$ of *regular curves* with $\xi_1 = f'(0) \neq 0$.

Theorem

X_k is a smooth compactification of $J_k V^{\mathrm{reg}} / \mathbb{G}_k$.

Semple bundles and reparametrization of curves

Consider the group $\mathbb{G}_k$ of k-jets of germs of biholomorphisms $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, i.e.

$$\varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k + O(t^{k+1})$$

and the natural $\mathbb{G}_k$ action on the right:

$$J^k V \times \mathbb{G}_k \to J^k V, \quad (f, \varphi) \mapsto f \circ \varphi.$$

The action is free on germs $J^k V^{\mathrm{reg}}$ of *regular curves* with $\xi_1 = f'(0) \neq 0$.

Theorem

X_k is a smooth compactification of $J_k V^{\mathrm{reg}} / \mathbb{G}_k$.

Now we want to deal with possibly singular directed varieties (X, V), i.e. X and V both possibly singular.

Singular directed varieties

Definition

A singular directed variety is a pair (X, V) where X is a reduced complex space, and $V \subset T_X$ is a closed linear subspace of T_X.

Singular directed varieties

Definition

A singular directed variety is a pair (X, V) where X is a reduced complex space, and $V \subset T_X$ is a closed linear subspace of T_X.

This means that we have an irreducible component V_j lying over each irreducible component X_j of X.

Singular directed varieties

Definition

A singular directed variety is a pair (X, V) where X is a reduced complex space, and $V \subset T_X$ is a closed linear subspace of T_X.

This means that we have an irreducible component V_j lying over each irreducible component X_j of X.

Assume X to be irreducible, $\dim X = n$. Every point $x \in X$ has a neighborhood U with an embedding $U \hookrightarrow \Omega$ as a closed analytic subset in a smooth open set $\Omega \subset \mathbb{C}^N$. Then $T_{X|U}$ is taken to be the closure of $T_{U_{\rm reg}}$ in T_Ω, and V is always assumed to be the closure of $V_{\rm reg|U}$ (part of V that is a subbundle of $T_{X_{\rm reg}|U}$).

Singular directed varieties

Definition

A singular directed variety is a pair (X, V) where X is a reduced complex space, and $V \subset T_X$ is a closed linear subspace of T_X.

This means that we have an irreducible component V_j lying over each irreducible component X_j of X.

Assume X to be irreducible, $\dim X = n$. Every point $x \in X$ has a neighborhood U with an embedding $U \hookrightarrow \Omega$ as a closed analytic subset in a smooth open set $\Omega \subset \mathbb{C}^N$. Then $T_{X|U}$ is taken to be the closure of $T_{U_{\rm reg}}$ in T_Ω, and V is always assumed to be the closure of $V_{\rm reg|U}$ (part of V that is a subbundle of $T_{X_{\rm reg}|U}$).

If X is *non singular* and $V \subset T_X$ is *singular*, V is a subbundle of T_X over a Zariski open set $X' = X \smallsetminus Y$, and we have at least an absolute Semple tower (X^a_k, V^a_k) associated with (X, T_X).

Singular directed varieties

Definition

A singular directed variety is a pair (X, V) where X is a reduced complex space, and $V \subset T_X$ is a closed linear subspace of T_X.

This means that we have an irreducible component V_j lying over each irreducible component X_j of X.

Assume X to be irreducible, $\dim X = n$. Every point $x \in X$ has a neighborhood U with an embedding $U \hookrightarrow \Omega$ as a closed analytic subset in a smooth open set $\Omega \subset \mathbb{C}^N$. Then $T_{X|U}$ is taken to be the closure of $T_{U_{\rm reg}}$ in T_Ω, and V is always assumed to be the closure of $V_{\rm reg|U}$ (part of V that is a subbundle of $T_{X_{\rm reg}|U}$).

If X is *non singular* and $V \subset T_X$ is *singular*, V is a subbundle of T_X over a Zariski open set $X' = X \smallsetminus Y$, and we have at least an absolute Semple tower (X^a_k, V^a_k) associated with (X, T_X).

We then define (X_k, V_k) to be the closure of (X'_k, V'_k) [associated with (X', V'), $V' = V_{|X'}$] in (X^a_k, V^a_k).

Base resolution of singularities

Let (X, V) be singular pair. By Hironaka, there exists a modification $\mu : \widetilde{X} \to X$ (in the form of a composition of blow-ups with smooth centers), such that $\widetilde{X}$ is non singular.

Base resolution of singularities

Let (X, V) be singular pair. By Hironaka, there exists a modification $\mu : \widetilde{X} \to X$ (in the form of a composition of blow-ups with smooth centers), such that $\widetilde{X}$ is non singular.

Let $\mu_* : T_{\widetilde{X}} \to \mu^* T_X$ be the differential $d\mu$. We define $\widetilde{V} = \mu^{-1} V \subset T_{\widetilde{X}}$ to be the closure of $(\mu_*)^{-1}(V_{|X'})$, where $X' \subset X_{\rm reg}$ is a Zariski open set over which $V_{|X'}$ is a subbundle of $T_{X_{\rm reg}}$ and $\mu : \mu^{-1}(X') \to X'$ is a biregular.

Base resolution of singularities

Let (X, V) be singular pair. By Hironaka, there exists a modification $\mu : \widetilde{X} \to X$ (in the form of a composition of blow-ups with smooth centers), such that $\widetilde{X}$ is non singular.

Let $\mu_* : T_{\widetilde{X}} \to \mu^* T_X$ be the differential $d\mu$. We define $\widetilde{V} = \mu^{-1} V \subset T_{\widetilde{X}}$ to be the closure of $(\mu_*)^{-1}(V_{|X'})$, where $X' \subset X_{\mathrm{reg}}$ is a Zariski open set over which $V_{|X'}$ is a subbundle of $T_{X_{\mathrm{reg}}}$ and $\mu : \mu^{-1}(X') \to X'$ is a biregular.

We can then construct a Semple tower $(\widetilde{X}_k, \widetilde{V}_k)$ by taking the closure over regular points of (X'_k, V'_k) in the (regular) absolute tower $(\widetilde{X}^a_k, \widetilde{V}^a_k)$, where $\widetilde{V}^a_0 = T_{\widetilde{X}}$.

Big caution !

In general, for $\dim X \geq 2$, one can never make $\widetilde{V}$ non singular, even by blowing up further !

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. We first look at the $\mathbb{C}^*$-action induced by *dilations* $\varphi(t) = \eta_\lambda(t) = \lambda t$.

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. We first look at the $\mathbb{C}^*$-action induced by *dilations* $\varphi(t) = \eta_\lambda(t) = \lambda t$.

Putting $\xi_s = \Delta^s f(0)$, the $\mathbb{C}^*$ action is obtained by computing the derivatives of $f(\lambda t)$, hence it is given on $J^k V_x \simeq V_x^{\oplus k}$ by

$$(*) \qquad \lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. We first look at the $\mathbb{C}^*$-action induced by *dilations* $\varphi(t) = \eta_\lambda(t) = \lambda t$.

Putting $\xi_s = \Delta^s f(0)$, the $\mathbb{C}^*$ action is obtained by computing the derivatives of $f(\lambda t)$, hence it is given on $J^k V_x \simeq V_x^{\oplus k}$ by

$$(*) \qquad \lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k).$$

We consider the Green-Griffiths bundle $E_{k,m}^{\mathrm{GG}} V^*$ of polynomials of weighted degree m on $J^k V_x$ written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(x)\, \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V_x.$$

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. We first look at the $\mathbb{C}^*$-action induced by *dilations* $\varphi(t) = \eta_\lambda(t) = \lambda t$.

Putting $\xi_s = \Delta^s f(0)$, the $\mathbb{C}^*$ action is obtained by computing the derivatives of $f(\lambda t)$, hence it is given on $J^k V_x \simeq V_x^{\oplus k}$ by

$$(*) \qquad \lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k).$$

We consider the Green-Griffiths bundle $E_{k,m}^{\mathrm{GG}} V^*$ of polynomials of weighted degree m on $J^k V_x$ written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(x)\, \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V_x.$$

Take P to be a holomorphic section in x. It can then be viewed as an algebraic differential operator $P(f_{[k]}) = P(f\,;\,f', f'', \ldots, f^{(k)})$,

$$P(f_{[k]})(t) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}.$$

Direct image formula for Green-Griffiths bundles

The homogeneity expressed by the $\mathbb{C}^*$ action $(*)$ means that $P((f\circ\eta_\lambda)_{[k]}) = \lambda^m P(f_{[k]})\circ\eta_\lambda$ for $\eta_\lambda(t)=\lambda t$, and our polynomials are taken over multi-indices $(\alpha_1,\ldots,\alpha_k)$ such that

$$|\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k| = m.$$

Green Griffiths bundles

Consider $X_k^{\rm GG} := J^k V^{\neq {\rm const}}/\mathbb{C}^*$. This defines a bundle $\pi_k : X_k^{\rm GG} \to X$ of weighted projective spaces and by definition

$$\mathcal{O}(E_{k,m}^{\rm GG} V^*) = (\pi_k)_*\mathcal{O}_{X_k^{\rm GG}}(m)$$

is the direct image of the m-th power of the tautological bundle (or sheaf) $\mathcal{O}_{X_k^{\rm GG}}(1)$ on $X_k^{\rm GG}$.

In case V is singular, we take *by definition* $\mathcal{O}_{X_k^{\rm GG}}(m)$ to be the sheaf of germs of polynomials $P(x;\xi_1,\ldots,\xi_k)$ that are locally bounded with respect to a smooth ambient hermitian metric h on T_X (and the induced metric on V_k).

Direct image formula for Semple bundles

Now, look instead at the direct image of $\mathcal{O}_{X_k}(m)$ on the Semple bundle $X_k = \overline{J^k V^{\rm reg}}/\mathbb{G}_k$, by the projection $\pi_{k,0} : X_k \to X_0$ from the Semple tower

$$X_k \to X_{k-1} \to \ldots X_1 \to X_0 = X \quad (X \text{ non singular}).$$

Direct image formula for Semple bundles

Now, look instead at the direct image of $\mathcal{O}_{X_k}(m)$ on the Semple bundle $X_k = \overline{J^k V^{\mathrm{reg}}}/\mathbb{G}_k$, by the projection $\pi_{k,0} : X_k \to X_0$ from the Semple tower

$$X_k \to X_{k-1} \to \ldots X_1 \to X_0 = X \quad (X \text{ non singular}).$$

Semple direct image formula

The direct image sheaf

$$(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m}V^*)$$

is the sheaf of sections of the bundle $E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$ of $\mathbb{G}_k$-invariant algebraic differential operators $f \mapsto P(f_{[k]})$ such that

$$P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi, \quad \forall \varphi \in \mathbb{G}_k.$$

(by definition, the sections are taken to be locally bounded with respect to an ambient smooth hermitian metric h on T_X).

Canonical sheaf of a singular pair (X,V)

When (X, V) is nonsingular, we simply set $K_V = \det(V^*)$.

Canonical sheaf of a singular pair (X,V)

When (X, V) is nonsingular, we simply set $K_V = \det(V^*)$.
When X is non singular and V singular, we first introduce the rank 1 sheaf ${}^bK_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X. One can show that ${}^bK_V$ is equal to the integral closure of the image of the natural morphism

$$\Lambda^r T_X^* \to \Lambda^r V^* \to \mathcal{L}_V := \mathrm{invert.\ sheaf}\ (\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V, \quad \mathcal{J}_V \subset \mathcal{O}_X$,

$${}^bK_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V, \quad \overline{\mathcal{J}}_V = \text{integral closure of } \mathcal{J}_V.$$

Canonical sheaf of a singular pair (X,V)

When (X, V) is nonsingular, we simply set $K_V = \det(V^*)$.
When X is non singular and V singular, we first introduce the rank 1 sheaf ${}^bK_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X. One can show that ${}^bK_V$ is equal to the integral closure of the image of the natural morphism

$$\Lambda^r T_X^* \to \Lambda^r V^* \to \mathcal{L}_V := \text{invert. sheaf } (\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\quad \mathcal{J}_V \subset \mathcal{O}_X$,

$${}^bK_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V, \qquad \overline{\mathcal{J}}_V = \text{integral closure of } \mathcal{J}_V.$$

Consequence

If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\tilde{\mu}^{-1}(V)}$, then

$${}^bK_V \subset \mu_*({}^bK_{\widetilde{V}}) \subset \mathcal{L}_V$$

and $\mu_*({}^bK_{\widetilde{V}})$ increases with μ.

Canonical sheaf of a singular pair (X,V) [cont.]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$K_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^bK_{\widetilde{V}})^{\otimes m}, \quad ({}^bK_V)^{\otimes m} \subset K_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

which we call the pluricanonical sheaf sequence of (X, V).

Canonical sheaf of a singular pair (X,V) [cont.]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$K_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^bK_{\widetilde{V}})^{\otimes m}, \quad ({}^bK_V)^{\otimes m} \subset K_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

which we call the pluricanonical sheaf sequence of (X, V).

Remark

The blow-up μ for which the limit is attained may depend on m. We do not know if there is a μ that works for all m.

Canonical sheaf of a singular pair (X,V) [cont.]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$K_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^bK_{\widetilde{V}})^{\otimes m}, \quad ({}^bK_V)^{\otimes m} \subset K_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

which we call the pluricanonical sheaf sequence of (X, V).

Remark

The blow-up μ for which the limit is attained may depend on m. We do not know if there is a μ that works for all m.

This generalizes the concept of reduced singularities of foliations, which is known to work only for surfaces.

Definition

We say that (X, V) is of general type if the pluricanonical sheaf sequence is big, i.e. $H^0(X, K_V^{[m]})$ provides a generic embedding of X for a suitable $m \gg 1$.

Generalized GGL conjecture

Generalized GGL conjecture

If (X, V) is directed manifold of general type, i.e. $K_V^\bullet$ is big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

Generalized GGL conjecture

Generalized GGL conjecture

If (X, V) is directed manifold of general type, i.e. $K_V^\bullet$ is big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

Remark. Elementary if $r = \operatorname{rk} V = 1$, and more generally if V^* itself is big, i.e. $\exists A$ ample such that $S^m V^* \otimes \mathcal{O}(-A)$ generated by sections on a Zariski open set $X \smallsetminus Y$.

Generalized GGL conjecture

Generalized GGL conjecture

If (X, V) is directed manifold of general type, i.e. $K_V^\bullet$ is big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

Remark. Elementary if $r = \mathrm{rk}\, V = 1$, and more generally if V^* itself is big, i.e. $\exists A$ ample such that $S^m V^* \otimes \mathcal{O}(-A)$ generated by sections on a Zariski open set $X \smallsetminus Y$.

Ahlfors-Schwarz lemma

Let $\gamma = i \sum \gamma_{jk} dt_j \wedge d\overline{t}_k \geq 0$ be an a.e. positive hermitian form on the ball $B(0, R) \subset \mathbb{C}^p$, such that $-\mathrm{Ricci}(\gamma) := i\,\partial\overline{\partial} \log \det \gamma \geq C\gamma$ in the sense of currents, for some constant $C > 0$. Then the γ-volume form is controlled by the Poincaré volume form :

$$\det(\gamma) \leq \left(\frac{p+1}{CR^2}\right)^p \frac{1}{(1 - |t|^2/R^2)^{p+1}}.$$

In particular one has a bound $R \leq \left(\frac{p+1}{C}\right)^{1/2} (\det(\gamma(0))^{-1/2p}$.

Fundamental vanishing theorem

Proof. Construct a (singular) Finsler metric on V by
$\|\xi\|^2_{V,h} := \Big(\sum_j |\sigma_j(x)\cdot\xi^m|^2_{h_A^*}\Big)^{1/m}$ with $\xi\in V_x$,
$\sigma_j\in H^0(X, S^mV^*\otimes\mathcal{O}(-A)$. Set $\gamma(t) = i\|f'(t)\|^2_{V,h}dt\wedge d\overline{t}$ on the disk $D(0,R)$. Then if $\gamma\not\equiv 0$, we have

$$-\mathrm{Ricci}(\gamma) = i\partial\overline{\partial}\|f'(t)\|^2_{V,h} \geq f^*\Theta_{V^*,h^*} \geq \frac{1}{m}f^*\Theta_{A,h_A} \geq C\gamma,$$

thus R is bounded and one cannot have $R=+\infty$.

Fundamental vanishing theorem

Proof. Construct a (singular) Finsler metric on V by
$\|\xi\|^2_{V,h} := \Big(\sum_j |\sigma_j(x)\cdot\xi^m|^2_{h_A^*}\Big)^{1/m}$ with $\xi \in V_x$,
$\sigma_j \in H^0(X, S^m V^* \otimes \mathcal{O}(-A)$. Set $\gamma(t) = i\|f'(t)\|^2_{V,h} dt \wedge d\overline{t}$ on the disk $D(0,R)$. Then if $\gamma \not\equiv 0$, we have

$$-\mathrm{Ricci}(\gamma) = i\partial\overline{\partial}\|f'(t)\|^2_{V,h} \geq f^*\Theta_{V^*,h^*} \geq \frac{1}{m} f^*\Theta_{A,h_A} \geq C\gamma,$$

thus R is bounded and one cannot have $R = +\infty$.

Fundamental vanishing theorem for jet differentials

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
$\forall P \in H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$: global diff. operator on X (A ample divisor), $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $P(f_{[k]}) \equiv 0$.

Geometric consequence of the fundamental vanishing theorem

Green-Griffiths locus.

Define $\mathrm{GG}_k(X,V)$ to be the base locus of sections in

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(-A)) = H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-A)).$$

Geometric consequence of the fundamental vanishing theorem

Green-Griffiths locus.

Define $\mathrm{GG}_k(X, V)$ to be the base locus of sections in

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(-A)) = H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-A)).$$

Then all entires curves $f(\mathbb{C})$ are contained in

$$\mathrm{GG}(X, V) := \bigcap_{k>0} \pi_{k,0}(\mathrm{GG}_k(X, V)).$$

Geometric consequence of the fundamental vanishing theorem

Green-Griffiths locus.

Define $\mathrm{GG}_k(X,V)$ to be the base locus of sections in

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(-A)) = H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-A)).$$

Then all entires curves $f(\mathbb{C})$ are contained in

$$\mathrm{GG}(X,V) := \bigcap_{k>0} \pi_{k,0}(\mathrm{GG}_k(X,V)).$$

The problem is then to compute these base loci and show possibly that $Y := \mathrm{GG}(X,V) \subsetneq X$. However this is (unfortunately) only a *sufficient condition* (Diverio-Rousseau, 2013).

Geometric consequence of the fundamental vanishing theorem

Green-Griffiths locus.
Define $\mathrm{GG}_k(X,V)$ to be the base locus of sections in

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(-A)) = H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-A)).$$

Then all entires curves $f(\mathbb{C})$ are contained in

$$\mathrm{GG}(X,V) := \bigcap_{k>0} \pi_{k,0}(\mathrm{GG}_k(X,V)).$$

The problem is then to compute these base loci and show possibly that $Y := \mathrm{GG}(X,V) \subsetneq X$. However this is (unfortunately) only a *sufficient condition* (Diverio-Rousseau, 2013).

A simple example for which $\mathrm{GG}(X,V) = X$ is the case of a product $X = X' \times X''$ of hyperbolic manifolds, with $V = T_X$.

Proof of fundamental vanishing theorem

Simple case. First assume that f is a Brody curve, i.e. $\|f'\|_\omega$ bounded for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

Proof of fundamental vanishing theorem

Simple case. First assume that f is a Brody curve, i.e. $\|f'\|_\omega$ bounded for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

The Cauchy inequalities imply that all derivatives $f^{(s)}$ are bounded in any relatively compact coordinate chart. Hence $u_A(t) = P(f_{[k]})(t)$ is bounded, and must thus be constant by Liouville's theorem.

Proof of fundamental vanishing theorem

Simple case. First assume that f is a Brody curve, i.e. $\|f'\|_\omega$ bounded for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

The Cauchy inequalities imply that all derivatives $f^{(s)}$ are bounded in any relatively compact coordinate chart. Hence $u_A(t) = P(f_{[k]})(t)$ is bounded, and must thus be constant by Liouville's theorem.

Since A is very ample, we can move $A \in |A|$ such that A hits $f(\mathbb{C}) \subset X$. Bu then u_A vanishes somewhere and so $u_A \equiv 0$.

Case of an invariant jet differential. Assume $P \in H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-A))$ is $\mathbb{G}_k$-invariant. This is the same as a section $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(-A))$.

Proof of fundamental vanishing theorem (cont.)

From the existence of σ and the fact that $\mathcal{O}_{X_k}(1)$ is relatively ample over X_{k-1}, we infer the existence of a singular hermitian metric h_σ on $\mathcal{O}_{X_k}(-1)$ (essentially given by $|\xi^m \cdot \sigma|^{2/m}$ corrected with relatively ample terms), such that $i\partial\overline{\partial} \log h_\sigma$ is bounded below by a positive definite Kähler form ω on X_k, and the zeroes of h_σ coincide with the zero divisor Z_σ.

Proof of fundamental vanishing theorem (cont.)

From the existence of σ and the fact that $\mathcal{O}_{X_k}(1)$ is relatively ample over X_{k-1}, we infer the existence of a singular hermitian metric h_σ on $\mathcal{O}_{X_k}(-1)$ (essentially given by $|\xi^m \cdot \sigma|^{2/m}$ corrected with relatively ample terms), such that $i\partial\overline{\partial}\log h_\sigma$ is bounded below by a positive definite Kähler form ω on X_k, and the zeroes of h_σ coincide with the zero divisor Z_σ.

Now $f_{[k-1]} : \mathbb{C} \to X_{k-1}$ has a derivative $f'_{[k-1]}$ that can be viewed as a section of the pull-back line bundle $f^*_{[k]}\mathcal{O}_{X_k}(-1)$.

Proof of fundamental vanishing theorem (cont.)

From the existence of σ and the fact that $\mathcal{O}_{X_k}(1)$ is relatively ample over X_{k-1}, we infer the existence of a singular hermitian metric h_σ on $\mathcal{O}_{X_k}(-1)$ (essentially given by $|\xi^m \cdot \sigma|^{2/m}$ corrected with relatively ample terms), such that $i\partial\overline{\partial}\log h_\sigma$ is bounded below by a positive definite Kähler form ω on X_k, and the zeroes of h_σ coincide with the zero divisor Z_σ.

Now $f_{[k-1]} : \mathbb{C} \to X_{k-1}$ has a derivative $f'_{[k-1]}$ that can be viewed as a section of the pull-back line bundle $f^*_{[k]}\mathcal{O}_{X_k}(-1)$.

If we put $\gamma(t) = i\,\|f'_{[k-1]}\|^2_{h_\sigma} dt \wedge d\overline{t}$, then assuming $f(\mathbb{C}) \not\subset Z_\sigma$, we get $\gamma \not\equiv 0$ on $\mathbb{C}$ and

$$-\mathrm{Ricci}(\gamma) = i\partial\overline{\partial}\log\gamma \geq f^*_{[k]}\Theta_{\mathcal{O}_{X_k}(1),h^*_\sigma} \geq Cf^*_{[k]}\omega \geq C'\gamma.$$

This is a contradiction, hence $f(\mathbb{C}) \subset Z_\sigma$, as desired.

Existence theorem for jet differentials

Relation between invariant and non invariant jet differentials. On a non invariant polynomial P one can define in a natural way a $\mathbb{G}_k$-action by putting $(\varphi^* P)(f_{[k]}) := P((f \circ \varphi)_{[k]})(0)$.

Relation between invariant and non invariant jet differentials. On a non invariant polynomial P one can define in a natural way a $\mathbb{G}_k$-action by putting $(\varphi^* P)(f_{[k]}) := P((f \circ \varphi)_{[k]})(0)$.
By expanding the derivatives, one finds

$$(\varphi^* P)(f_{[k]}) = \sum_{\alpha \in \mathbb{N}^k,\, |\alpha|_w = m} \varphi^{(\alpha)}(0)\, P_\alpha(f_{[k]})$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, $\varphi^{(\alpha)} = (\varphi')^{\alpha_1} (\varphi'')^{\alpha_2} \ldots (\varphi^{(k)})^{\alpha_k}$, $|\alpha|_w = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k$ is the weighted degree of α, and if one puts $\deg P = m$, P_α is a homogeneous polynomial of degree $\deg P_\alpha = m - (\alpha_2 + 2\alpha_3 + \ldots + (k-1)\alpha_k) = \alpha_1 + \alpha_2 + \ldots + \alpha_k$.

Existence theorem for jet differentials

Relation between invariant and non invariant jet differentials. On a non invariant polynomial P one can define in a natural way a $\mathbb{G}_k$-action by putting $(\varphi^* P)(f_{[k]}) := P((f \circ \varphi)_{[k]})(0)$. By expanding the derivatives, one finds

$$(\varphi^* P)(f_{[k]}) = \sum_{\alpha \in \mathbb{N}^k,\, |\alpha|_w = m} \varphi^{(\alpha)}(0)\, P_\alpha(f_{[k]})$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, $\varphi^{(\alpha)} = (\varphi')^{\alpha_1}(\varphi'')^{\alpha_2} \ldots (\varphi^{(k)})^{\alpha_k}$, $|\alpha|_w = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k$ is the weighted degree of α, and if one puts $\deg P = m$, P_α is a homogeneous polynomial of degree

$\deg P_\alpha = m - (\alpha_2 + 2\alpha_3 + \ldots + (k-1)\alpha_k) = \alpha_1 + \alpha_2 + \ldots + \alpha_k$.

Fundamental existence theorem (D-, 2010)

Let (X, V) be of general type, such that ${}^b K_V$ is a big rank 1 sheaf. Then $\exists$ many $P \in H^0(X, E_{k,m} V^* \otimes \mathcal{O}(-A))$, $m \gg k \gg 1 \Rightarrow \exists$ algebr. hypersurface $Z \subsetneq X_k$ such that $f_{[k]}(\mathbb{C}) \subset Z$, $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$

Holomorphic Morse inequalities

Theorem (D, 1985, L. Bonavero 1996)

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold. Assume L equipped with a *singular hermitian metric* $h = e^{-\varphi}$ with analytic singularities in $\Sigma \subset X$, and $\theta = \frac{i}{2\pi}\Theta_{L,h}$.

Holomorphic Morse inequalities

Theorem (D, 1985, L. Bonavero 1996)

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold. Assume L equipped with a *singular hermitian metric* $h = e^{-\varphi}$ with analytic singularities in $\Sigma \subset X$, and $\theta = \frac{i}{2\pi}\Theta_{L,h}$. Let

$$X(\theta, q) := \big\{x \in X \smallsetminus \Sigma\,;\ \theta(x) \text{ has signature } (n-q, q)\big\}$$

be the q-index set of the $(1,1)$-form θ, and

$$X(\theta, J) = \bigcup_{j\in J} X(\theta, j), \quad \text{for } J \subset \{0, \ldots, n\}.$$

Holomorphic Morse inequalities

Theorem (D, 1985, L. Bonavero 1996)

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold. Assume L equipped with a *singular hermitian metric* $h = e^{-\varphi}$ with analytic singularities in $\Sigma \subset X$, and $\theta = \frac{i}{2\pi}\Theta_{L,h}$. Let

$$X(\theta, q) := \big\{x \in X \smallsetminus \Sigma\,;\ \theta(x) \text{ has signature } (n-q, q)\big\}$$

be the q-index set of the $(1,1)$-form θ, and

$$X(\theta, J) = \bigcup\nolimits_{j \in J} X(\theta, j), \qquad \text{for } J \subset \{0, \ldots, n\}.$$

Then

(i) $h^q(X, L^{\otimes m} \otimes \mathcal{I}(m\varphi)) \le \frac{m^n}{n!} \int_{X(\theta,q)} (-1)^q \theta^n + o(m^n)$,

(ii) $h^q(X, L^{\otimes m} \otimes \mathcal{I}(m\varphi)) \ge \frac{m^n}{n!} \int_{X(\theta,\{q-1,q,q+1\})} (-1)^q \theta^n - o(m^n)$,

where $\mathcal{I}(m\varphi) \subset \mathcal{O}_X$ denotes the multiplier ideal sheaf

$$\mathcal{I}(m\varphi)_x = \Big\{f \in \mathcal{O}_{X,x}\,;\ \exists U \ni x \text{ s.t. } \int_U |f|^2 e^{-m\varphi} dV < +\infty\Big\}.$$

Finsler metric on the k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$

Finsler metric on the k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$

Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1 \le s \le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Finsler metric on the k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$

Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1 \le s \le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \le s \le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$.

Finsler metric on the k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$

Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1 \le s \le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\rm GG}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{{\rm FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \le s \le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} \, dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{{\rm FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\rm GG} \to X$.

The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

Probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.

Probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha} \overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.

The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s}\gamma(u_s)$ where u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) \int_{u\in SV} \gamma(u)\, du.$$

As γ is quadratic here, $\int_{u\in SV} \gamma(u)\, du = \frac{1}{r}\,\mathrm{Tr}(\gamma)$.

Main cohomological estimate

$\Rightarrow$ the leading term only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*, \det h^*)$, that can be taken > 0 if $\det V^*$ is big.

Corollary (D-, 2010)

Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big),$$

$$\eta = \Theta_{\det V^*,\det h^*} + \Theta_{F,h_F}.$$

Then $\forall q \ge 0$ [$q = 0$ most useful!], $\forall m \gg k \gg 1$ with m sufficiently divisible, the sheaf $\mathcal{G}_{k,m} = \mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{I}(h_k^m)$ satisfies bounds

$$h^q(X_k^{\rm GG}, \mathcal{G}_{k,m}) \le \frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)}(-1)^q\eta^n + \frac{C}{\log k}\Big)$$

Main cohomological estimate

$\Rightarrow$ the leading term only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*, \det h^*)$, that can be taken > 0 if $\det V^*$ is big.

Corollary (D-, 2010)

Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big),$$

$$\eta = \Theta_{\det V^*,\det h^*} + \Theta_{F,h_F}.$$

Then $\forall q \geq 0$ [$q = 0$ most useful!], $\forall m \gg k \gg 1$ with m sufficiently divisible, the sheaf $\mathcal{G}_{k,m} = \mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{I}(h_k^m)$ satisfies bounds

$$h^q(X_k^{\mathrm{GG}}, \mathcal{G}_{k,m}) \leq \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)}(-1)^q\eta^n + \frac{C}{\log k}\Big)$$

$$h^q(X_k^{\mathrm{GG}}, \mathcal{G}_{k,m}) \geq \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,\{q,\,q\pm1\})}(-1)^q\eta^n - \frac{C'}{\log k}\Big).$$

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\rm reg}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$ (so as to deal only with nonsingular ambient Semple bundles).

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\rm reg}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$ (so as to deal only with nonsingular ambient Semple bundles).

This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that $\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \mathrm{rk}\, W < \mathrm{rk}\, V_k = \mathrm{rk}\, V$.

Partial solution of GGL conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto X, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. ${}^bK_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$, after a suitable blow-up.

Partial solution of GGL conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto X, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. ${}^bK_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$, after a suitable blow-up.

Theorem (D-, 2014)

If (X, V) is strongly of general type, the Green-Griffiths-Lang conjecture holds true for (X, V), namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.

Partial solution of GGL conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto X, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. ${}^bK_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$, after a suitable blow-up.

Theorem (D-, 2014)

If (X, V) is strongly of general type, the Green-Griffiths-Lang conjecture holds true for (X, V), namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.

Proof: Induction on rank V, using existence of jet differentials.

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

$$\frac{\inf\left\{\lambda \in \mathbb{Q}\,;\ \exists m \in \mathbb{Q}_+,\ {}^bK_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(\lambda A)\big)_{|Z} \text{ big on } Z\right\}}{\mathrm{rank}\, W}.$$

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

$$\frac{\inf\left\{\lambda \in \mathbb{Q}\,;\ \exists m \in \mathbb{Q}_+,\ {}^bK_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(\lambda A)\big)_{|Z} \text{ big on } Z\right\}}{\operatorname{rank} W}.$$

Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z\subset X_k$ that projects onto X_{k-1} for $k\geq 1$, $Z\not\subset D_k$, and $Z=X=X_0$ for $k=0$, we define the slope of the corresponding directed variety (Z,W) to be $\mu_A(Z,W)=$

$$\frac{\inf\left\{\lambda\in\mathbb{Q}\,;\ \exists m\in\mathbb{Q}_+,\ {}^bK_W\otimes\big(\mathcal{O}_{X_k}(m)\otimes\pi_{k,0}^*\mathcal{O}(\lambda A)\big)_{|Z}\ \text{big on}\ Z\right\}}{\mathrm{rank}\,W}.$$

Notice that (X,V) is of general type iff $\mu_A(X,V)<0$.

We say that (X,V) is A-jet-stable (resp. A-jet-semi-stable) if $\mu_A(Z,W)<\mu_A(X,V)$ (resp. $\mu_A(Z,W)\leq\mu_A(X,V)$) for all $Z\subsetneq X_k$ as above.

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

$$\frac{\inf \{\lambda \in \mathbb{Q}\,;\ \exists m \in \mathbb{Q}_+,\ {}^bK_W \otimes (\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A))_{|Z} \text{ big on } Z\}}{\mathrm{rank}\, W}.$$

Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

We say that (X, V) is A-jet-stable (resp. A-jet-semi-stable) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above.

Observation. If (X, V) is of general type and A-jet-semi-stable, then (X, V) is strongly of general type.

Approach of the Kobayashi conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

Approach of the Kobayashi conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

Theorem (D-, 2014)

If (X, V) is algebraically jet-hyperbolic, then (X, V) is Kobayashi (or Brody) hyperbolic, i.e. there are no entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

Approach of the Kobayashi conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

Theorem (D-, 2014)

If (X, V) is algebraically jet-hyperbolic, then (X, V) is Kobayashi (or Brody) hyperbolic, i.e. there are no entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

Now, the hope is that a (very) generic complete intersection $X = H_1 \cap \ldots \cap H_c \subset \mathbb{P}^{n+c}$ of codimension c and degrees $(d_1, \ldots, d_c)$ s.t. $\sum d_j \geq 2n + c$ yields (X, T_X) algebraically jet-hyperbolic.

Invariance of “directed” plurigenera (?)

One way to check the above property, at least with non optimal bounds, would be to show some sort of Zariski openness of the properties “strongly of general type” or “algebraically jet-hyperbolic”.

Invariance of "directed" plurigenera (?)

One way to check the above property, at least with non optimal bounds, would be to show some sort of Zariski openness of the properties "strongly of general type" or "algebraically jet-hyperbolic". One would need e.g. to know the answer to

Question

Let $(\mathcal{X}, \mathcal{V}) \to S$ be a proper family of directed varieties over a base S, such that $\pi : \mathcal{X} \to S$ is a nonsingular deformation and the directed structure on $X_t = \pi^{-1}(t)$ is $V_t \subset T_{X_t}$, possibly singular. Under which conditions is

$$t \mapsto h^0(X_t, K_{V_t}^{[m]})$$

locally constant over S ?

This would be very useful since one can easily produce jet sections for hypersurfaces $X \subset \mathbb{P}^{n+1}$ admitting meromorphic connections with low pole order (Siu, Nadel).

Related work

In 1993, Masuda and Noguchi gave examples of hyperbolic hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geq 2$, of degree $d \geq d_n$ large enough.

Related work

In 1993, Masuda and Noguchi gave examples of hyperbolic hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geq 2$, of degree $d \geq d_n$ large enough.

In 2012, Y.T. Siu announced the generic hyperbolicity of of hyperbolic hypersurfaces of $\mathbb{P}^n$ of degree $d \geq d_n$ large enough.

Related work

In 1993, Masuda and Noguchi gave examples of hyperbolic hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geq 2$, of degree $d \geq d_n$ large enough.

In 2012, Y.T. Siu announced the generic hyperbolicity of of hyperbolic hypersurfaces of $\mathbb{P}^n$ of degree $d \geq d_n$ large enough.

In 2015, Dinh Tuan Huynh, showed that the complement of a small deformation of the union of $2n$ hyperplanes in general position in $\mathbb{P}^n$ is hyperbolic: the resulting degree $d_n = 2n$ is extremely close to optimality (if not optimal).

Related work

In 1993, Masuda and Noguchi gave examples of hyperbolic hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geq 2$, of degree $d \geq d_n$ large enough.

In 2012, Y.T. Siu announced the generic hyperbolicity of of hyperbolic hypersurfaces of $\mathbb{P}^n$ of degree $d \geq d_n$ large enough.

In 2015, Dinh Tuan Huynh, showed that the complement of a small deformation of the union of $2n$ hyperplanes in general position in $\mathbb{P}^n$ is hyperbolic: the resulting degree $d_n = 2n$ is extremely close to optimality (if not optimal).

Very recently, Gergely Berczi stated a positivity conjecture for Thom polynomials of Morin singularities, and showed that it would imply a polynomial bound $d_n = 2\,n^{10}$ for the generic hyperbolicity of hypersurfaces.