Invariant manifolds and semi-conjugacy

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Indo-French Conference
Chennai, January 13, 2016

A few references

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R. McGehee, E. A. Sander, A new proof of the stable manifold theorem, *Z. Angew. Math. Phys.* **47**, no. 4 (1996), 497–513.

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——, Modest proposal for preventing functional analysis from being a burden to invariant manifold theory, to appear.

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"Hyperbolic" example. $E = E_s \times E_u$ (Banach spaces), $H = (f, g) : (E, 0) \to (E, 0)$ local C^1 map, $DH(0) = A_s \times A_u$, A_u invertible, $|A_s| < 1 < |A_u^{-1}|^{-1}$.

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$\mathrm{dom}\, h^\ell = \{z \in Z : h^\ell(z) \neq \emptyset\} = \big\{(x, G_\ell(x,y)) : (x,y) \in Z\big\}$ satisfies $\mathrm{diam}\{G_\ell(x,y) : y \in Y\} \leq 2r\mu^\ell$ for each $x \in X$. Therefore, the inclusion $\mathrm{dom}\, h^\ell \supset \mathrm{dom}\, h^m$ for $\ell \leq m$ implies $\bigcap \mathrm{dom}\, h^\ell = \mathrm{graph}\,\varphi$ and $\lim_{\ell\to\infty} G_\ell(x, y_\ell) = \varphi(x)$ uniformly with respect to $x \in X$ and $(y_\ell)_{\ell\in\mathbb{N}}$ in Y, hence $\mathrm{Lip}\,\varphi \leq \mu$ by (2).

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$\mathrm{Im}\, h^\ell := \bigcup_{z\in Z} h^\ell(z) = \{(F_\ell(x,y), y) : (x,y) \in Z\}$ satisfies $\mathrm{diam}\{F_\ell(x,y) : x \in X\} \le 2r\lambda^\ell$; as $\mathrm{Im}\, h^\ell \supset \mathrm{Im}\, h^m$ for $\ell \le m$, this yields $\bigcap \mathrm{Im}\, h^\ell = \{(\psi(y), y) : y \in Y\}$ and $\lim_{\ell\to\infty} F_\ell(x_\ell, y) = \psi(y)$ uniformly with respect to $y \in Y$ and $(x_\ell)_{\ell\in\mathbb{N}}$ in X, hence $\mathrm{Lip}\,\psi \le \lambda$ by (1).

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this holds for non-integer k as well.

Theorem

Let X, Y be metric spaces one of which at least is complete, and let h be a correspondence of $Z = X \times Y$ into itself admitting a Lipschitzian generating map $(F, G) : Z \to Z$ such that

$$\lambda := \mathrm{Lip}\, F \quad \textit{and} \quad \mu := \mathrm{Lip}\, G \quad \textit{satisfy} \quad \lambda\mu < 1.$$

Then, for each $\ell \in \mathbb{N}$ and each $z = (x, y) \in Z$, there is exactly one orbit $(z_0, \ldots, z_\ell)$ of h such that the X-component of z_0 equals x and the Y-component of z_ℓ equals y; setting then $z_i = \big(F_i^\ell(z), G_i^\ell(z)\big)$, $0 \le i \le \ell$, the correspondence h^ℓ clearly admits the generating map (F_ℓ^ℓ, G_0^ℓ) and one has

$$\begin{array}{rcl} (\lambda^i, \lambda\mu^{\ell-i}) & \in & \mathrm{Lip}_2\, F_i^\ell \\ (\mu\lambda^i, \mu^{\ell-i}) & \in & \mathrm{Lip}_2\, G_i^\ell \end{array} \qquad \textit{for} \quad 0 \le i \le \ell$$

$$\begin{array}{rcl} F_i^m(x, y) & = & F_i^\ell\big(x, G_\ell^m(x, y)\big) \\ G_i^m(x, y) & = & G_i^\ell\big(x, G_\ell^m(x, y)\big) \end{array} \qquad \textit{for} \quad 0 \le i \le \ell \le m.$$

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i) *For each $x \in X$, there is precisely one orbit $(z_i)_{i \geq 0}$ of h such that the X-component of z_0 equals x; it is given by $z_i = \lim_{\ell \to \infty} \big(F_i^\ell(x, y_\ell), G_i^\ell(x, y_\ell)\big)$, limit independent of $(y_\ell)_{\ell \geq 0}$; in particular, $z_0 = \Big(x, \lim_{\ell \to \infty} G_0^\ell(x, y_\ell)\Big) =: \big(x, \varphi(x)\big)$.*

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ii) *Thus, the graph W_s of the map $\varphi : X \to Y$ so defined consists of those $z \in Z$ such that there exists an orbit $(z_i)_{i \geq 0}$ of h with $z_0 = z$; it is invariant by h and $\mathrm{Lip}\, \varphi \leq \mu$.*

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iii) *The correspondence $W_s \ni z \mapsto h(z) \cap W_s$ is a* map *$h_s : W_s \to W_s$ and therefore $h_s\big(x, \varphi(x)\big) = \Big(f_s(x), \varphi\big(f_s(x)\big)\Big)$; thus, in* i), *$z_i = h_s^i\big(x, \varphi(x)\big) = \Big(f_s^i(x), \varphi\big(f_s^i(x)\big)\Big)$, hence $f_s(x) = \lim_{\ell\to\infty} F_1^\ell(x, y_\ell)$ and $\operatorname{Lip} h_s = \operatorname{Lip} f_s \leq \lambda$.*

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The second corollary follows from the first, applied to the dual correspondence of h, i. e., the correspondence h^* of $Z^* = Y \times X$ into itself obtained from h^{-1} by exchanging X and Y, which admits the generating map $(F^*, G^*) : (y, x) \mapsto (G, F)(x, y)$.

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Assume the hypotheses of both the previous corollaries satisfied, i.e. Z bounded, complete and λ, μ less than 1. Then:

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Proof. The fixed points of h are those of the strict contraction (F, G). The constant sequence $z_i = p$ is an orbit of both h and h^{-1}, hence $p \in W_s \cap W_u$; now, $W_s \cap W_u$ consists of all $\big(x, \varphi(x)\big)$ with $x = \psi \circ \varphi(x)$ and therefore contains only p since $\psi \circ \varphi$ is a strict contraction of X. $\square$

Absolute normal hyperbolicity

Here, Z is the product $Z_s \times Z_c \times Z_u$ of three complete metric spaces with $\#Z_c \geq 2$, equipped with the product space metric, and h is a correspondence of Z into itself, satisfying both

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ii) *The* z_0 *terms of orbits* $(z_n)_{n\leq 0}$ *of* h *form the* h^{-1}*-invariant "graph"* W_u *of a map* $\psi : Z_c \times Z_u \to Z_s$ *with* $\operatorname{Lip}\psi \leq \lambda_u$*, and* h^{-1} *restricts to a map* h_u^- *of* W_u *into itself, which writes* $h_u^-\big(\psi(\theta, y), \theta, y\big) = \big(\psi \circ g_u(\theta, y), g_u(\theta, y)\big)$ *with* $\operatorname{Lip} g_u \leq \mu_u$*.*

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iii) *The subspace* $W_c := W_s \cap W_u$ *of all* z_0 *terms of orbits* $(z_n)_{n\in\mathbb{Z}}$ *of* h *is invariant by* h_s *and* h_u^-*; the restriction* $h_c : W_c \to W_c$ *of* h_s *is bijective and* h_c^{-1} *is the restriction of* h_u^-.

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Two lemmas

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Basic Lemma

Given metric spaces $U_1, \ldots, U_n, E$, *set* $U := U_1 \times \cdots \times U_n$ *and assume that* $\Phi : U \times E \to E$ *satisfies* $(c_1, \ldots, c_n, c) \in \mathrm{Lip}_{n+1} \Phi$ *for nonnegative real numbers* $c_1, \ldots, c_n, c$. *If* E *is complete and* $c < 1$ *then each* $x \mapsto \Phi(u, x)$ *has a unique fixed point* $B(u)$, *which defines a map* $B : U \to E$ *with* $(c_1, \ldots, c_n) \in \mathrm{Lip}_n B$.

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Proof. Setting $x = B(u)$ and $x' = B(u')$, one has

$$\begin{aligned} d(x, x') &= d\big(\Phi(u, x), \Phi(u', x')\big) \\ &\leq \max\{c_1 d(u_1, u_1'), \ldots, c_n d(u_n, u_n'), c\, d(x, x')\}; \end{aligned}$$

if the maximum on the right-hand side were strictly $c\, d(x, x')$, we would get the absurdity $0 < (1 - c) d(x, x') \leq 0$, hence $d\big(B(u), B(u')\big) \leq \max\{c_1 d(u_1, u_1'), \ldots, c_n d(u_n, u_n')\}$. $\square$

Two lemmas

Composition Lemma. Given metric spaces X_0, Y_0, X_1, Y_1, X_2, Y_2 and, for $m = 1, 2$, a correspondence h_m of $X_{m-1} \times Y_{m-1}$ into $X_m \times Y_m$ having a generating map (F_m, G_{m-1}) with $(\alpha_m, \beta_m) \in \mathrm{Lip}_2\, F_m$ and $(\gamma_{m-1}, \delta_{m-1}) \in \mathrm{Lip}_2\, G_{m-1}$, assume Y_1 or X_1 complete and $\beta_1\gamma_1 < 1$. Then:

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i) $h_2 \circ h_1$ has a generating map $(\bar{F}, \bar{G}) : X_0 \times Y_2 \to X_2 \times Y_0$ with $(\alpha_2\alpha_1, \max\{\alpha_2\beta_1\delta_1, \beta_2\}) \in \mathrm{Lip}_2\, \bar{F}$ and $(\max\{\gamma_0, \delta_0\gamma_1\alpha_1\}, \delta_0\delta_1) \in \mathrm{Lip}_2\, \bar{G}$.

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ii) For all $(x_0, y_2) \in X_0 \times Y_2$ there is one orbit (z_0, z_1, z_2) of (h_1, h_2) with $(x_0, y_2) = (\mathrm{pr}_1 z_0, \mathrm{pr}_2 z_2)$; setting $z_1 =: \big(A(x_0, y_2), B(x_0, y_2)\big)$, one has $(\gamma_1\alpha_1, \delta_1) \in \mathrm{Lip}_2\, B$ and $(\alpha_1, \beta_1\delta_1) \in \mathrm{Lip}_2\, A$.

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Proof. $\big((x_0, y_0), (x_1, y_1), (x_2, y_2)\big)$ is an orbit of (h_1, h_2) iff $x_1 = F_1(x_0, y_1)$, $y_0 = G_0(x_0, y_1)$, $x_2 = F_2(x_1, y_2)$, $y_1 = G_1(x_1, y_2)$; replace the last equation by $y_1 = G_1\big(F_1(x_0, y_1), y_2\big) =: \Phi(x_0, y_2, y_1)$; as $\mathrm{Lip}_3\, \Phi \ni (\gamma_1\alpha_1, \delta_1, \gamma_1\beta_1)$, $\gamma_1\beta_1 < 1$, this reads $y_1 = B(x_0, y_2)$ with $(\gamma_1\alpha_1, \delta_1) \in \mathrm{Lip}_2\, B$ when Y_1 is complete. The rest follows. $\square$

Proof of the theorem

$(\lambda^i, \lambda\mu^{\ell-i}) \in \mathrm{Lip}_2\, F_i^\ell$ is obvious if $i = 0$ since $F_0^\ell(x, y) = x$.
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- Otherwise, assume the theorem proved for all smaller values of ℓ and apply the composition lemma to $h_1 = h^i$ and $h_2 = h^{\ell-i}$ for $0 < i < \ell$.
- Finally, the relations $F_i^m(x, y) = F_i^\ell\big(x, G_\ell^m(x, y)\big)$ and $G_i^m(x, y) = G_i^\ell\big(x, G_\ell^m(x, y)\big)$ reflect the fact that $(z_i)_{0 \le i \le m} := \big(F_i^m(x, y), G_i^m(x, y)\big)_{0 \le i \le m}$ is the sole orbit such that the X-component of z_0 equals x and the Y-component of z_m equals y, and $(z_i)_{0 \le i \le \ell}$ is the sole orbit such that the X-component of z_0 equals x and the Y-component of z_ℓ equals $G_\ell^m(x, y)$.