Motives with modulus

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What are motives?

Grothendieck's idea: universal cohomology theory for algebraic varieties, made out of algebraic cycles.

- **Pure motives:** for smooth projective varieties.
- **Mixed motives:** for all varieties.
- **More generally:** why not replace base field by base scheme… (not covered in this talk).

1. Review: pure motives (Grothendieck)

Smooth projective varieties

Weil cohomology theory: cohomology theory on smooth projective k-varieties (k a field) satisfying certain axioms: Künneth formula, Poincaré duality...

Classical examples: l-adic cohomology for $l \neq \operatorname{char} k$ (coefficients $\mathbb{Q}_l$), Betti cohomology (coefficients $\mathbb{Q}$), de Rham cohomology (coefficients k) in characteristic 0, crystalline cohomology (coefficients quotient field of Witt vectors on k) in characteristic > 0.

Grothendieck: wants universal Weil cohomology with values in suitable abelian category!

Algebraic cycles

X variety over field k (or more generally, any scheme): algebraic cycle over X = linear combination of closed irreducible subsets Z_α of X (with coefficients in a ring R)

Cycle of dimension (codimension) i: all Z_α are of dimension (codimension) i. Notation: $Z_i(X, R)$, $Z^i(X, R)$.
To intersect cycles, need in general to mod out by some "adequate" equivalence relation $\sim$ (notation $A_i^\sim(X, R)$, $A^i_\sim(X, R)$):

- **Rational equivalence:** parametrize cycles by lines ($\mathbf{P}^1$ or $\mathbf{A}^1$).
- **Algebraic equivalence:** parametrize cycles by smooth curves or smooth algebraic varieties.
- **Homological equivalence:** take image of cycle map with value in some cohomology theory.
- **Numerical equivalence (on smooth proper varieties):** mod out by kernel of intersection pairing.

Pure motives

$\sim$ adequate equivalence relation on algebraic cycles, R commutative ring of coefficients:

$\mathbf{Corr}_\sim(k,R)$ category of algebraic correspondences:

Objects: smooth projective varieties.

Morphisms: $\mathrm{Hom}(X,Y) = A^\sim_{\dim X}(X \times Y, R)$.

Composition of correspondences: $X, Y, Z \in \mathbf{Corr}_\sim(k,R)$, $\alpha \in \mathrm{Hom}(X,Y)$, $\beta \in \mathrm{Hom}(Y,Z)$:

$$\beta \circ \alpha = p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\beta)$$

$$\begin{array}{ccc} & X \times Y \times Z & \\ {\scriptstyle p_{12}}\swarrow & \downarrow{\scriptstyle p_{13}} & \searrow{\scriptstyle p_{23}} \\ X \times Y & X \times Z & Y \times Z \\ \\ \alpha & \beta \circ \alpha & \beta \end{array}$$

Graph functor: $\gamma : \mathbf{Sm}^{\mathrm{proj}}(k) \to \mathbf{Corr}_\sim(k,R)$

$$X \mapsto X, \quad (f : X \to Y) \mapsto \Gamma_f \in A^\sim_{\dim X}(X \times Y, R) \quad \text{(graph of } f\text{)}$$

Get from correspondences to motives by string of functors

$$\begin{array}{rcl} \mathbf{Sm}^{\mathrm{proj}}(k) \to \mathbf{Corr}_{\sim}(k,R) & \xrightarrow{\natural} \mathcal{M}^{\mathrm{eff}}_{\sim}(k,R) & \xrightarrow{\mathbb{L}^{-1}} \mathcal{M}_{\sim}(k,R) \\ X \mapsto X & \mapsto h(X) & \mapsto h(X) \\ f \mapsto \Gamma_f & & \end{array}$$

$\natural$: adjoin kernels to idempotents (called *Karoubian envelope* or *idempotent completion*). $\natural$ and $\mathbb{L}^{-1}$ fully faithful.

To pass from $\mathcal{M}^{\mathrm{eff}}_{\sim}(k,R)$ (effective motives) to $\mathcal{M}_{\sim}(k,R)$ (all motives), *invert the Lefschetz motive*:

$\otimes$-structures on $\mathbf{Sm}^{\mathrm{proj}}(k), \mathbf{Corr}_{\sim}(k,R), \mathcal{M}^{\mathrm{eff}}_{\sim}(k,R)$ induced by $(X,Y) \mapsto X \times Y$. Unit object in $\mathcal{M}^{\mathrm{eff}}_{\sim}(k,R)$: $\mathbf{1} = h(\mathrm{Spec}\, k)$. Then $h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbb{L}$, $\mathbb{L}$ the Lefschetz motive: quasi-invertible ($M \mapsto M \otimes \mathbb{L}$ fully faithful).

$$\mathcal{M}_{\sim}(k,R) = \mathcal{M}^{\mathrm{eff}}_{\sim}(k,R)[\mathbb{L}^{-1}].$$

Basic results on pure motives

Theorem 1 (easy). *$\mathcal{M}_\sim(k, A)$ rigid $\otimes$-category: every object has a dual and every object is isomorphic to its double dual.*

Dual of $h(X)$: $h(X) \otimes \mathbb{L}^{-\dim X}$.

Theorem 2 (Jannsen, 1991). *$\mathcal{M}_{\text{num}}(k, \mathbb{Q})$ is abelian semi-simple.*

$\mathcal{M}_{\text{num}}(k, \mathbb{Q})$ Grothendieck's candidate for receptacle of a universal cohomology theory.

The problem

H Weil cohomology theory with coefficients in K:

$$\begin{array}{ccc} \mathcal{M}_H(k,\mathbb{Q}) & \xrightarrow{H^*} & \mathrm{Vec}_K^* \\ \downarrow & & \\ \mathcal{M}_{\mathrm{num}}(k,\mathbb{Q}) & & \end{array}$$

Vec_K^* finite-dimensional graded K-vector spaces; horizontal functor faithful, vertical functor full. If want H^* to factor through $\mathcal{M}_{\mathrm{num}}(k,\mathbb{Q})$, need vertical functor to be an equivalence of categories:

$$\text{Homological equivalence} = \text{numerical equivalence}$$

The main standard conjecture of Grothendieck: still open after almost 50 years!

(Then grander vision: motivic Galois group…)

2. Mixed motives?

Grothendieck: no construction but a vision: there should be an abelian rigid $\otimes$-category $\mathcal{MM}(k,\mathbb{Q})$ of "mixed motives" such that (at least)

- The socle (semi-simple part) of $\mathcal{MM}(k,\mathbb{Q})$ is $\mathcal{M}_{\mathrm{num}}(k,\mathbb{Q})$; every object is of finite length.
- Any k-variety X has cohomology objects $h^i(X) \in \mathcal{MM}(k,\mathbb{Q})$ and cohomology objects with compact supports $h^i_c(X) \in \mathcal{MM}(k,\mathbb{Q})$, which are "universal" for suitable cohomology theories.

No construction of $\mathcal{MM}(k, \mathbb{Q})$ yet (apart from 1-motives), but two ideas to get towards it:

(1) (Deligne, Jannsen, André, Nori; Bloch-Kriz): add a few homomorphisms which are not (known to be) algebraic.
(2) (suggested by Deligne and Beilinson; Hanamura, Levine, Voevodsky): might be easier to construct a *triangulated category* out of algebraic cycles, and to look for a "motivic t-structure" with heart $\mathcal{MM}(k, \mathbb{Q})$.

3. What is K_2?

Steinberg: k field,

$$K_2(k) = k^* \otimes_{\mathbb{Z}} k^* / \langle x \otimes (1 - x) \mid x \neq 0, 1 \rangle.$$

Conceptual definition?

First answer: **Tate** (1970es) for K_2/n:

$$K_2/n = (\mathbf{G}_m \otimes \mathbf{G}_m)/n + \text{ transfers } + \text{ projection formula.}$$

How about K_2 itself?
Two answers: Suslin, Kato (1980es).

Suslin:

$K_2 = \mathbf{G}_m \otimes \mathbf{G}_m +$ transfers + projection formula + *homotopy invariance*.

$\longrightarrow$ Suslin-Voevodsky's motivic cohomology, Voevodsky's homotopy invariant motives.

Kato:

$$K_2 = \mathbf{G}_m \otimes \mathbf{G}_m + \text{ transfers } + \text{ projection formula } + \textit{ Weil reciprocity}.$$

$\longrightarrow$ reciprocity sheaves and motives with modulus.

4. Review: Voevodsky's triangulated categories of motives (over a field)

k base field

Goal: two $\otimes$-triangulated categories $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \hookrightarrow \mathbf{DM}^{\mathrm{eff}}$

$X \in \mathbf{Sm}(k) \mapsto M(X) \in \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (covariant) with

Mayer-Vietoris: $X = U \cup V$ open cover $\mapsto$ exact triangle

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \xrightarrow{+1}$$

Homotopy invariance: $M(X \times \mathbf{A}^1) \xrightarrow{\sim} M(X)$.

$\mathbf{DM}^{\mathrm{eff}}$ "large" category allowing to compute Hom groups via Nisnevich hypercohomology.

Construction:

Cor additive $\otimes$-category:

Objects: Smooth varieties

Morphisms: finite correspondences

$$\mathbf{Cor}(X,Y) = \mathbb{Z}[Z \subset X \times Y \mid Z \text{ integral}, \\ Z \to X \text{ finite and surjective over a component of } X].$$

Graph functor $\mathbf{Sm} \to \mathbf{Cor}$.
$\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ = pseudo-abelian envelope of

$$K^b(\mathbf{Cor})/\langle MV + HI \rangle$$

(MV = Mayer-Vietoris, HI = homotopy invariance as on previous page).

Any $X \in \mathbf{Sm}$ has a motive $M(X) \in \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$; $M(\operatorname{Spec} k) =: \mathbb{Z}$, $M(\mathbf{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$, $\mathbb{Z}(1)$ *Tate (or Lefschetz) object.*

Product of varieties induces $\otimes$-structure on $\mathbf{Sm}, \mathbf{Cor}, \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$: get $\mathbf{DM}_{\mathrm{gm}}$ from $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ by $\otimes$-inverting $\mathbb{Z}(1)$.

Similar to construction of effective motives:

$$
\begin{array}{ccccccccc}
\mathbf{Sm}^{\mathrm{proj}} & \longrightarrow & \mathbf{Corr}_{\mathrm{rat}} & & & \xrightarrow{\natural} & \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}} & \xrightarrow{\mathbb{L}^{-1}} & \mathcal{M}_{\mathrm{rat}} \\
\downarrow & & & & & & \downarrow & & \downarrow \\
\mathbf{Sm} & \longrightarrow & \mathbf{Cor} & \longrightarrow & \dfrac{K^b(\mathbf{Cor})}{\langle MV + HI \rangle} & \xrightarrow{\natural} & \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}} & \xrightarrow{\mathbb{Z}(1)^{-1}} & \mathbf{DM}_{\mathrm{gm}}
\end{array}
$$

($\mathbb{Z}(1)^{-1}$ and vertical functors fully faithful if k perfect!)

To define $\mathbf{DM}^{\text{eff}}$:
$\mathbf{PST} = \text{Mod}-\mathbf{Cor} = \{\text{additive contravariant functors } \mathbf{Cor} \to \mathbf{Ab}\}$
$\mathbf{NST} = \{F \in \mathbf{PST} \mid F \text{ is a Nisnevich sheaf}\}$

$\mathbf{Cor} \ni X \mapsto \mathbb{Z}_{\text{tr}}(X) \in \mathbf{PST}$ the presheaf with transfers represented by X: it belongs to $\mathbf{NST}$.

$$\mathbf{DM}^{\text{eff}} = D(\mathbf{NST})/\langle HI \rangle.$$

Natural functor $\mathbf{DM}^{\text{eff}}_{\text{gm}} \to \mathbf{DM}^{\text{eff}}$, fully faithful if k perfect (non-trivial theorem!)

To avoid perfectness hypothesis: strengthen Mayer-Vietoris to "Nisnevich Mayer-Vietoris" (using elementary distinguished squares) in definition of $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$.

$\mapsto$ variant of definition of $\mathbf{DM}^{\mathrm{eff}}$:

$$\mathbf{DM}^{\mathrm{eff}} = D(\mathbf{PST})/\langle MV_{\mathrm{Nis}} + HI \rangle$$

Completely parallel to $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$!

$$\begin{array}{ccccc}
\mathbf{Corr}_{\mathrm{rat}} & & \xrightarrow{\natural} & & \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}} \\
 & & & & \downarrow \\
\mathbf{Cor} & \longrightarrow & \dfrac{K^b(\mathbf{Cor})}{\langle MV + HI \rangle} \xrightarrow{\natural} & & \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}} \\
\downarrow & & & & \downarrow \\
\mathbf{PST} & & \longrightarrow & & \dfrac{D(\mathbf{PST})}{\langle MV + HI \rangle} = \mathbf{DM}^{\mathrm{eff}}
\end{array}$$

$$\mathbf{HI} = \{F \in \mathbf{NST} \mid F(X) \xrightarrow{\sim} F(X \times \mathbf{A}^1)\ \forall X \in \mathbf{Sm}\}.$$

If k perfect: $\mathbf{DM}^{\text{eff}}$ has a t-structure with heart $\mathbf{HI}$, the *homotopy t-structure* (here, not known how to avoid perfectness hypothesis).

This is not the searched-for motivic t-structure! But very useful nevertheless.

5. Rosenlicht's theorems

(See Serre's Groupes algébriques et corps de classes.)

C smooth projective curve over $k = \bar{k}$, $U \subset C$ affine open subset.

Theorem 3. *$f : U \to G$ k-morphism with G commutative algebraic group; $\exists$ effective divisor $\mathfrak{m}$ with support $C - U$ such that*

$$f(\mathrm{div}(g)) = 0 \textit{ if } g \in k(C)^*, g \equiv 1 \pmod{\mathfrak{m}}.$$

Here, extended f to homomorphism $Z_0(U) \to G(k)$ by linearity; hypothesis on $g \Rightarrow$ support of $\mathrm{div}(g) \subset U$.

Theorem 4. *Given* $\mathfrak{m}$ *and* $u_0 \in U$*, the functor*

$$G \mapsto \{f : U \to G \mid f(u_0) = 0 \textit{ and } f \textit{ has modulus } \mathfrak{m}\}$$

from commutative algebraic groups to abelian groups is corepresentable by the generalized Jacobian $J(C, \mathfrak{m})$.

If $\mathfrak{m}$ reduced, get connected component of *relative Picard group*

$$J(C, \mathfrak{m}) = \mathrm{Pic}^0(C, C - U).$$

In general, extension of this by unipotent group.

6. Reciprocity sheaves

A reciprocity sheaf is a **NST** satisfying a reciprocity condition inspired by Rosenlicht's modulus condition. (Definition skipped!)

Examples 5.

- HI sheaves have reciprocity.
- G commutative algebraic group: the sheaf represented by G has reciprocity (e.g. $G = \mathbf{G}_a$).
- The modulus condition is "representable":

Definition 6. A *modulus pair* is a pair $M = (\overline{M}, \overline{M}^\infty)$ with

(i) $\overline{M}^\infty \subset \overline{M}$ the closed immersion of an effective Cartier divisor;

(ii) $M^o := \overline{M} - \overline{M}^\infty$ is smooth.

Theorem 7 (K-S-Y, 2014)**.** *M modulus pair with $\overline{M}$ proper and M^o quasi-affine. There exists a quotient $h(M)$ of $\mathbb{Z}_{\mathrm{tr}}(M^o)$ which represents the functor*

$$\mathbf{PST} \ni F \mapsto \{\alpha \in F(M^o) \mid \alpha \textit{ has modulus } M\}.$$

Moreover, $h(M)$ has reciprocity and

$$h(M)(k) = CH_0(M)$$

$CH_0(M)$ Kerz-Saito group of 0-cycles with modulus.

$\mathbf{Rec} \subset \mathbf{PST}$ full subcategory of reciprocity PST:

- closed under subobjects and quotients (in particular abelian)
- not clearly closed under extensions
- inclusion functor does not have a left adjoint (it has a right adjoint)

So-so category…

Idea: take modulus pairs seriously, try and make a triangulated category out of them.

7. Motives with modulus

7.1. Categories of modulus pairs.

Definition 8. $\underline{\mathbf{M}}\mathbf{Cor}$:

Objects: Modulus pairs M.

Morphisms:

$$\underline{\mathbf{M}}\mathbf{Cor}(M,N) = \langle Z \in \mathbf{Cor}(M^{\mathrm{o}}, N^{\mathrm{o}}) \mid Z \text{ integral}, p^*M^\infty \geq q^*N^\infty,$$
$$p, q \text{ projections } \overline{Z}^N \to \overline{Z} \to \overline{M}, \overline{Z}^N \to \overline{Z} \to \overline{N}; \overline{Z} \to \overline{M} \text{ proper}\rangle$$

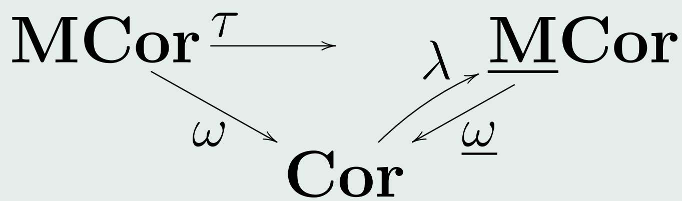
$\overline{Z}$ closure of Z in $\overline{M} \times \overline{N}$, $\overline{Z}^N$ normalisation of $\overline{Z}$. Properness on $\overline{M}$ necessary for composition!

$\mathbf{MCor}$: full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ where $\overline{M}$ is proper (proper modulus pairs).

Tensor structure on $\underline{\mathbf{M}}\mathbf{Cor}$ and $\mathbf{MCor}$:

$$(\overline{M}, M^\infty) \otimes (\overline{N}, N^\infty) = (\overline{M} \times \overline{N}, M^\infty \times \overline{N} + \overline{M} \times N^\infty).$$

Diagram of $\otimes$-additive categories and $\otimes$-functors:

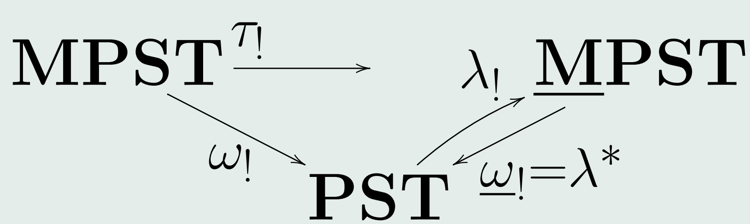


$\tau(M) = M$, $\underline{\omega}(M) = M^{\mathrm{o}}$, $\omega = \underline{\omega} \circ \tau$, $\lambda(X) = (X, \emptyset)$ (λ is left adjoint to $\underline{\omega}$).

Categories of presheaves

$$\mathbf{MPST} = \mathrm{Mod}-\mathbf{MCor}, \quad \underline{\mathbf{M}}\mathbf{PST} = \mathrm{Mod}-\underline{\mathbf{M}}\mathbf{Cor}\,.$$

Diagram of $\otimes$-functors



$?_!$ left adjoint to $?^*$.

Theorem 9. *ω and τ have pro-left adjoints. In particular, $\omega_!$ and $\tau_!$ are exact.*

7.2. Topologies. To make sense of topologies on modulus pairs, need to use $\underline{\mathbf{M}}\mathbf{Cor}$ ($\mathbf{MCor}$ not sufficient), plus some subtleties (skipped). Get category of σ-sheaves $\underline{\mathbf{M}}\mathbf{PST}_\sigma$, $\sigma \in \{\mathrm{Zar}, \mathrm{Nis}, \text{ét}\}$ and pair of *exact* adjoint functors:

$$\begin{array}{c} \underline{\mathbf{M}}\mathbf{PST}_\sigma \\ \underline{\omega}_\sigma \downarrow \uparrow \underline{\omega}^\sigma \\ \mathbf{PST}_\sigma \end{array}$$

Will use mainly $\sigma = \mathrm{Nis}$, $\underline{\mathbf{M}}\mathbf{NST} := \underline{\mathbf{M}}\mathbf{PST}_{\mathrm{Nis}}$.

7.3. Voevodsky's abstract homotopy theory. $(\mathcal{C}, \otimes)$ $\otimes$-category: an *interval* in $\mathcal{C}$ is a quintuple (I, i_0, i_1, p, μ):

- $I \in \mathcal{C}$
- $i_0, i_1 : \mathbf{1} \to I$, $p : I \to \mathbf{1}$ ($\mathbf{1}$ unit object)
- $\mu : I \otimes I \to I$

with conditions

- $p \circ i_0 = p \circ i_1 = 1_{\mathbf{1}}$
- $\mu \circ (1_I \otimes i_0) = i_0 \circ p$, $\mu \circ (1_I \otimes i_1) = 1_I$.

$(\mathcal{C}, \otimes, I)$ $\otimes$-category with interval: a presheaf F on $\mathcal{C}$ is I*-invariant* if $F(X) \xrightarrow{\sim} F(X \otimes I)$ for any $X \in \mathcal{C}$ (via the morphism $1_X \otimes p$).

Main example: $\mathcal{C} = \mathbf{Sm}$, $I = \mathbf{A}^1$, $i_t : \operatorname{Spec} k \to \mathbf{A}^1$ inclusion of point t, $\mu : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ multiplication map. Get $\mathbf{A}^1$-*invariance* (= homotopy invariance).

7.4. The affine line with modulus.

Proposition 10. *The object* $\overline{\square} = (\mathbf{P}^1, \infty)$ *has the structure of an interval, given by the interval structure on* $\overline{\square}^{\mathrm{o}} = \mathbf{A}^1$.

In fact, more convenient to "put 1 at ∞", i.e. redefine $\overline{\square}$ as $(\mathbf{P}^1, 1)$.

Theorem 11. *If* $F \in \mathbf{MPST}$ *is* $\overline{\square}$*-invariant,* $\omega_! F$ *has reciprocity.*

Consequence: $\otimes$-structure on **Rec** (cf. Ivorra-Rülling: *the* K*-groups of reciprocity functors*).

Definition 12.

$$\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} = (K^b(\underline{\mathbf{M}}\mathbf{Cor})/\langle MV_{\mathrm{Nis}} + CI\rangle)^\natural.$$

$$\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} = D(\underline{\mathbf{M}}\mathbf{PST})/\langle MV_{\mathrm{Nis}} + CI\rangle = D(\underline{\mathbf{M}}\mathbf{NST})/\langle CI\rangle.$$

CI: $\overline{\square}$-invariance.

Naturally commutative diagram

$$\begin{array}{ccccc}
\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} & \xrightarrow{M\Phi} & \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} & \xrightarrow{M\iota} & \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} \\
\| & & \downarrow{\scriptstyle \underline{\omega}_{\mathrm{eff}}} & & {\scriptstyle \underline{\omega}_{\mathrm{eff}}}\downarrow \uparrow{\scriptstyle \underline{\omega}^{\mathrm{eff}}} \\
\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} & \xrightarrow{\Phi} & \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} & \xrightarrow{\iota} & \mathbf{DM}^{\mathrm{eff}}
\end{array}$$

$\iota, M\iota$ $\otimes$ and fully faithful, $\underline{\omega}_{\mathrm{eff}}$ $\otimes$ and localisations, $\underline{\omega}^{\mathrm{eff}}$ right adjoint to $\underline{\omega}_{\mathrm{eff}}$ (hence fully faithful), Φ fully faithful if k perfect (Voevodsky).

Main theorem. *X smooth proper: $\underline{\omega}^{\mathrm{eff}} M(X) = M(X, \emptyset)$.*

Corollary 13. *a) p exponential characteristic of k:* $\omega^{\mathrm{eff}}(\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]) \subset \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$.
b) k perfect: $M\Phi$ *fully faithful.*
c) k perfect, X smooth proper, $\mathcal{Y} = (\overline{\mathcal{Y}}, \mathcal{Y}^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$, $j \in \mathbb{Z}$*: canonical isomorphism*

$$\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(M(\mathcal{Y}), M(X,\emptyset)[j]) \simeq H^{2d+j}((\overline{\mathcal{Y}} - \mathcal{Y}^\infty) \times X, \mathbb{Z}(d))$$

right hand side = Voevodsky's motivic cohomology. In particular, vanishes for $j > 0$.

Next steps in the programme (not exhaustive):

- (In progress:) get homotopy t-structure on $\mathbf{MDM}^{\mathrm{eff}} \subset \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ when k perfect, by extending Voevodsky's theorems on homotopy invariant presheaves with transfers.
- Construct realisation functors for interesting non homotopy invariant cohomology theories.