Commutative algebraic groups up to isogeny

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Abstract

Consider the abelian category $\mathcal{C}_k$ of commutative algebraic groups over a field k. By results of Serre and Oort, $\mathcal{C}_k$ has homological dimension 1 (resp. 2) if k is algebraically closed of characteristic 0 (resp. positive). In this article, we explore the abelian category of commutative algebraic groups up to isogeny, defined as the quotient of $\mathcal{C}_k$ by the full subcategory $\mathcal{F}_k$ of finite k-group schemes. We show that $\mathcal{C}_k/\mathcal{F}_k$ has homological dimension 1, and we determine its projective objects. We also obtain structure results for $\mathcal{C}_k/\mathcal{F}_k$, which take a simpler form in positive characteristics.

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1 Introduction

There has been much recent progress on the structure of algebraic groups over an arbitrary field, with the classification of pseudo-reductive groups (see [CGP15, CP15]). Yet commutative algebraic groups over an imperfect field remain somewhat mysterious, e.g., extensions with unipotent quotients are largely unknown; see [To13] for interesting results, examples, and conjectures.

In this article, we develop a categorical approach to commutative algebraic groups up to isogeny, which bypasses the problems raised by imperfect fields, and yields rather simple and uniform results.

More specifically, denote by $\mathcal{C}_k$ the category with objects the group schemes of finite type over the ground field k, and with morphisms, the homomorphisms of k-group schemes (all group schemes under consideration will be assumed commutative). By a result of Grothendieck (see [SGA3, VIA, Thm. 5.4.2]), $\mathcal{C}_k$ is an abelian category. We define the category of 'algebraic groups up to isogeny' as the quotient category of $\mathcal{C}_k$ by the Serre subcategory of finite group schemes; then $\mathcal{C}_k/\mathcal{F}_k$ is obtained from $\mathcal{C}_k$ by inverting all isogenies, i.e., all morphisms with finite kernel and cokernel.

It will be easier to deal with the full subcategory $\underline{\mathcal{C}}_k$ of $\mathcal{C}_k/\mathcal{F}_k$ with objects the smooth connected algebraic groups, since these categories turn out to be equivalent, and morphisms in $\underline{\mathcal{C}}_k$ admit a simpler description.

As a motivation for considering the 'isogeny category' $\underline{\mathcal{C}}_k$, note that some natural constructions involving algebraic groups are only exact up to isogeny; for example, the formations of the maximal torus or of the largest abelian variety quotient, both of which are not exact in $\mathcal{C}_k$. Also, some structure theorems for algebraic groups take a simpler form when working up to isogeny. A classical example is the Poincaré complete reducibility theorem, which is equivalent to the semi-simplicity of the isogeny category of abelian varieties, i.e., the full subcategory $\underline{\mathcal{A}}_k$ of $\underline{\mathcal{C}}_k$ with objects abelian varieties. Likewise, the isogeny category of tori, $\underline{\mathcal{T}}_k$, is semi-simple.

We gather our main results in the following:

Theorem. (i) *The category $\underline{\mathcal{C}}_k$ is artinian and noetherian. Its non-zero simple objects are exactly the additive group $\mathbb{G}_{a,k}$, the simple tori, and the simple abelian varieties.*

(ii) *The product functor $\underline{\mathcal{T}}_k \times \underline{\mathcal{U}}_k \to \underline{\mathcal{L}}_k$ yields an equivalence of categories, where $\underline{\mathcal{U}}_k$ (resp. $\underline{\mathcal{L}}_k$) denotes the isogeny category of unipotent (resp. linear) algebraic groups.*

(iii) *If* $\mathrm{char}(k) > 0$, *then the product functor $\underline{\mathcal{S}}_k \times \underline{\mathcal{U}}_k \to \underline{\mathcal{C}}_k$ yields an equivalence of categories, where $\underline{\mathcal{S}}_k$ denotes the isogeny category of semi-abelian varieties. If in addition k is locally finite, then the product functor $\underline{\mathcal{T}}_k \times \underline{\mathcal{A}}_k \to \underline{\mathcal{S}}_k$ yields an equivalence of categories as well.*

(iv) *The base change under any purely inseparable field extension k' of k yields an equivalence of categories $\underline{\mathcal{C}}_k \to \underline{\mathcal{C}}_{k'}$.*

(v) *The homological dimension of $\underline{\mathcal{C}}_k$ is* 1.

We also describe the projective objects in $\underline{\mathcal{C}}_k$ (Theorem 5.13), and obtain a structure result for $\underline{\mathcal{C}}_k$ in characteristic 0 (Proposition 5.12), which turns out to be more technical than in positive characteristics.

Let us now compare the above statements with known results on $\mathcal{C}_k$ and its full subcategories $\mathcal{A}_k$ (resp. $\mathcal{T}_k$, $\mathcal{U}_k$, $\mathcal{L}_k$, $\mathcal{S}_k$) of abelian varieties (resp. tori, unipotent groups, linear groups, semi-abelian varieties).

About (i): $\mathcal{C}_k$ is artinian and not noetherian. Also, every algebraic group is an iterated extension of 'elementary' groups; these are the simple objects of $\underline{\mathcal{C}}_k$ and the simple finite group schemes.

About (ii): the product functor $\mathcal{T}_k \times \mathcal{U}_k \to \mathcal{L}_k$ yields an equivalence of categories if k is perfect. But over an imperfect field, there exist non-zero extensions of unipotent groups by tori, which are only partially understood (see [To13] again).

About (iii): the first assertion follows from recent structure results for algebraic groups (see [Br15b, §5]), together with a lifting property for extensions of such groups with finite quotients (see [Br15a, LA15]). The second assertion is a direct consequence of the Weil-Barsotti formula (see e.g. [Oo66, §III.18]).

About (iv): this is a weak version of a result of Chow on abelian varieties, which asserts (in categorical language) that base change yields a fully faithful functor $\mathcal{A}_k \to \mathcal{A}_{k'}$ for any primary field extension k' of k (see [Ch55], and [Co06, §3] for a modern proof).

About (v), the main result of this article: recall that the homological dimension of an abelian category $\mathcal{D}$ is the smallest integer, $\mathrm{hd}(\mathcal{D})$, such that $\mathrm{Ext}^n_{\mathcal{D}}(A, B) = 0$ for all objects A, B of $\mathcal{D}$ and all $n > \mathrm{hd}(\mathcal{D})$; these Ext groups are defined via Yoneda extensions. In particular, $\mathrm{hd}(\mathcal{D}) = 0$ if and only if $\mathcal{D}$ is semi-simple.

It follows from work of Serre (see [Se60, 10.1 Thm. 1] together with [Oo66, §I.4]) that $\mathrm{hd}(\mathcal{C}_k) = 1$ if k is algebraically closed of characteristic 0. Also, by a result of Oort (see [Oo66, Thm. 14.1]), $\mathrm{hd}(\mathcal{C}_k) = 2$ if k is algebraically closed of positive characteristic. Building on these results, Milne determined $\mathrm{hd}(\mathcal{C}_k)$ when k is perfect (see [Mi70, Thm. 1]); then the homological dimension takes arbitrary large values. In the approach of Serre and Oort, the desired vanishing of higher extension groups is obtained by constructing projective resolutions of all elementary groups, in the category of pro-algebraic groups. The latter category contains $\mathcal{C}_k$ as a full subcategory, and has enough projectives.

In contrast, to show that $\mathrm{hd}(\underline{\mathcal{C}}_k) = 1$ over an arbitrary field k, we do not need to go to a larger category. We rather observe that tori are projective objects in $\underline{\mathcal{C}}_k$, and abelian varieties are injective objects there. This yields the vanishing of all but three extension groups between simple objects of $\underline{\mathcal{C}}_k$; two of the three remaining cases are handled directly, and the third one reduces to the known vanishing of $\mathrm{Ext}^2_{\mathcal{C}_k}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ when k is perfect.

Abelian categories of homological dimension 1 are called hereditary. The most studied hereditary categories consist either of finite-dimensional modules over a finite-dimensional hereditary algebra, or of coherent sheaves on a weighted projective line (see e.g. [Ha01]). Such categories are k-linear and Hom-finite, i.e., all spaces of morphisms are finite-dimensional vector spaces over the ground field k. But this seldom holds for the above isogeny categories. More specifically, $\underline{\mathcal{A}}_k$ and $\underline{\mathcal{T}}_k$ are both $\mathbb{Q}$-linear and Hom-finite, but not $\underline{\mathcal{C}}_k$ unless k is a number field. In fact, $\underline{\mathcal{C}}_k$ may be viewed as a mixture of k-linear and $\mathbb{Q}$-linear categories, when k has characteristic 0. This is already displayed by the full subcategory $\underline{\mathcal{V}}_k$ with objects the vector extensions of abelian varieties: as shown in §5.1,

$\underline{\mathcal{V}}_k$ has enough projectives, and these are either the unipotent groups (k-linear objects), or the vector extensions of simple abelian varieties ($\mathbb{Q}$-linear objects).

When k has characteristic $p > 0$, one may also consider the quotient category of $\mathcal{C}_k$ by the Serre subcategory $\mathcal{I}_k$ of infinitesimal group schemes. This yields the abelian category of 'algebraic groups up to purely inseparable isogeny', which is equivalent to that introduced by Serre in [Se60]; as a consequence, it has homological dimension 1 if k is algebraically closed. For any arbitrary field k, the category $\mathcal{C}_k/\mathcal{I}_k$ turns out to be unchanged by purely inseparable field extensions; its homological properties may be worth investigating.

Notation and conventions. We will use the book [DG70] as a general reference, especially for affine algebraic groups, and the expository text [Br15b] for some further results.

Throughout this text, we fix a ground field k and an algebraic closure $\bar{k}$; the characteristic of k is denoted by $\mathrm{char}(k)$. We denote by k_s the separable closure of k in $\bar{k}$, and by Γ_k the Galois group of k_s over k. We say that k is *locally finite*, if it is algebraic over $\mathbb{F}_p$ for some prime p; equivalently, k is either finite or the algebraic closure of a finite field.

By an *algebraic k-group*, we mean a commutative group scheme G of finite type over k; we denote by G^0 the neutral component of G. The group law of G will be denoted additively: $(x, y) \mapsto x + y$.

By a *k-subgroup* of G, we mean a closed k-subgroup scheme. *Morphisms* are understood to be homomorphisms of k-group schemes. The (scheme-theoretic) image of a morphism $f : G \to H$ will be denoted by $\mathrm{Im}(f)$ or $f(G)$, and the (scheme-theoretic) pull-back of a k-subgroup $H' \subset H$, by $G \times_H H'$ or $f^{-1}(H')$.

Recall that a k-group scheme G is an affine algebraic k-group if and only if G is isomorphic to a k-subgroup of the general linear group $\mathrm{GL}_{n,k}$ for some n. We will thus call affine algebraic k-groups *linear.* Also, we say that an algebraic k-group G is *of multiplicative type* if G is isomorphic to a k-subgroup of some k-torus.

To simplify the notation, we will suppress the mention of the ground field k whenever this yields no confusion. For example, the category $\mathcal{C}_k$ will be denoted by $\mathcal{C}$, except when we use base change by a field extension.

Given an algebraic group G and two subgroups G_1, G_2, we denote by $G_1 + G_2$ the subgroup of G generated by G_1 and G_2. Thus, $G_1 + G_2$ is the image of the morphism $G_1 \times G_2 \to G$, $(x_1, x_2) \mapsto x_1 + x_2$.

An *isogeny* is a morphism with finite kernel and cokernel. Two algebraic groups G_1, G_2 are *isogenous* if they can be connected by a chain of isogenies.

We say that two subgroups G_1, G_2 of an algebraic group G are *commensurable* if both quotients $G_1/G_1 \cap G_2$ and $G_2/G_1 \cap G_2$ are finite; then G_1 and G_2 are isogenous.

Given an algebraic group G and a non-zero integer n, the multiplication by n yields a morphism $n_G : G \to G$. We denote its kernel by $G[n]$, and call it the *n-torsion subgroup.* We say that G is *divisible* if n_G is an epimorphism for all $n \neq 0$; then n_G is an isogeny for all such n. On the other hand, when $\mathrm{char}(k) = p > 0$, we say that G is a *p-group* if $p_G^n = 0$ for $n \gg 0$. Examples of p-groups include the unipotent groups and the connected finite algebraic groups, also called *infinitesimal.*

2 Structure of algebraic groups

2.1 Preliminary results

We will use repeatedly the following simple observation:

Lemma 2.1. *Let G be a smooth connected algebraic group.*

(i) *If G' is a subgroup of G such that G/G' is finite, then $G' = G$.*

(ii) *Any isogeny $f : H \to G$ is an epimorphism.*

Proof. (i) The quotient G/G' is smooth, connected and finite, hence zero.

(ii) This follows from (i) applied to $\mathrm{Im}(f)$. □

The following lifting result for finite quotients will also be frequently used:

Lemma 2.2. *Let G be an algebraic group, and H a subgroup such that G/H is finite.*

(i) *There exists a finite subgroup $F \subset G$ such that $G = H + F$.*

(ii) *If G/H is infinitesimal (resp. a finite p-group), then F may be chosen infinitesimal (resp. a finite p-group) as well.*

Proof. (i) This is a special case of [Br15a, Thm. 1.1].

(ii) Assume G/H infinitesimal. Then the quotient $G/H + F^0$ is infinitesimal (as a quotient of G/H) and étale (as a quotient of F/F^0), hence zero. Thus, we may replace F with F^0, an infinitesimal subgroup.

Next, assume that G/H is a finite p-group. Denote by $F[p^\infty]$ the largest p-subgroup of F. Then the quotient $G/H + F[p^\infty]$ is a finite p-group and is killed by the order of $F/F[p^\infty]$. Since the latter order is prime to p, $G/H + F[p^\infty]$ must be zero. Thus, we may replace F with $F[p^\infty]$. □

Next, we recall a version of a theorem of Chevalley:

Theorem 2.3. (i) *Every algebraic group G contains a linear subgroup L such that G/L is an abelian variety. Moreover, L is unique up to commensurability in G, and G/L is unique up to isogeny.*

(ii) *If G is connected, then there exists a smallest such subgroup, $L = L(G)$, and this subgroup is connected.*

(iii) *If in addition G is smooth, then every morphism from G to an abelian variety factors uniquely through the quotient map $G \to G/L(G)$.*

Proof. The assertion (ii) follows from [Ra70, Lem. IX 2.7] (see also [BLR90, 9.2 Thm. 1]).

To prove (i), note that G contains a finite subgroup F such that G/F is connected (as follows from Lemma 2.2). Then we may take for L the pull-back of a linear subgroup of G/F with quotient an abelian variety. If L' is another linear subgroup of G such that G/L' is an abelian variety, then $L + L'$ is linear, as a quotient of $L \times L'$. Moreover, the natural map $q : G/L \to G/L + L'$ is the quotient by $L + L'/L$, a linear subgroup of the abelian variety G/L. It follows that $L + L'/L$ is finite. Thus, q is an isogeny and $L'/L \cap L'$ is finite. Likewise, $q' : G/L' \to G/L + L'$ is an isogeny and $L/L \cap L'$ is finite; this completes the proof of (i).

Finally, the assertion (iii) is a consequence of [Br15b, Thm. 4.3.4]. □

The linear algebraic groups may be described as follows (see [DG70, Thm. IV.3.3.1]):

Theorem 2.4. *Let G be a linear algebraic group. Then G has a largest subgroup of multiplicative type, M; moreover, G/M is unipotent. If k is perfect, then $G = M \times U$, where U denotes the largest unipotent subgroup of G.*

Also, note the following orthogonality relations:

Proposition 2.5. (i) *Let M be a group of multiplicative type, and U a unipotent group. Then* $\mathrm{Hom}_{\mathcal{C}}(M, U) = 0 = \mathrm{Hom}_{\mathcal{C}}(U, M)$.

(ii) *Let L be a linear algebraic group, and A an abelian variety. Then* $\mathrm{Hom}_{\mathcal{C}}(A, L) = 0$, *and every morphism from L to A has finite image. Moreover,* $\mathrm{Hom}_{\mathcal{C}}(L, A)$ *is n-torsion for some positive integer n.*

Proof. (i) This follows from [DG70, IV.2.2.4].

(ii) The image of a morphism from A to L is proper, smooth, connected and affine, hence zero. Likewise, the image of a morphism from L to A is affine and proper, hence finite.

To show the final assertion, we may replace k with any field extension, and hence assume that k is perfect. Then the reduced neutral component L^0_{red} is a smooth connected subgroup of L, the quotient L/L^0_{red} is finite, and $\mathrm{Hom}_{\mathcal{C}}(L^0_{\mathrm{red}}, A) = 0$ by the above argument. Thus, $\mathrm{Hom}_{\mathcal{C}}(L, A) = \mathrm{Hom}_{\mathcal{C}}(L/L^0_{\mathrm{red}}, A)$ is killed by the order of L/L^0_{red}. □

Next, we obtain a key preliminary result. To state it, recall that a unipotent group G is *split* if it admits a finite increasing sequence of subgroups $0 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that $G_i/G_{i-1} \cong \mathbb{G}_a$ for $i = 1, \ldots, n$.

Proposition 2.6. *Let G be an algebraic group.*

(i) *There exists a finite subgroup $F \subset G$ such that G/F is smooth and connected.*

(ii) *If G is unipotent, then we may choose F such that G/F is split.*

Proof. (i) By Lemma 2.2, we have $G = G^0 + F$ for some finite subgroup $F \subset G$. Thus, $G/F \cong G^0/F \cap G^0$ is connected; this completes the proof when $\mathrm{char}(k) = 0$.

When $\mathrm{char}(k) = p > 0$, we may assume G connected by the above step. Consider the relative Frobenius morphism $F_{G/k} : G \to G^{(p)}$ and its iterates $F^n_{G/k} : G \to G^{(p^n)}$, where $n \geq 1$. Then $\mathrm{Ker}(F^n_{G/k})$ is finite for all n; moreover, $G/\mathrm{Ker}(F^n_{G/k})$ is smooth for $n \gg 0$ (see [SGA3, VIIA, Prop. 8.3]), and still connected.

(ii) We argue by induction on the dimension of G. The statement is obvious if $\dim(G) = 0$. In the case where $\dim(G) = 1$, we may assume that G is smooth and connected in view of Lemma 2.2 again; then G is a k-form of $\mathbb{G}_a$. By [Ru70, Thm. 2.1], there is an exact sequence

$$0 \longrightarrow G \longrightarrow \mathbb{G}_a^2 \xrightarrow{f} \mathbb{G}_a \longrightarrow 0,$$

where $f \in \mathcal{O}(\mathbb{G}_a^2) \cong k[x,y]$ satisfies $f(x,y) = y^{p^n} - a_0\, x - a_1\, x^p - \cdots - a_m\, x^{p^m}$ for some integers $m, n \geq 0$ and some $a_0, \ldots, a_m \in k$ with $a_0 \neq 0$. Thus, the projection

$$p_1 : G \longrightarrow \mathbb{G}_a, \quad (x,y) \longmapsto x$$

lies in an exact sequence

$$0 \longrightarrow \alpha_{p^n} \longrightarrow G \xrightarrow{p_1} \mathbb{G}_a \longrightarrow 0,$$

where α_{p^n} denotes the kernel of the endomorphism $x \mapsto x^{p^n}$ of $\mathbb{G}_a$. This yields the assertion in this case.

If $\dim(G) \geq 2$, then we may choose a subgroup $G_1 \subset G$ such that $0 < \dim(G_1) < \dim(G)$ (as follows from [DG70, Prop. IV.2.2.5]). By the induction assumption for G/G_1, there exists a subgroup $G_2 \subset G$ such that $G_1 \subset G_2$, G_2/G_1 is finite, and G/G_2 is split. Next, the induction assumption for G_2 yields a finite subgroup $F \subset G_2$ such that G_2/F is split. Then G/F is split as well. □

Remark 2.7. By Proposition 2.6, every algebraic group G admits an isogeny $u : G \to H$, where H is smooth and connected. If k is perfect, then there also exists an isogeny $v : K \to G$, where K is smooth and connected: just take v to be the inclusion of the reduced neutral component G^0_{red}. But this fails over any imperfect field k. Indeed, if such an isogeny v exists, then its image must be G^0_{red}. On the other hand, by [SGA3, VIA, Ex. 1.3.2], there exists a connected algebraic group G such that G_{red} is not a subgroup.

By combining Lemma 2.2, Theorems 2.3 and 2.4, and Proposition 2.6, we obtain readily:

Proposition 2.8. *Every algebraic group G admits a finite increasing sequence of subgroups $0 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that each G_i/G_{i-1}, $i = 1, \ldots, n$, is finite or isomorphic to $\mathbb{G}_a$, a simple torus, or a simple abelian variety. Moreover, G is linear if and only if no abelian variety occurs.*

2.2 Characteristic zero

In this subsection, we assume that $\mathrm{char}(k) = 0$. Recall that every unipotent group is isomorphic to the additive group of its Lie algebra, via the exponential map; this yields an equivalence between the category $\mathcal{U}$ of unipotent groups and the category of finite-dimensional k-vector spaces (see [DG70, Prop. IV.2.4.2]).

Next, consider a connected algebraic group G. By Theorem 2.3, there is a unique exact sequence $0 \to L \to G \to A \to 0$, where A is an abelian variety, and L is connected and linear. Moreover, in view of Theorem 2.4, we have $L = T \times U$, where T is a torus and U is the unipotent radical, $R_u(G)$.

We now extend this result to possibly non-connected groups:

Theorem 2.9. (i) *Every algebraic group G lies in an exact sequence*

$$0 \longrightarrow M \times U \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where M is of multiplicative type, U is unipotent, and A is an abelian variety. Moreover, $U = R_u(G)$ is unique, M is unique up to commensurability in G, and A is unique up to isogeny.

(ii) *The formation of the unipotent radical commutes with base change under field extensions, and yields an exact functor*

$$R_u : \mathcal{C} \longrightarrow \mathcal{U},$$

right adjoint to the inclusion $\mathcal{U} \to \mathcal{C}$.

(iii) *The projective objects of $\mathcal{C}$ are exactly the unipotent groups.*

Proof. (i) By Theorem 2.3, there exists an exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where L is linear (possibly non-connected), and A is an abelian variety. By Theorem 2.4, we have $L = M \times U$, where M is of multiplicative type and U is unipotent.

Since M and A have no non-trivial unipotent subgroups, we have $U = R_u(G)$; in particular, U is unique. Given another exact sequence

$$0 \longrightarrow M' \times U \longrightarrow G \longrightarrow A' \longrightarrow 0$$

satisfying the same assumptions, the image of M' in $A \cong G/(M \times U)$ is finite by Proposition 2.5. In other words, the quotient $M'/(M \times U) \cap M'$ is finite. Likewise, $M/M \cap (M' \times U)$ is finite as well. Since $(M \times U) \cap M' = M \cap M' = M \cap (M' \times U)$, we see that M, M' are commensurable in G. Then $A = G/M \times U$ and $A' = G/M' \times U$ are both quotients of $G/(M \cap M') \times U$ by finite subgroups, and hence are isogenous.

(ii) In view of (i), $G/R_u(G)$ is an extension of an abelian variety by a group of multiplicative type. Since these two classes of algebraic groups are stable under base change by any field extension k' of k, it follows that $(G/R_u(G))_{k'}$ has zero unipotent radical. Thus, $R_u(G)_{k'} = R_u(G_{k'})$.

Next, note that every morphism $f : G \to H$ sends $R_u(G)$ to $R_u(H)$. Moreover, every exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

induces an exact sequence

$$0 \to R_u(G_1) \to R_u(G_2) \to R_u(G_3) \to G_1/R_u(G_1) \to G_2/R_u(G_2) \to G_3/R_u(G_3) \to 0$$

by the snake lemma. Since $G_1/R_u(G_1)$ has trivial unipotent radical, the sequence

$$0 \longrightarrow R_u(G_1) \longrightarrow R_u(G_2) \longrightarrow R_u(G_3) \longrightarrow 0$$

is exact as well.

The assertion about adjointness follows from the fact that every morphism $U \to G$, where U is unipotent and G arbitrary, factors through a unique morphism $U \to R_u(G)$.

(iii) Consider an epimorphism $\varphi : G \to H$, a unipotent group U, and a morphism $\psi : U \to H$. Then ψ factors through $R_u(H)$. Also, by (ii), φ restricts to an epimorphism $R_u(G) \to R_u(H)$, which admits a section as unipotent groups are just vector spaces. Thus, ψ lifts to a morphism $U \to G$. This shows that U is projective in $\mathcal{C}$.

Conversely, let G be a projective object in $\mathcal{C}$. By Theorem 2.3, there exists an exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where L is linear, and A is an abelian variety. Let n be a positive integer; then the exact sequence

$$0 \longrightarrow A[n] \longrightarrow A \xrightarrow{n_A} A \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(G, A[n]) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(G, A) \xrightarrow{\times n} \mathrm{Hom}_{\mathcal{C}}(G, A) \longrightarrow 0,$$

since G is projective. Thus, the (abstract) group $\mathrm{Hom}_{\mathcal{C}}(G, A)$ is divisible. But there is an exact sequence

$$0 \longrightarrow \mathrm{End}_{\mathcal{C}}(A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(G, A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(L, A),$$

where the abelian group $\mathrm{End}_{\mathcal{C}}(A)$ is free of finite rank, and $\mathrm{Hom}_{\mathcal{C}}(L, A)$ is killed by some positive integer (Proposition 2.5). It follows that $\mathrm{End}_{\mathcal{C}}(A)$ is zero, and hence so is A. Thus, G is linear, and hence $G = M \times U$ as above. Since U is projective, so is M. Thus, we may assume that G is of multiplicative type, i.e., contained in some torus T. By the above argument, the group $\mathrm{Hom}_{\mathcal{C}}(G, T)$ is divisible; since this group is also finitely generated, it must be zero. Thus, $T = 0 = G$. □

Remark 2.10. With the notation of the above theorem, we have a natural map

$$G \longrightarrow G/M \times_A G/U,$$

which is a morphism of $M \times U$-torsors over A, and hence an isomorphism. Moreover, G/M is an extension of an abelian variety by a unipotent group; such 'vector extensions' will be studied in detail in §5.1. Also, G/U is an extension of an abelian variety by a group of multiplicative type, and hence of a semi-abelian variety by a finite group. The semi-abelian varieties will be considered in §5.2.

2.3 Positive characteristics

In this subsection, we assume that $\mathrm{char}(k) = p > 0$. We obtain a variant of [Br15b, Thm. 5.6.3]:

Theorem 2.11. *Let G be an algebraic group.*

(i) *G has a smallest subgroup H such that $U := G/H$ is unipotent. Moreover, H is an extension of an abelian variety A by a group of multiplicative type M. Also, M is unique up to commensurability in G, and A is unique up to isogeny.*

(ii) *Every morphism $H \to U$ is zero; every morphism $U \to H$ has finite image.*

(iii) *The formation of U commutes with base change under field extensions, and yields an exact functor*

$$U : \mathcal{C} \longrightarrow \mathcal{U},$$

which is left adjoint to the inclusion of $\mathcal{U}$ in $\mathcal{C}$.

(iv) *There exists a subgroup $V \subset G$ such that $G = H + V$ and $H \cap V$ is a finite p-group.*

Proof. (i) Since the underlying topological space of G is noetherian, we may choose a subgroup $H \subset G$ such that G/H is unipotent, and H is minimal for this property. Let $H' \subset G$ be another subgroup such that G/H' is unipotent. Then so is $G/H \cap H'$ in view of the exact sequence

$$0 \longrightarrow G/H \cap H' \longrightarrow G/H \times G/H'.$$

By minimality of H, it follows that $H \cap H' = H$, i.e., $H \subset H'$. Thus, H is the smallest subgroup with unipotent quotient.

Since the class of unipotent groups is stable under extensions, every unipotent quotient of H is zero. Also, by the affinization theorem (see [Br15b, Thm. 1, Prop. 5.5.1]), H is an extension of a linear algebraic group L by a semi-abelian variety S. Thus, every unipotent quotient of L is zero, and hence L must be of multiplicative type in view of Theorem 2.4. By [DG70, Cor. IV.1.3.9], the reduced neutral component of L is its maximal torus, T; the quotient L/T is a finite group of multiplicative type. Denote by S' the preimage of T in H; then S' is an abelian variety (extension of T by S) and we have an exact sequence

$$0 \longrightarrow S' \longrightarrow H \longrightarrow L/T \longrightarrow 0.$$

By Lemma 2.2, there exists a finite subgroup $F \subset H$ such that $H = S' + F$; equivalently, the quotient map $H \to L/T$ restricts to an epimorphism $F \to L/T$. Also, by Theorem 2.4 again, F has a largest subgroup of multiplicative type, M_F, and the quotient F/M_F is unipotent. Since L/T is of multiplicative type, it follows that the composition $M_F \to F \to L/T$ is an epimorphism as well. Thus, we may replace F with M_F, and assume that F is of multiplicative type. Let T' be the maximal torus of the semi-abelian variety S'. Then the group $T' + F$ is of multiplicative type; moreover, $H/T' + F$ is a quotient of S'/T', and hence is an abelian variety. This yields an exact sequence

$$0 \longrightarrow M \longrightarrow H \longrightarrow A \longrightarrow 0,$$

where M is of multiplicative type, and A an abelian variety.

The uniqueness assertions may be checked as in the proof of Theorem 2.9.

(ii) This follows readily from Proposition 2.5.

(iii) The assertion on base change under field extensions follows from the stability of the classes of unipotent groups, abelian varieties, and groups of multiplicative type, under such base changes.

To show the exactness assertion, consider an exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

Then the snake lemma yields an exact sequence

$$0 \to H_1 \to H_2 \to H_3 \to U_1 \to U_2 \to U_3 \to 0$$

with an obvious notation. By Proposition 2.5 again, every morphism $H_3 \to U_1$ is zero; thus, the sequence

$$0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow U_3 \longrightarrow 0$$

is exact as well.

Finally, the adjointness assertion may be checked as in the proof of Theorem 2.9.

(iv) Consider the subgroups $\mathrm{Ker}(p_G^n) \subset G$, where n is a positive integer. Since they form a decreasing sequence, there exists a positive integer m such that $\mathrm{Ker}(p_G^n) = \mathrm{Ker}(p_G^m)$ for all $n \geq m$. Let $\overline{G} := G/\mathrm{Ker}(p_G^m)$, then $\mathrm{Ker}(p_{\overline{G}}) = \mathrm{Ker}(p_G^{m+1})/\mathrm{Ker}(p_G^m)$ is zero, and hence $p_{\overline{G}}$ is an isogeny. Next, let $H \subset G$ be as in (i) and put $\overline{H} := H/\mathrm{Ker}(p_H^m)$, $\overline{U} = \overline{G}/\overline{H}$. Then $\overline{U}$ is unipotent (as a quotient of U) and $p_{\overline{U}}$ has finite cokernel (since this holds for $p_{\overline{G}}$). Thus, $\overline{U}$ is a finite p-group. By Lemma 2.2, there exists a finite p-subgroup $F \subset G$ such that $\overline{G} = \overline{H} + \overline{F}$ with an obvious notation. Thus, $G = H + V$, where $V := \mathrm{Ker}(p_G^m) + F$. Also, $H \cap \mathrm{Ker}(p_G^m) = \mathrm{Ker}(p_H^m)$ is finite, since H is an extension of A by M, and $\mathrm{Ker}(p_A^m)$ and $\mathrm{Ker}(p_M^m)$ are finite. As F is finite, it follows that $H \cap V$ is finite as well. Moreover, V is a p-group, since so are F and $\mathrm{Ker}(p_G^m)$; we conclude that $H \cap V$ is a finite p-group. □

Corollary 2.12. *Let G be an algebraic group.*

(i) *There exists a finite subgroup $F \subset G$ such that $G/F \cong S \times U$, where S is a semi-abelian variety, and U a split unipotent group. Moreover, S and U are unique up to isogeny.*

(ii) *If k is locally finite, then we may choose F so that $S \cong T \times A$, where T is a torus, and A an abelian variety. Moreover, T and A are unique up to isogeny.*

Proof. (i) With the notation of Theorem 2.11, we have isomorphisms

$$G/H \cap V \cong G/V \times G/H \cong (H/H \cap V) \times U.$$

Also, $H/H \cap V$ is an extension of an abelian variety, $H/(H \cap V) + M$, by a group of multiplicative type, $M/M \cap V$. Moreover, U is an extension of a split unipotent group by a finite group (Proposition 2.6). Thus, we may assume that $G = H$. Then G^0_{red} is a semi-abelian variety, as follows from [Br15b, Lem. 5.6.1]. Since G/G^0_{red} is finite, applying Lemma 2.2 yields that G is an extension of a semi-abelian variety by a finite group.

(ii) By [Br15b, Cor. 5.5.5], there exists an abelian subvariety $A \subset S$ such that $S = T + A$, where $T \subset S$ denotes the maximal torus. Then $T \cap A$ is finite, and $S/T \cap A \cong T/T \cap A \times A/T \cap A$.

This completes the proof of the existence assertions in (i) and (ii). The uniqueness up to isogeny follows from Proposition 2.5. □

3 The isogeny category of algebraic groups

3.1 Definition and first properties

Recall that $\mathcal{C}$ denotes the category of commutative algebraic groups, and $\mathcal{F}$ the full subcategory of finite groups. By [Ga62, III.1], we may form the quotient category $\mathcal{C}/\mathcal{F}$; it has the same objects as $\mathcal{C}$, and its morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{F}}(G,H) = \lim_{\rightarrow} \mathrm{Hom}_{\mathcal{C}}(G', H/H'),$$

where the direct limit is taken over all subgroups $G' \subset G$ such that G/G' is finite, and all finite subgroups $H' \subset H$. The category $\mathcal{C}/\mathcal{F}$ is abelian, and comes with an exact functor

$$Q : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{F},$$

which is the identity on objects and the natural map

$$\mathrm{Hom}_{\mathcal{C}}(G,H) \longrightarrow \lim_{\rightarrow} \mathrm{Hom}_{\mathcal{C}}(G', H/H'), \quad f \longmapsto \underline{f}$$

on morphisms. The quotient functor Q satisfies the following universal property: given an exact functor $R : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an abelian category, such that $R(F) = 0$ for any finite group F, there exists a unique exact functor $S : \mathcal{C}/\mathcal{F} \to \mathcal{D}$ such that $R = S \circ Q$ (see [Ga62, Cor. III.1.2, Cor. III.1.3]).

Alternatively, $\mathcal{C}/\mathcal{F}$ may be viewed as the localization of $\mathcal{C}$ at the multiplicative system of isogenies (see [GZ67, §I.2] and [SP16, §4.26] for localization of categories); this is easily checked by arguing as in the proof of [SP16, Lem. 12.9.6.].

We now show that $\mathcal{C}/\mathcal{F}$ is equivalent to a category with somewhat simpler objects and morphisms:

Lemma 3.1. *Let $\underline{\mathcal{C}}$ be the full subcategory of $\mathcal{C}/\mathcal{F}$ with objects the smooth connected algebraic groups.*

(i) *The inclusion of $\underline{\mathcal{C}}$ in $\mathcal{C}/\mathcal{F}$ is an equivalence of categories.*

(ii) $\mathrm{Hom}_{\underline{\mathcal{C}}}(G,H) = \lim \mathrm{Hom}_{\mathcal{C}}(G, H/H')$, *where the direct limit is taken over all finite subgroups $H' \subset H$.*

(iii) *Let $\underline{f} \in \mathrm{Hom}_{\underline{\mathcal{C}}}(G,H)$ be represented by a morphism $f : G \to H/H'$ in $\mathcal{C}$. Then $\underline{f}$ is zero (resp. a monomorphism, an epimorphism, an isomorphism) if and only if f is zero (resp. has a finite kernel, is an epimorphism, is an isogeny).*

Proof. (i) This follows from Proposition 2.6.

(ii) This follows from Lemma 2.1.

(iii) By [Ga62, Lem. III.1.2], $\underline{f}$ is zero (resp. a monomorphism, an epimorphism) if and only if $\mathrm{Im}(f)$ (resp. $\mathrm{Ker}(f)$, $\mathrm{Coker}(f)$) is finite. By Lemma 2.1 again, the finiteness of $\mathrm{Im}(f)$ is equivalent to $f = 0$, and the finiteness of $\mathrm{Coker}(f)$ is equivalent to f being an epimorphism. As a consequence, $\underline{f}$ is an isomorphism if and only if f is an isogeny. □

The abelian category $\underline{\mathcal{C}}$ will be called the *isogeny category of (commutative) algebraic groups*. Every exact functor $R : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an abelian category and $R(f)$ is an isomorphism for any isogeny f, factors uniquely through $Q : \mathcal{C} \to \underline{\mathcal{C}}$ (indeed, such a functor R must send any finite group to zero).

We may now prove the assertion (i) of the main theorem:

Proposition 3.2. (i) *The category $\underline{\mathcal{C}}$ is noetherian and artinian.*

(ii) *The non-zero simple objects of $\underline{\mathcal{C}}$ are exactly $\mathbb{G}_a$, the simple tori, and the simple abelian varieties.*

Proof. (i) Let G be a smooth connected algebraic group, and $(G_n)_{n\geq 0}$ an increasing sequence of subobjects of G in $\underline{\mathcal{C}}$, i.e., each G_n is smooth, connected, and equipped with a $\mathcal{C}$-morphism

$$\varphi_n : G_n \longrightarrow G/G'_n,$$

where $\mathrm{Ker}(\varphi_n)$ and G'_n are finite; moreover, we have $\mathcal{C}$-morphisms

$$\psi_n : G_n \longrightarrow G_{n+1}/G''_{n+1},$$

where $\mathrm{Ker}(\psi_n)$ and G''_{n+1} are finite. Thus, $\dim(G_n) \leq \dim(G_{n+1}) \leq \dim(G)$. It follows that $\dim(G_n) = \dim(G_{n+1})$ for $n \gg 0$, and hence ψ_n is an isogeny. So $G_n \cong G_{n+1}$ in $\underline{\mathcal{C}}$ for $n \gg 0$. This shows that $\underline{\mathcal{C}}$ is noetherian. One may check likewise that $\underline{\mathcal{C}}$ is artinian.

(ii) This follows from Proposition 2.8. □

Next, we relate the short exact sequences in $\mathcal{C}$ with those in $\underline{\mathcal{C}}$:

Lemma 3.3. *Consider a short exact sequence in $\mathcal{C}$,*

$$\xi : \quad 0 \longrightarrow G_1 \xrightarrow{u} G_2 \xrightarrow{v} G_3 \longrightarrow 0,$$

where G_1, G_2, G_3 are smooth and connected. Then ξ splits in $\underline{\mathcal{C}}$ if and only if the push-out $f_*\xi$ splits in $\mathcal{C}$ for some isogeny $f : G_1 \to H$, where H is smooth and connected.

Proof. Recall that ξ splits in $\underline{\mathcal{C}}$ if and only if there exists a $\underline{\mathcal{C}}$-morphism $\underline{f} : G_2 \to G_1$ such that $\underline{f} \circ u = \mathrm{id}$ in $\underline{\mathcal{C}}$. Equivalently, there exists a finite subgroup $G'_1 \subset G_1$ and a $\mathcal{C}$-morphism $f : G_2 \to G_1/G'_1$ such that $f \circ u$ is the quotient map $q_1 : G_1 \to G_1/G'_1$.

If such a pair (G'_1, f) exists, then f factors through a morphism $G_2/u(G'_1) \to G_1/G'_1$, which splits the bottom exact sequence in the push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_1 & \xrightarrow{u} & G_2 & \xrightarrow{v} & G_3 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle q_1} & & \Big\downarrow{\scriptstyle q_2} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & G_1/G'_1 & \xrightarrow{u'} & G_2/u(G'_1) & \xrightarrow{v'} & G_3 & \longrightarrow & 0.
\end{array}$$

Replacing G'_1 by a larger finite subgroup, we may assume that G_1/G'_1 is smooth and connected (Lemma 2.2).

Conversely, a splitting of the bottom exact sequence in the above diagram is given by a $\mathcal{C}$-morphism $f' : G_2/u(G'_1) \to G_1/G'_1$ such that $f' \circ u' = \mathrm{id}$ in $\mathcal{C}$. Let $f : G_2 \to G_1/G'_1$ denote the composition $G_2 \xrightarrow{q_2} G_2/u(G'_1) \xrightarrow{f'} G_1/G'_1$. Then $f \circ u = f' \circ q_2 \circ u = f' \circ u' \circ q_1 = q_1$ as desired. □

We may now produce examples of non-split exact sequences in $\underline{\mathcal{C}}$, thereby showing that $\mathrm{hd}(\underline{\mathcal{C}}) \geq 1$:

Examples 3.4. (i) Consider an exact sequence

$$\xi : \quad 0 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where A is an abelian variety. Then ξ, viewed as an extension of A by $\mathbb{G}_a$ in $\mathcal{C}$, is classified by an element $\eta \in H^1(A, \mathcal{O}_A)$ (see [Ro58] or [MM74, §1.9]).

If $\mathrm{char}(k) = 0$, then every isogeny $f : \mathbb{G}_a \to H$, where H is connected, may be identified with the multiplication by some $t \in k^*$, viewed as an endomorphism of $\mathbb{G}_a$; then the push-out $f_*\xi$ is classified by $t\eta$. In view of Lemma 3.3, it follows that ξ is non-split in $\underline{\mathcal{C}}$ whenever $\eta \neq 0$.

In contrast, if $\mathrm{char}(k) = p > 0$, then ξ splits in $\underline{\mathcal{C}}$, as the multiplication map p_A yields an isomorphism in $\underline{\mathcal{C}}$, and $p_{\mathbb{G}_a} = 0$.

(ii) Assume that $\mathrm{char}(k) = p > 0$ and consider the algebraic group W_2 of Witt vectors of length 2. This group comes with an exact sequence

$$\xi : \quad 0 \longrightarrow \mathbb{G}_a \longrightarrow W_2 \longrightarrow \mathbb{G}_a \longrightarrow 0,$$

see e.g. [DG70, §V.1.1.6]. Every isogeny $f : \mathbb{G}_a \to H$, where H is smooth and connected, may be identified with a non-zero endomorphism of $\mathbb{G}_a$. In view of [DG70, Cor. V.1.5.2], it follows that the push-forward $f_*\xi$ is non-split. Thus, ξ does not split in $\underline{\mathcal{C}}$.

Proposition 3.5. *Consider an exact sequence*

$$0 \longrightarrow G_1 \xrightarrow{\underline{u}} G_2 \xrightarrow{\underline{v}} G_3 \longrightarrow 0$$

in $\underline{\mathcal{C}}$. Then there exists an exact sequence

$$0 \longrightarrow H_1 \xrightarrow{u'} H_2 \xrightarrow{v'} H_3 \longrightarrow 0$$

in $\mathcal{C}$, and epimorphisms with finite kernels $f_i : G_i \to H_i$ ($i = 1, 2, 3$), such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & G_1 \arrow[r, "\underline{u}"] \arrow[d, "\underline{f}_1"'] & G_2 \arrow[r, "\underline{v}"] \arrow[d, "\underline{f}_2"'] & G_3 \arrow[r] \arrow[d, "\underline{f}_3"'] & 0 \\
0 \arrow[r] & H_1 \arrow[r, "\underline{u}'"] & H_2 \arrow[r, "\underline{v}'"] & H_3 \arrow[r] & 0
\end{tikzcd}$$

commutes in $\underline{\mathcal{C}}$.

Proof. The $\underline{\mathcal{C}}$-morphism $\underline{v}$ is represented by an epimorphism $v : G_2 \to G_3/G_3'$ in $\mathcal{C}$, where G_3' is a finite subgroup of G_3. We may thus replace G_3 with G_3/G_3', and assume that v is an epimorphism in $\mathcal{C}$.

Next, $\underline{u}$ is represented by a morphism $u : G_1 \to G_2/G_2'$ with finite kernel, where G_2' is a finite subgroup of G_2. We may thus replace G_1 (resp. G_2, G_3) with $G_1/\mathrm{Ker}(u)$ (resp. G_2/G_2', $G_3/v(G_2')$) and assume that u is a monomorphism in $\mathcal{C}$. Then $v \circ u$ has finite image, and hence is zero since G_1 is smooth and connected.

We now have a complex in $\mathcal{C}$

$$0 \longrightarrow G_1 \xrightarrow{u} G_2 \xrightarrow{v} G_3 \longrightarrow 0,$$

where u is a monomorphism, v an epimorphism, and $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is finite. By Lemma 2.2, we may choose a finite subgroup $F \subset \mathrm{Ker}(v)$ such that $\mathrm{Ker}(v) = \mathrm{Im}(u) + F$. This yields a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & G_1 \arrow[r, "u"] \arrow[d, "q_1"'] & G_2 \arrow[r, "v"] \arrow[d, "q_2"'] & G_3 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & G_1/u^{-1}(F) \arrow[r, "u'"] & G_2/F \arrow[r, "v'"] & G_3 \arrow[r] & 0,
\end{tikzcd}$$

where q_1, q_2 denote the quotient maps. Clearly, u' is a monomorphism, and v' an epimorphism. Also, $v' \circ u' = 0$, since q_1 is an epimorphism. Finally, q_2 restricts to an epimorphism $\mathrm{Ker}(v) \to \mathrm{Ker}(v')$, and hence $\mathrm{Ker}(v') = \mathrm{Im}(u')$. This completes the proof. $\square$

3.2 Divisible groups

Given a divisible algebraic group G and a positive integer n, the morphism $n_G : G \to G$ factors through an isomorphism $G/G[n] \xrightarrow{\cong} G$. We denote the inverse isomorphism by

$$u_n : G \xrightarrow{\cong} G/G[n].$$

By construction, we have a commutative triangle

$$\begin{tikzcd}
G \arrow[d, "n_G"'] \arrow[dr, "q"] & \\
G \arrow[r, "u_n"] & G/G[n],
\end{tikzcd}$$

where q denotes the quotient morphism. Since q yields the identity morphism in $\underline{\mathcal{C}}$, we see that u_n yields the inverse of the $\underline{\mathcal{C}}$-automorphism n_G of G. As a consequence, $\mathrm{End}_{\underline{\mathcal{C}}}(G)$ is a $\mathbb{Q}$-algebra.

More generally, we have the following:

Proposition 3.6. *Let G, H be smooth connected algebraic groups, and assume that H is divisible.*

(i) *Every extension group $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, H)$ is a $\mathbb{Q}$-vector space.*

(ii) *The natural map $Q : \mathrm{Hom}_{\mathcal{C}}(G, H) \to \mathrm{Hom}_{\underline{\mathcal{C}}}(G, H)$ is injective and induces an isomorphism*

$$\gamma : \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(G, H) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(G, H).$$

(iii) *If G is divisible as well, then the natural map $Q^1 : \mathrm{Ext}^1_{\mathcal{C}}(G, H) \to \mathrm{Ext}^1_{\underline{\mathcal{C}}}(G, H)$ induces an isomorphism*

$$\gamma^1 : \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{C}}(G, H) \longrightarrow \mathrm{Ext}^1_{\underline{\mathcal{C}}}(G, H).$$

Proof. (i) Just note that $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(G,H)$ is a module over the $\mathbb{Q}$-algebra $\mathrm{End}_{\underline{\mathcal{C}}}(H)$.

(ii) Let $f \in \mathrm{Hom}_{\mathcal{C}}(G,H)$, and n a positive integer. If $\gamma(\frac{1}{n} \otimes f) = 0$, then of course $\gamma(f) = 0$, i.e., $\underline{f} = 0$. Thus, $f = 0$ by Lemma 3.1. This shows the injectivity of f.

For the surjectivity, consider a $\underline{\mathcal{C}}$-morphism $\underline{f} : G \to H$ represented by a $\mathcal{C}$-morphism $f : G \to H/H'$, where H' is a finite subgroup of H. Then $H' \subset H[n]$ for some positive integer n, which we may take to be the order of H'. Thus, we may assume that $H' = H[n]$. Then the $\underline{\mathcal{C}}$-morphism $\varphi := \underline{u}_n^{-1} \circ \underline{f} : G \to H$ satisfies $\underline{f} = \underline{u}_n \circ \varphi$, i.e., $\underline{f} = \gamma(\frac{1}{n} \otimes \varphi)$.

(iv) Consider $\eta \in \mathrm{Ext}^1_{\mathcal{C}}(G,H)$ such that $\gamma^1(\frac{1}{n} \otimes \eta) = 0$ for some positive integer n. Then of course $\gamma^1(\eta) = 0$, i.e., η is represented by an exact sequence in $\mathcal{C}$

$$0 \longrightarrow H \xrightarrow{u} E \xrightarrow{v} G \longrightarrow 0,$$

which splits in $\underline{\mathcal{C}}$. By Lemma 3.3 and the divisibility of H, it follows that the push-out by m_H of the above extension splits in $\mathcal{C}$ for some $m > 0$. But $(m_H)_*\eta = m\eta$ (see e.g. [Oo66, Lem. I.3.1]), and hence $m\eta = 0$. This shows the injectivity of γ^1.

For the surjectivity, we adapt the argument of Proposition 3.5. Let $\eta \in \mathrm{Ext}^1_{\underline{\mathcal{C}}}(G,H)$ be represented by an exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow H \xrightarrow{\underline{u}} E \xrightarrow{\underline{v}} G \longrightarrow 0.$$

Since G is divisible, $\underline{v}$ is represented by a $\mathcal{C}$-morphism $v : E \to G/G[m]$ for some positive integer m. Replacing η with its pull-back $u_m^*\eta = (m_G^*)^{-1}\eta = \gamma^1(\frac{1}{m} \otimes \eta)$, we may thus assume that $\underline{v}$ is represented by a $\mathcal{C}$-epimorphism $v : E \to G$.

Likewise, since H is divisible, $\underline{u}$ is represented by some $\mathcal{C}$-morphism $u : H \to E/E[n]$. Then η is represented by the exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow H \xrightarrow{\underline{u}} E/E[n] \xrightarrow{\underline{v_n}} G \longrightarrow 0,$$

where $v_n : E/E[n] \to G/G[n]$ is the $\mathcal{C}$-epimorphism induced by v. So we may further assume that u is represented by a $\mathcal{C}$-morphism $u : H \to E$. By Lemma 3.1, we then have $v \circ u = 0$; moreover, $\mathrm{Ker}(u)$ and $\mathrm{Ker}(v)/\mathrm{Im}(u)$ are finite. In view of Lemma 2.2, we have $\mathrm{Ker}(v) = \mathrm{Im}(u) + E'$ for some finite subgroup $E' \subset E$. This yields a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "u"] \arrow[d] & E \arrow[r, "v"] \arrow[d] & G \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & H/u^{-1}(E') \arrow[r, "u_1"] & E/E' \arrow[r, "v_1"] & G \arrow[r] & 0,
\end{tikzcd}$$

where the bottom sequence is exact, and $u^{-1}(E')$ is finite.

We may thus choose a positive integer r such that $u^{-1}(E') \subset H[r]$. Taking the push-out by the quotient map $H/u^{-1}(E') \to H/H[r]$ yields a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "u"] \arrow[d] & E \arrow[r, "v"] \arrow[d] & G \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & H/H[r] \arrow[r, "u_2"] & E/E' + u(H[r]) \arrow[r, "v_2"] & G \arrow[r] & 0,
\end{tikzcd}$$

where the bottom sequence is exact again. Thus, $r\eta = (r_H)_*\eta$ is represented by an exact sequence in $\mathcal{C}$. □

Remarks 3.7. (i) Given two divisible groups G, H, the map

$$Q^1 : \operatorname{Ext}^1_{\mathcal{C}}(G, H) \longrightarrow \operatorname{Ext}^1_{\underline{\mathcal{C}}}(G, H)$$

is not necessarily injective. Indeed, the group $\operatorname{Ext}^1_{\mathcal{C}}(A, \mathbb{G}_m)$ has non-zero torsion for any non-zero abelian variety A over (say) a separably closed field.

(ii) We may also consider the natural maps $Q^n : \operatorname{Ext}^n_{\mathcal{C}}(G, H) \to \operatorname{Ext}^n_{\underline{\mathcal{C}}}(G, H)$ for $n \geq 2$. But these maps turn out to be zero for any algebraic groups G, H, since $\operatorname{Ext}^n_{\underline{\mathcal{C}}}(G, H) = 0$ (Lemma 4.10) and $\operatorname{Ext}^n_{\mathcal{C}}(G, H)$ is torsion (Remark 4.11).

As a first application of Proposition 3.6, we obtain:

Proposition 3.8. *Assume that* $\operatorname{char}(k) = 0$.

(i) *The composition of the inclusion* $\mathcal{U} \to \mathcal{C}$ *with the quotient functor* $Q : \mathcal{C} \to \underline{\mathcal{C}}$ *identifies* $\mathcal{U}$ *with a full subcategory of* $\underline{\mathcal{C}}$.

(ii) *The unipotent radical functor yields an exact functor*

$$\underline{R}_u : \underline{\mathcal{C}} \longrightarrow \mathcal{U},$$

which is right adjoint to the inclusion. Moreover, $\underline{R}_u$ *commutes with base change under field extensions.*

(iii) *Every unipotent group is a projective object in* $\underline{\mathcal{C}}$.

Proof. (i) Recall that a morphism of unipotent groups is just a linear map of the associated k-vector spaces. In view of Proposition 3.6, it follows that the natural map $\operatorname{Hom}_{\mathcal{C}}(U, V) \to \operatorname{Hom}_{\underline{\mathcal{C}}}(U, V)$ is an isomorphism for any unipotent groups U, V.

(ii) The functor $R_u : \mathcal{C} \to \mathcal{U}$ is exact by Theorem 2.9, and sends every finite group to 0. By the universal property of Q, there exists a unique exact functor $S : \mathcal{C}/\mathcal{F} \to \mathcal{U}$ such that $R_u = S \circ Q$. Since R_u commutes with base change under field extensions (Theorem 2.9 again), so does S by uniqueness. Thus, composing S with the inclusion $\underline{\mathcal{C}} \to \mathcal{C}/\mathcal{F}$ yields the desired functor.

For any unipotent group U and any algebraic group G, the natural map

$$\operatorname{Hom}_{\mathcal{U}}(U, R_u(G)) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(U, G)$$

is an isomorphism. By Proposition 3.6 again, the natural map

$$Q : \operatorname{Hom}_{\mathcal{C}}(U, G) \longrightarrow \operatorname{Hom}_{\underline{\mathcal{C}}}(U, G)$$

is an isomorphism as well. It follows that $\underline{R}_u$ is right adjoint to the inclusion.

(iii) Let U be a unipotent group. Then the functor on $\mathcal{C}$ defined by

$$G \longmapsto \operatorname{Hom}_{\mathcal{U}}(U, R_u(G))$$

is exact, since the unipotent radical functor is exact and the category $\mathcal{U}$ is semi-simple. Thus, $G \mapsto \operatorname{Hom}_{\underline{\mathcal{C}}}(U, G)$ is exact as well; this yields the assertion. $\square$

3.3 Field extensions

Let k' be a field extension of k. Then the assignement $G \mapsto G_{k'} := G \otimes_k k'$ yields the *base change functor*

$$\otimes_k k' : \mathcal{C}_k \longrightarrow \mathcal{C}_{k'}.$$

Clearly, this functor is faithful and exact; also, note that G is connected (resp. smooth, finite, linear, unipotent, a torus, an abelian variety, a semi-abelian variety) if and only if so is $G_{k'}$.

Lemma 3.9. *With the above notation, the functor $\otimes_k k'$ yields an exact functor*

$$\underline{\otimes}_k k' : \underline{\mathcal{C}}_k \longrightarrow \underline{\mathcal{C}}_{k'}.$$

Proof. The composite functor $Q_{k'} \circ \otimes_k k' : \mathcal{C}_k \to \underline{\mathcal{C}}_{k'}$ is exact and sends every finite k-group to 0; hence it factors through a unique exact functor $\mathcal{C}_k/\mathcal{F}_k \to \underline{\mathcal{C}}_{k'}$. This yields the existence and exactness of $\underline{\otimes}_k k'$. □

Lemma 3.10. *Let k' be a purely inseparable field extension of k, and G' a k'-group.*

(i) *There exists a smooth k-group G and an epimorphism $f : G' \to G_{k'}$ such that* $\operatorname{Ker}(f)$ *is infinitesimal.*

(ii) *If $G' \subset H_{k'}$ for some k-group H, then there exists a k-subgroup $G \subset H$ such that $G' \subset G_{k'}$ and $G_{k'}/G'$ is infinitesimal.*

Proof. (i) Let n be a positive integer and consider the nth relative Frobenius morphism

$$F^n_{G'/k'} : G' \longrightarrow G'^{(p^n)}.$$

Recall that the quotient $G'/\operatorname{Ker}(F^n_{G'/k'})$ is smooth for $n \gg 0$. Since $\operatorname{Ker}(F^n_{G'/k'})$ is infinitesimal, we may assume that G' is smooth. Then $F^n_{G'/k'}$ is an epimorphism in view of [SGA3, VIIA, Cor. 8.3.1].

Next, note that G' is defined over some finite subextension k'' of k', i.e., there exists a k''-subgroup G'' such that $G' = G'' \otimes_{k''} k'$. By transitivity of base change, we may thus assume that k' is finite over k. Let $q := [k' : k]$, then $q = p^n$, where $p = \operatorname{char}(k)$ and n is a positive integer; also, $k'^q \subset k$. Consider again the morphism $F^n_{G'/k'}$; then by construction, $G'^{(p^n)} \cong G' \otimes_{k'} k'$, where k' is sent to itself via the qth power map. Thus, $G'^{(p^n)} \cong G_{k'}$, where G denotes the k-group $G' \otimes_{k'} k$; here k' is sent to k via the qth power map again. So the induced map $G' \to G_{k'}$ is the desired morphism.

(ii) As above, we may reduce to the case where k' is finite over k. Then the statement follows from [Br15b, Lem. 4.3.5]. □

We now are in a position to prove Theorem 1 (iv):

Theorem 3.11. *Let k' be a purely inseparable field extension of k. Then the base change functor $\underline{\otimes}_k k' : \underline{\mathcal{C}}_k \to \underline{\mathcal{C}}_{k'}$ is an equivalence of categories.*

Proof. By Lemma 3.10, every k'-group G' is isogenous to $G_{k'}$ for some smooth k-group G. It follows that $\underline{\otimes}_k k'$ is essentially surjective.

Next, let G, H be smooth connected k-groups, and $\underline{f} : G \to H$ a $\underline{\mathcal{C}}_k$-morphism, represented by a $\mathcal{C}_k$-morphism $f : G \to H/H'$ for some finite k-subgroup $H' \subset H$. If $\underline{f}_{k'} : G_{k'} \to H_{k'}$ is zero in $\underline{\mathcal{C}}_{k'}$, then the image of $f_{k'} : G_{k'} \to H_{k'}/H'_{k'}$ is finite. By Lemma 2.1, it follows that $f_{k'} = 0$. This shows that $\underline{\otimes}_k k'$ is faithful.

Finally, we check that $\underline{\otimes}_k k'$ is full. Let again G, H be smooth connected k-groups, and let $\underline{f} \in \mathrm{Hom}_{\underline{\mathcal{C}}_{k'}}(G_{k'}, H_{k'})$. We show that there exists a finite k-subgroup $H' \subset H$ and a k-morphism $\varphi : G \to H/H'$ such that $\varphi_{k'}$ represents $\underline{f}$. For this, we may replace H with its quotient by any finite k-subgroup.

Choose a representative $f : G_{k'} \to H_{k'}/H''$ of $\underline{f}$, where $H'' \subset H_{k'}$ is a finite k'-subgroup. By Lemma 3.10, there exists a k-subgroup $I \subset H$ such that $H'' \subset I_{k'}$ and $I_{k'}/H''$ is finite; then $I_{k'}$ is finite as well, and hence so is I. We may thus replace H by H/I, and f by its composition with the quotient morphism $H_{k'}/H'' \to H_{k'}/I_{k'} = (H/I)_{k'}$. Then $\underline{f}$ is represented by a morphism $f : G_{k'} \to H_{k'}$.

Consider the graph $\Gamma(f) \subset G_{k'} \times_{k'} H_{k'}$. By Lemma 3.9 again, there exists a k-subgroup $\Delta \subset G \times H$ such that $\Gamma(f) \subset \Delta_{k'}$ and $\Delta_{k'}/\Gamma(f)$ is finite. Then the intersection $\Delta_{k'} \cap (e_G \times H_{k'})$ is finite, since $\Gamma(f) \cap (e_G \times H_{k'})$ is zero. Thus, $\Delta \cap (e_G \times H)$ is finite as well; equivalently, the k-group $H' := H \cap (e_G \times \mathrm{id})^{-1}(\Delta)$ is finite. Denoting by Γ the image of Δ in $G \times H/H'$, we have a cartesian square

$$\begin{array}{ccc} \Delta & \longrightarrow & G \times H \\ \big\downarrow & & \big\downarrow \\ \Gamma & \longrightarrow & G \times H/H', \end{array}$$

where the horizontal arrows are closed immersions, and the left (resp. right) vertical arrow is the quotient by $\Delta \cap (e_G \times H)$ (resp. by H' acting on H via addition). So Γ is a k-subgroup of $G \times H/H'$, and $\Gamma \cap (e_G \times H/H')$ is zero; in other words, the projection $\pi : \Gamma \to G$ is a closed immersion. Since G is smooth and connected, and $\dim(\Gamma) = \dim(\Delta) = \dim \Gamma(f) = \dim(G)$, it follows that π is an isomorphism. In other words, Γ is the graph of a k-morphism $\varphi : G \to H/H'$. Since the above cartesian square lies in a push-out diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & G \times H & \xrightarrow{\varphi - q} & H/H' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & G \times H/H' & \xrightarrow{\varphi - \mathrm{id}} & H/H' & \longrightarrow & 0, \end{array}$$

where $q : H \to H/H'$ denotes the quotient morphism, it follows that $\Delta = \ker(\varphi - q)$. As $\Gamma(f) \subset \Delta_{k'}$, we see that $\varphi_{k'} = q_{k'} \circ f$. Thus, $\underline{f}$ is represented by $\varphi_{k'}$; this completes the proof of the fullness assertion. $\square$

Remarks 3.12. (i) Likewise, the base change functor induces equivalences of categories $\underline{\mathcal{U}}_k \to \underline{\mathcal{U}}_{k'}$, $\underline{\mathcal{T}}_k \to \underline{\mathcal{T}}_{k'}$, $\underline{\mathcal{L}}_k \to \underline{\mathcal{L}}_{k'}$, and $\underline{\mathcal{A}}_k \to \underline{\mathcal{A}}_{k'}$. For tori, this follows much more directly from the anti-equivalence of $\underline{\mathcal{T}}_k$ with the category of rational representations of the absolute Galois group of k, see Proposition 4.1.

(ii) In particular, the category $\underline{\mathcal{U}}_k$ is equivalent to $\underline{\mathcal{U}}_{k_i}$, where k_i denotes the perfect closure of k in $\bar{k}$. Recall from [DG70, §V.1.4] that the category $\mathcal{U}_{k_i}$ is anti-equivalent to the category of those finitely generated modules over the Dieudonné ring $\mathbb{D}_{k_i}$ that are killed by some power of the Verschiebung map V. Moreover, the category $\underline{\mathcal{U}}_{k_i}$ is anti-equivalent to the category of those finitely generated modules over the localization $\mathbb{D}_{(V)}$ that are killed by some power of V; see [DG70, §V.6.7].

By work of Schoeller, Kraft, and Takeuchi (see [Sc72, Kr75, Tak75]), the category $\mathcal{U}_k$ is anti-equivalent to a category of finitely generated modules over a certain k-algebra, which generalizes the Dieudonné ring but seems much less tractable.

4 Tori, abelian varieties, and homological dimension

4.1 Tori

Denote by $\mathcal{M}$ (resp. $\mathcal{FM}$) the full subcategory of $\mathcal{C}$ with objects the groups of multiplicative type (resp. the finite groups of multiplicative type). Since $\mathcal{F}$ and $\mathcal{FM}$ are stable under taking subobjects, quotients and extensions, we may form the quotient abelian category $\mathcal{M}/\mathcal{FM}$, as in §3.1. One may readily check that $\mathcal{M}/\mathcal{FM}$ is a full subcategory of $\mathcal{C}/\mathcal{F}$.

Let $\underline{\mathcal{T}}$ be the full subcategory of $\mathcal{M}/\mathcal{FM}$ with objects the tori. Since these are the smooth connected objects of $\mathcal{M}$, one may check as in Lemma 3.1 that the inclusion of $\underline{\mathcal{T}}$ in $\mathcal{M}/\mathcal{FM}$ is an equivalence of categories. The remaining statements of Lemma 3.1 also adapt to this setting; note that we may replace the direct limits over all finite subgroups with those over all n-torsion subgroups, since tori are divisible. Also, Proposition 3.6 yields natural isomorphisms

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(T_1, T_2) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathcal{T}}}(T_1, T_2)$$

for any tori T_1, T_2.

By assigning with each group of multiplicative type G its character group,

$$X(G) := \mathrm{Hom}_{k_s}(G_{k_s}, \mathbb{G}_{m,k_s}),$$

one obtains an anti-equivalence between $\mathcal{M}$ (resp. $\mathcal{FM}$) and the category of finitely generated (resp. finite) commutative groups equipped with a continuous action of the Galois group Γ; see [DG70, Thm. IV.1.3.6]. Thus, the assignement

$$X_{\mathbb{Q}} : G \longmapsto \mathbb{Q} \otimes_{\mathbb{Z}} X(G) =: X(G)_{\mathbb{Q}}$$

yields a contravariant exact functor from $\mathcal{M}$ to the category $\mathrm{Rep}_{\mathbb{Q}}(\Gamma)$ of finite-dimensional $\mathbb{Q}$-vector spaces equipped with a continuous representation of Γ; moreover, every finite group of multiplicative type is sent to 0. This yields in turn a contravariant exact functor

$$\underline{X}_{\mathbb{Q}} : \underline{\mathcal{T}} \longrightarrow \mathrm{Rep}_{\mathbb{Q}}(\Gamma).$$

Proposition 4.1. *The functor $\underline{X}_{\mathbb{Q}}$ is an anti-equivalence of categories. In particular, the category $\underline{\mathcal{T}}$ is semi-simple, and $\mathrm{Hom}_{\underline{\mathcal{T}}}(T_1, T_2)$ is a finite-dimensional $\mathbb{Q}$-vector space for any tori T_1, T_2.*

Proof. Given a finite-dimensional $\mathbb{Q}$-vector space V equipped with a continuous action of Γ, there exists a finitely generated Γ-stable subgroup $M \subset V$ which spans V; thus, $V \cong X(T)_{\mathbb{Q}}$, where T denotes the torus with character group M. So $\underline{X}_{\mathbb{Q}}$ is essentially surjective.

Given two tori T_1, T_2, the natural isomorphism $\mathrm{Hom}_{\mathcal{M}}(T_1, T_2) \cong \mathrm{Hom}^{\Gamma}(X(T_2), X(T_1))$ yields an isomorphism

$$\mathrm{Hom}_{\underline{\mathcal{T}}}(T_1, T_2) \cong \mathrm{Hom}^{\Gamma}(X(T_2)_{\mathbb{Q}}, X(T_1)_{\mathbb{Q}}).$$

It follows that $\underline{X}_{\mathbb{Q}}$ is fully faithful. □

Lemma 4.2. (i) *Every algebraic group G has a unique maximal torus, $T(G)$.*

(ii) *Every morphism of algebraic groups $u : G \to H$ sends $T(G)$ to $T(H)$.*

(iii) *The formation of $T(G)$ commutes with base change under field extensions.*

Proof. (i) This follows from the fact that $T_1 + T_2$ is a torus for any subtori $T_1, T_2 \subset G$.

(ii) Just note that the image of a torus under any morphism is still a torus.

(iii) Consider an algebraic group G, its maximal torus T, and a field extension K of k. If $\mathrm{char}(k) = 0$, then Theorem 2.9 implies that G/T is a an extension of an abelian variety by a product $M \times U$, where M is finite and U unipotent. As a consequence, a similar assertion holds for G_K/T_K; it follows that G_K/T_K contains no nonzero torus, and hence T_K is the maximal torus of G_K. On the other hand, if $\mathrm{char}(k) > 0$, then G/T is a 3-step extension of a unipotent group by an abelian variety by a finite group, in view of Theorem 2.11. It follows similarly that T_K is the maximal torus of G_K. □

By Lemma 4.2, the assignement $G \mapsto T(G)$ yields a functor

$$T : \mathcal{C} \longrightarrow \mathcal{T},$$

the *functor of maximal tori.* This functor is neither left exact, nor right exact, as seen from the exact sequence

$$0 \longrightarrow G[n] \longrightarrow G \xrightarrow{n_G} G \longrightarrow 0,$$

where G is a nonzero torus and n a nonzero integer. But T is exact up to finite groups, as shown by the following:

Lemma 4.3. *Every exact sequence in $\mathcal{C}$*

$$0 \longrightarrow G_1 \xrightarrow{u} G_2 \xrightarrow{v} G_3 \longrightarrow 0$$

yields a complex in $\mathcal{C}$

$$0 \longrightarrow T(G_1) \xrightarrow{T(u)} T(G_2) \xrightarrow{T(v)} T(G_3) \longrightarrow 0,$$

where $T(u)$ is a monomorphism, $T(v)$ an epimorphism, and $\mathrm{Ker}\, T(v)/\mathrm{Im}\, T(u)$ *is finite.*

Proof. Clearly, $T(u)$ is a monomorphism. Also, the group

$$\operatorname{Ker} T(v)/\operatorname{Im} T(u) = T(G_2) \cap u(G_1)/u(T(G_1))$$

is the quotient of a group of multiplicative type by its maximal torus, and hence is finite.

To show that $T(v)$ is an epimorphism, we may replace G_2 with $v^{-1}(T(G_3))$, and hence assume that G_3 is a torus. Next, we may replace G_2 with $G_2/T(G_2)$, and hence assume that $T(G_2)$ is zero. We then have to check that G_3 is zero.

If $\operatorname{char}(k) = 0$, then there is an exact sequence

$$0 \longrightarrow M \times U \longrightarrow G_2 \longrightarrow A \longrightarrow 0$$

as in Theorem 2.9, where M is finite. Thus, every morphism $G_2 \to G_3$ has finite image. Since $v : G_2 \to G_3$ is an epimorphism, it follows that $G_3 = 0$. On the other hand, if $\operatorname{char}(k) > 0$, then there are exact sequences

$$0 \longrightarrow H \longrightarrow G_2 \longrightarrow U \longrightarrow 0, \quad 0 \longrightarrow M \longrightarrow H \longrightarrow A \longrightarrow 0$$

as in Theorem 2.11, where M is finite. This implies again that every morphism $G_2 \to G_3$ has finite image, and hence that $G_3 = 0$. □

Proposition 4.4. (i) *The functor of maximal tori yields an exact functor*

$$\underline{T} : \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{T}},$$

right adjoint to the inclusion $\underline{\mathcal{T}} \to \underline{\mathcal{C}}$. Moreover, $\underline{T}$ commutes with base change under field extensions.

(ii) *Every torus is a projective object in $\underline{\mathcal{C}}$.*

Proof. (i) Composing T with the functor $\mathcal{T} \to \underline{\mathcal{T}}$ induced by the quotient functor Q, we obtain an exact functor $\mathcal{C} \to \underline{\mathcal{T}}$ (Lemma 4.3), which sends every finite group to 0. So this functor factors through an exact functor $\underline{T} : \underline{\mathcal{C}} \to \underline{\mathcal{T}}$. The adjointness assertion follows from the natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(T, G) \cong \operatorname{Hom}_{\mathcal{T}}(T, T(G))$$

for any torus T and any algebraic group G, which yields a natural isomorphism

$$\operatorname{Hom}_{\underline{\mathcal{C}}}(T, G) \cong \operatorname{Hom}_{\underline{\mathcal{T}}}(T, T(G))$$

by using Lemmas 3.1 and 4.3. Finally, the assertion on field extensions is a direct consequence of Lemma 4.2.

(ii) This follows by arguing as in the proof of Proposition 3.8 (iii). □

4.2 Abelian varieties

Denote by $\mathcal{P}$ the full subcategory of $\mathcal{C}$ with objects the proper groups (i.e., those algebraic groups G such that the structure map $G \to \mathrm{Spec}(k)$ is proper). Clearly, $\mathcal{P}$ is stable by subobjects, quotients and extensions; it also contains the category $\mathcal{F}$ of finite groups. We may thus form the quotient abelian category $\mathcal{P}/\mathcal{F}$, which is a full subcategory of $\mathcal{C}/\mathcal{F}$.

Next, let $\underline{\mathcal{A}}$ be the full subcategory of $\mathcal{P}/\mathcal{F}$ with objects the abelian varieties. As in §4.1, the inclusion of $\underline{\mathcal{A}}$ in $\mathcal{P}/\mathcal{F}$ is an equivalence of categories, and the remaining statements of Lemma 3.1 adapt to this setting. Also, Proposition 3.6 yields natural isomorphisms

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A_1, A_2) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathcal{A}}}(A_1, A_2)$$

for any abelian varieties A_1, A_2. As a consequence, $\mathrm{Hom}_{\underline{\mathcal{A}}}(A_1, A_2)$ is a finite-dimensional $\mathbb{Q}$-vector space. Moreover, the category $\underline{\mathcal{A}}$ is semi-simple, in view of the Poincaré complete reducibility theorem (which holds over an arbitrary field, see [Co06, Cor. 3.20] or [Br15b, Cor. 4.2.6]).

Lemma 4.5. (i) *Every smooth connected algebraic group G has a largest abelian variety quotient,*

$$\alpha = \alpha_G : G \longrightarrow A(G).$$

(ii) *Every morphism $u : G \to H$, where H is smooth and connected, induces a unique morphism $A(u) : A(G) \to A(H)$ such that the square*

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[d, "\alpha_G"'] & H \arrow[d, "\alpha_H"] \\ A(G) \arrow[r, "A(u)"] & A(H) \end{tikzcd}$$

commutes.

(iii) *For any field extension K of k, the natural morphism $A(G_K) \to A(G)_K$ is an isomorphism if* $\mathrm{char}(k) = 0$, *and an isogeny if* $\mathrm{char}(k) > 0$.

Proof. (i) and (ii) Both assertions are obtained by standard arguments, already used at the beginning of the proof of Theorem 2.11.

(iii) By Theorem 2.3, we have an exact sequence

$$0 \longrightarrow L(G) \longrightarrow G \xrightarrow{\alpha} A(G) \longrightarrow 0,$$

where $L(G)$ is linear and connected. This yields an exact sequence

$$0 \longrightarrow L(G)_K \longrightarrow G_K \longrightarrow A(G)_K \longrightarrow 0.$$

Thus, $L(G_K) \subset L(G)_K$ and we obtain an exact sequence

$$0 \longrightarrow L(G)_K/L(G_K) \longrightarrow A(G_K) \longrightarrow A(G)_K \longrightarrow 0.$$

Sinc $L(G)_K$ is linear, the quotient $L(G)_K/L(G_K)$ must be finite; this yields the assertion when $\mathrm{char}(k) > 0$.

When $\mathrm{char}(k) = 0$, we may characterize $L(G)$ as the largest connected linear subgroup of G. It follows that $L(G)_K \subset L(G_K)$; hence equality holds, and $A(G_K) \xrightarrow{\cong} A(G)_K$. □

In view of Lemma 4.5, the assignement $G \mapsto A(G)$ yields a functor

$$A : \mathcal{C} \longrightarrow \mathcal{A},$$

the *Albanese functor*. Like the functor of maximal tori, A is neither left exact nor right exact, but it is exact up to finite groups:

Lemma 4.6. *Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence in $\mathcal{C}$, where G_1, G_2, G_3 are smooth and connected. Then we have a commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_1 & \xrightarrow{u} & G_2 & \xrightarrow{v} & G_3 & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & \\
0 & \longrightarrow & A(G_1) & \xrightarrow{A(u)} & A(G_2) & \xrightarrow{A(v)} & A(G_3) & \longrightarrow & 0,
\end{array}$$

where $A(v)$ is an epimorphism, and $\operatorname{Ker} A(u)$, $\operatorname{Ker} A(v)/\operatorname{Im} A(u)$ *are finite.*

Proof. Clearly, $A(v)$ is an epimorphism. Let $L_i := \operatorname{Ker}(\alpha_i)$ for $i = 1, 2, 3$; then each L_i is connected and linear by Theorem 2.3. We have isomorphisms $\operatorname{Ker} A(u) \cong u^{-1}(L_2)/L_1$, $\operatorname{Im} A(u) \cong u(G_1)/u(G_1) \cap L_2$ and $\operatorname{Ker} A(v) \cong v^{-1}(L_3)/L_2$. Since $u^{-1}(L_2)$ is linear, $\operatorname{Ker} A(u)$ is linear as well; it is also proper, and hence finite. Also,

$$\operatorname{Ker} A(v)/\operatorname{Im} A(u) \cong v^{-1}(L_3)/L_2 + u(G_1)$$

is a quotient of $v^{-1}(L_3)/u(G_1) \cong L_3$. It follows similarly that $\operatorname{Ker} A(v)/\operatorname{Im} A(u)$ is finite. □

We may now state a dual version of Proposition 4.4 for the Albanese functor:

Proposition 4.7. (i) *The Albanese functor yields an exact functor*

$$\underline{A} : \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{A}},$$

which is left adjoint to the inclusion $\underline{\mathcal{A}} \to \underline{\mathcal{C}}$. Moreover, $\underline{A}$ commutes with base change under field extensions.

(ii) *Every abelian variety is an injective object in $\underline{\mathcal{C}}$.*

The proof is entirely similar to that of Proposition 4.4, and will be omitted.

Remarks 4.8. (i) Denote by $\widehat{A}$ the dual of an abelian variety A. Then the assignement $A \mapsto \widehat{A}$ yields a contravariant endofunctor of $\mathcal{A}$, which is involutive and preserves isogenies and finite products. As an easy consequence, we obtain a contravariant endofunctor of $\underline{\mathcal{A}}$, which is involutive and exact. Note that each abelian variety is (non-canonically) $\underline{\mathcal{A}}$-isomorphic to its dual, via the choice of a polarization.

(ii) Let K be a field extension of k. Then the assignement

$$A \longmapsto \mathbb{Q} \otimes_{\mathbb{Z}} A(K) =: A(K)_{\mathbb{Q}}$$

yields a functor from $\mathcal{A}$ to the category of $\mathbb{Q}$-vector spaces (possibly of infinite dimension), which preserves finite products. Moreover, each isogeny $f : A \to B$ yields an isomorphism $A(K)_{\mathbb{Q}} \xrightarrow{\cong} B(K)_{\mathbb{Q}}$, since this holds for the multiplication maps n_A. Thus, we obtain an exact functor from $\underline{\mathcal{A}}$ to the category of $\mathbb{Q}$-vector spaces.

4.3 Vanishing of extension groups

In this subsection, we prove the assertion (v) of Theorem 1. We first collect general vanishing results for extension groups in $\underline{\mathcal{C}}$:

Lemma 4.9. *Let G be a smooth connected algebraic group, U a smooth connected unipotent group, A an abelian variety, and T a torus.*

(i) $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(T, G) = 0 = \mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, A) = 0$ *for all* $n \geq 1$.

(ii) *If* $\mathrm{char}(k) = 0$, *then* $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(U, G) = 0$ *for all* $n \geq 1$.

(iii) *If* $\mathrm{char}(k) > 0$ *and G is divisible, then* $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(U, G) = 0 = \mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, U)$ *for all* $n \geq 0$.

(iv) *If G is linear, then* $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, T) = 0$ *for all* $n \geq 1$. *If in addition* $\mathrm{char}(k) = 0$, *then* $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, U) = 0$ *for all* $n \geq 1$ *as well.*

Proof. (i) Just recall that T is projective in $\underline{\mathcal{C}}$ (Proposition 4.4), and A is injective in $\underline{\mathcal{C}}$ (Proposition 4.7).

(ii) Likewise, U is projective in $\underline{\mathcal{C}}$ by Proposition 3.8.

(iii) Since U is unipotent, there exists a positive integer m such that $p^m_G = 0$. It follows that both groups $\mathrm{Ext}^m_{\underline{\mathcal{C}}}(U, G)$ and $\mathrm{Ext}^m_{\underline{\mathcal{C}}}(G, U)$ are p^m-torsion (see e.g. [Oo66, Lem. I.3.1]). But these groups are also modules over $\mathrm{End}_{\underline{\mathcal{C}}}(G)$, and hence $\mathbb{Q}$-vector spaces by Proposition 3.6. This yields the assertion.

(iv) By Proposition 2.8 and the long exact sequence for Ext groups, we may assume that G is unipotent or a torus. In the latter case, both assertions follows from (i); in the former case, the first assertion follows from (ii) and (iii), and the second assertion, from the fact that unipotent groups are just vector spaces. □

Next, recall that $\mathrm{hd}(\underline{\mathcal{C}}) \geq 1$ (Examples 3.4). So, to complete the proof of the assertion (v) of Theorem 1, it suffices to show the following:

Lemma 4.10. *For any smooth connected algebraic groups G, H and any integer $n \geq 2$, we have* $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, H) = 0$.

Proof. Let $\eta \in \mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, H)$, where $n \geq 3$. Then η is represented by an exact sequence

$$0 \longrightarrow H \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_n \longrightarrow G \longrightarrow 0,$$

which we may cut into two exact sequences

$$0 \longrightarrow H \longrightarrow G_1 \longrightarrow G_2 \longrightarrow K \longrightarrow 0, \quad 0 \longrightarrow K \longrightarrow G_3 \longrightarrow \cdots \longrightarrow G_n \longrightarrow G \longrightarrow 0.$$

Thus, η can be written as a Yoneda product $\eta_1 \cup \eta_2$, where $\eta_1 \in \mathrm{Ext}^2(G, K)$ and $\eta_2 \in \mathrm{Ext}^{n-2}(K, H)$. So it suffices to show the assertion when $n = 2$.

Using the long exact sequences for Ext groups, we may further reduce to the case where G, H are simple objects in $\underline{\mathcal{C}}$, i.e., $\mathbb{G}_a$, simple tori T, or simple abelian varieties A (Proposition 3.2). In view of Lemma 4.9, it suffices in turn to check that

(i) $\mathrm{Ext}^2_{\underline{\mathcal{C}}}(A, T) = 0$,

(ii) $\mathrm{Ext}^2_{\underline{\mathcal{C}}}(A, \mathbb{G}_a) = 0$ when $\mathrm{char}(k) = 0$,

(iii) $\mathrm{Ext}^2_{\underline{\mathcal{C}}}(\mathbb{G}_a, \mathbb{G}_a) = 0$ when $\mathrm{char}(k) > 0$.

For (i), we adapt the argument of [Oo66, Prop. II.12.3]. Let $\eta \in \mathrm{Ext}^2_{\underline{\mathcal{C}}}(A, T)$ be represented by an exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow T \longrightarrow G_1 \longrightarrow G_2 \longrightarrow A \longrightarrow 0.$$

As above, $\eta = \eta_1 \cup \eta_2$, where η_1 denotes the class of the extension

$$0 \longrightarrow T \longrightarrow G_1 \longrightarrow K \longrightarrow 0,$$

and η_2 that of the extension

$$0 \longrightarrow K \longrightarrow G_2 \longrightarrow A \to 0.$$

Since K is smooth and connected (as a quotient of G_1), Theorem 2.3 yields an exact sequence in $\mathcal{C}$

$$0 \longrightarrow L \longrightarrow K \longrightarrow B \to 0,$$

where B is an abelian variety, and L is connected and linear (but not necessarily smooth). Using Proposition 2.6, we obtain a surjective morphism with finite kernel $L \to L_1$, where L_1 is smooth, connected and linear. This yields an exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow L_1 \longrightarrow K \longrightarrow B \to 0.$$

As $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, B) = 0$, the natural map $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, L_1) \to \mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, K)$ is surjective. Thus, there exists a push-out diagram of exact sequences in $\underline{\mathcal{C}}$

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & K \arrow[r] & G_2 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Consider the pull-back diagram of exact sequences in $\underline{\mathcal{C}}$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\mathrm{id}"'] & H_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & G_1 \arrow[r] & K \arrow[r] & 0.
\end{tikzcd}$$

This yields a commutative diagram of exact sequences in $\underline{\mathcal{C}}$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\mathrm{id}"'] & H_1 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & T \arrow[r] & G_1 \arrow[r] & G_2 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Thus, we have $\eta = \kappa_1 \cup \kappa_2$ in $\mathrm{Ext}^2_{\underline{\mathcal{C}}}(A, T)$, where κ_1 denotes the class of the extension

$$0 \longrightarrow T \longrightarrow H_1 \longrightarrow L_1 \longrightarrow 0,$$

and κ_2, that of the extension

$$0 \longrightarrow L_1 \longrightarrow H_2 \longrightarrow A \longrightarrow 0.$$

But $\kappa_1 = 0$ in view of Lemma 4.9. Thus, $\eta = 0$; this completes the proof of (i).

For (ii), we adapt the above argument: just replace T with $\mathbb{G}_a$ and use the vanishing of $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(L, \mathbb{G}_a)$ for L linear (Lemma 4.9).

Finally, for (iii), it suffices to show that $\mathrm{Ext}^2_{\underline{\mathcal{U}}}(\mathbb{G}_a, \mathbb{G}_a) = 0$. Also, we may assume that k is perfect, in view of Theorem 3.11. Then we conclude by the vanishing of $\mathrm{Ext}^2_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a)$ (see [DG70, V.1.5.1, V.1.5.2]). □

Remark 4.11. When k is perfect, the groups $\mathrm{Ext}^n_{\mathcal{C}}(G, H)$ are torsion for all $n \geq 2$ and all algebraic groups G, H, in view of [Mi70, Cor., p. 439]. In fact, this assertion extends to an arbitrary field k: indeed, it clearly holds when G or H is finite, or more generally m-torsion for some positive integer m. Using Proposition 2.8, one may thus reduce to the case when G, H are simple objects of $\underline{\mathcal{C}}$. Then the assertion is obtained by combining Proposition 3.6, Lemma 4.9, and the proof of Lemma 4.10.

5 Structure of isogeny categories

5.1 Vector extensions of abelian varieties

In this subsection, we assume that $\mathrm{char}(k) = 0$. Recall that a *vector extension* of an abelian variety A is an algebraic group G that lies in an extension

$$\xi : \quad 0 \longrightarrow U \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where U is unipotent. Then $U = R_u(G)$ and $A = A(G)$ are uniquely determined by G; also, the extension ξ has no non-trivial automorphisms, since $\mathrm{Hom}_{\mathcal{C}}(A, U) = 0$. Thus, the data of the algebraic group G and the extension ξ are equivalent.

We denote by $\mathcal{V}$ the full subcategory of $\mathcal{C}$ with objects the vector extensions (of all abelian varieties). By Theorems 2.3 and 2.4, the objects of $\mathcal{V}$ are exactly those smooth connected algebraic groups that admit no non-zero subtorus. In view of Lemmas 4.2 and 4.3, this implies readily:

Lemma 5.1. (i) *Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence in $\mathcal{C}$, where G_2 is connected. Then G_2 is an object of $\mathcal{V}$ if and only if so are G_3 and G_1^0.*

(ii) *Let $f : G \to H$ be an isogeny of connected algebraic groups. Then G is an object of $\mathcal{V}$ if and only if so is H.*

(iii) *Let K be a field extension of k, and G an algebraic k-group. Then G is an object of $\mathcal{V}_k$ if and only if G_K is an object of $\mathcal{V}_K$.*

Next, recall from [Ro58] or [MM74, §1.9] that every abelian variety A has a universal vector extension,

$$\xi(A) : \quad 0 \longrightarrow U(A) \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

where $U(A)$ is the additive group of the vector space $H^1(A, \mathcal{O}_A)^*$; moreover, $\dim U(A) = \dim A$. Also, $E(A)$ is anti-affine, i.e., every morphism from $E(A)$ to a linear algebraic group is zero (see e.g. [Br15b, Prop. 5.5.8]).

Proposition 5.2. (i) *The assignements $A \to E(A)$, $A \mapsto U(A)$ yield exact functors*

$$E : \mathcal{V} \longrightarrow \mathcal{A}, \quad U : \mathcal{V} \longrightarrow \mathcal{U},$$

which commute with base change under field extensions.

(ii) *For any morphism $f : A \to B$ of abelian varieties, the map $U(f) : U(A) \to U(B)$ is the dual of the pull-back morphism $f^* : H^1(B, \mathcal{O}_B) \to H^1(A, \mathcal{O}_A)$. Moreover, $U(f)$ is zero (resp. an isomorphism) if and only if f is zero (resp. an isogeny).*

(iii) *E is left adjoint to the inclusion of $\mathcal{A}$ in $\mathcal{V}$.*

Proof. We prove (i) and (ii) simultaneously. Let $f : A \to B$ be a morphism of abelian varieties. Consider the pull-back diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & U(B) \arrow[r] \arrow[d, "\mathrm{id}"'] & F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & U(B) \arrow[r] & E(B) \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

By the universal property of $\xi(A)$, we have a commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & U(A) \arrow[r] \arrow[d] & E(A) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\ 0 \arrow[r] & U(B) \arrow[r] & F \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

and hence another such diagram,

$$\begin{tikzcd} 0 \arrow[r] & U(A) \arrow[r] \arrow[d] & E(A) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & U(B) \arrow[r] & E(B) \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

This yields morphisms $E(f)$ and $U(f)$.

Next, let $\eta \in H^1(B, \mathcal{O}_B)$, so that we have a push-out diagram of extensions

$$\begin{tikzcd} 0 \arrow[r] & U(B) \arrow[r] \arrow[d, "\eta"'] & E(B) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\ 0 \arrow[r] & \mathbb{G}_a \arrow[r] & E_\eta \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

By construction, the pull-back of E_η by f is the push-out of $\xi(A)$ by $\eta \circ U(f) : U(A) \to \mathbb{G}_a$. Hence $U(g \circ f) = U(g) \circ U(f)$, and in turn $E(g \circ f) = E(g) \circ E(f)$. This yields the functors E and U; also, note that $U(f)$ is the dual of $f^* : H^1(B, \mathcal{O}_B) \to H^1(A, \mathcal{O}_A)$.

Since the formation of the universal vector extension commutes with base change under field extensions, the functors E and U commute with such base change as well. We now show that these functors are exact. Clearly, they are additive. Consider an exact sequence of abelian varieties $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. By the Poincaré complete reducibility theorem, there exists a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f'"] \arrow[d, "\mathrm{id}"'] & B' \arrow[r, "g'"] \arrow[d, "u"'] & C' \arrow[r] \arrow[d, "v"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0,
\end{tikzcd}$$

where u, v are isogenies, and the top exact sequence splits in $\mathcal{A}$. Since $U(u), U(v)$ are isomorphisms, this yields the exactness of U, and in turn the exactness of E, completing the proof of (i).

To complete the proof of (ii), recall the canonical isomorphism $H^1(A, \mathcal{O}_A) \cong \mathrm{Lie}(\widehat{A})$, where the right-hand side denotes the Lie algebra of the dual abelian variety. This isomorphism identifies f^* with $\mathrm{Lie}(\hat{f})$, where $\hat{f} : \widehat{B} \to \widehat{A}$ denotes the dual morphism of f. As a consequence,

$$U(f) = 0 \Leftrightarrow f^* = 0 \Leftrightarrow \hat{f} = 0 \Leftrightarrow f = 0,$$

where the second equivalence holds since $\mathrm{char}(k) = 0$, and the third one follows from biduality of abelian varieties. Likewise, $U(f)$ is an isomorphism if and only if f^* is an isomorphism; equivalently, $\hat{f}$ is an isogeny, i.e., f is an isogeny.

(iii) Given a vector extension $0 \to U \to G \to A(G) \to 0$, we need to show that the map

$$\alpha : \mathrm{Hom}_{\mathcal{V}}(E(A), G) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, A(G)), \quad u \longmapsto A(u)$$

is an isomorphism.

Consider a morphism $u : E(A) \to G$ such that $A(u) = 0$. Then u factors through a morphism $E(A) \to R_u(G)$, and hence $u = 0$ as $E(A)$ is anti-affine.

Next, consider a morphism $v : A \to A(G)$. By (i), we have a commutative square

$$\begin{tikzcd}
E(A) \arrow[r, "E(v)"] \arrow[d] & E(A(G)) \arrow[d] \\
A \arrow[r, "v"] & A(G).
\end{tikzcd}$$

Also, the universal property of $\xi(A)$ yields a commutative square

$$\begin{tikzcd}
E(A(G)) \arrow[r, "\delta"] \arrow[d] & G \arrow[d, "\alpha_G"'] \\
A(G) \arrow[r, "\mathrm{id}"] & A(G).
\end{tikzcd}$$

Thus, $w := E(v) \circ \delta \in \mathrm{Hom}_{\mathcal{V}}(E(A), G)$ satisfies $\alpha(v) = u$. □

Denote by $\underline{\mathcal{V}}$ the isogeny category of vector extensions, that is, the full subcategory of $\underline{\mathcal{C}}$ with the same objects as $\mathcal{V}$. Then $\underline{\mathcal{V}}$ is an abelian category in view of Lemma 5.1. Also, Proposition 3.6 yields natural isomorphisms

$$\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(G_1, G_2) \xrightarrow{\cong} \operatorname{Hom}_{\underline{\mathcal{V}}}(G_1, G_2), \quad \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Ext}^1_{\mathcal{C}}(G_1, G_2) \xrightarrow{\cong} \operatorname{Ext}^1_{\underline{\mathcal{V}}}(G_1, G_2)$$

for any objects G_1, G_2 of $\underline{\mathcal{V}}$.

Corollary 5.3. (i) *The functors $E : \mathcal{A} \to \mathcal{V}$, $U : \mathcal{A} \to \mathcal{U}$ yield exact functors*

$$\underline{E} : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{V}}, \quad \underline{U} : \underline{\mathcal{A}} \longrightarrow \mathcal{U},$$

which commute with base change under field extensions. Moreover, $\underline{E}$ is left adjoint to the Albanese functor $\underline{A} : \underline{\mathcal{V}} \to \underline{\mathcal{A}}$.

(ii) *The universal vector extension of any abelian variety is a projective object of $\underline{\mathcal{V}}$.*

Proof. (i) This follows from Proposition 5.2.

(ii) We have canonical isomorphisms for any vector extension G:

$$\operatorname{Hom}_{\underline{\mathcal{V}}}(E(A), G) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{V}}(E(A), G) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(A, G) \cong \operatorname{Hom}_{\underline{\mathcal{A}}}(A, G),$$

where the first and third isomorphisms follow from Proposition 3.6, and the second one from Proposition 5.2 again. Since $\underline{\mathcal{A}}$ is semi-simple, it follows that the functor $G \mapsto \operatorname{Hom}_{\underline{\mathcal{V}}}(E(A), G)$ is exact. □

Next, let G be an object of $\mathcal{V}$. Form and label the commutative diagram of exact sequences in $\mathcal{C}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(A) & \xrightarrow{\iota} & E(A) & \longrightarrow & A & \longrightarrow & 0 \\ & & \gamma\big\downarrow & & \delta\big\downarrow & & \mathrm{id}\big\downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

where $U = U(G)$, $A = A(G)$, and $\gamma = \gamma_G$ classifies the bottom extension. This yields an exact sequence in $\mathcal{C}$

$$\xi : \quad 0 \longrightarrow U(A) \xrightarrow{\gamma - \iota} U \times E(A) \longrightarrow G \longrightarrow 0.$$

Proposition 5.4. *Keep the above notation.*

(i) *ξ yields a projective resolution of G in $\underline{\mathcal{V}}$.*

(ii) *For any object H of $\mathcal{V}$, we have an exact sequence*

$$0 \longrightarrow \operatorname{Hom}_{\underline{\mathcal{V}}}(G, H) \xrightarrow{\varphi} \operatorname{Hom}_{\mathcal{U}}(U(G), U(H)) \times \operatorname{Hom}_{\underline{\mathcal{A}}}(A(G), A(H))$$

$$\xrightarrow{\psi} \operatorname{Hom}_{\mathcal{U}}(U(A(G)), U(H)) \longrightarrow \operatorname{Ext}^1_{\underline{\mathcal{V}}}(G, H) \longrightarrow 0,$$

where $\varphi(\underline{f}) := (U(\underline{f}), \underline{A}(\underline{f}))$, and $\psi(u, v) := u \circ \gamma_G - \gamma_H \circ v$.

Proof. (i) This holds as $U, U(A)$ are projective in $\mathcal{C}$ (Theorem 2.9), and $E(A)$ is projective in $\underline{\mathcal{C}}$ (Proposition 5.2).

(ii) In view of (i), this is a direct consequence of the long exact sequence of extension groups

$$0 \longrightarrow \mathrm{Hom}_{\underline{\mathcal{V}}}(G,H) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{V}}}(U \times E(A), H) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{V}}}(U,H) \longrightarrow \mathrm{Ext}^1_{\underline{\mathcal{V}}}(G,H) \longrightarrow 0$$

associated with the short exact sequence ξ. □

Corollary 5.5. (i) *The indecomposable projective objects of* $\underline{\mathcal{V}}$ *are exactly* $\mathbb{G}_a$ *and the universal vector extensions of simple abelian varieties.*

(ii) *The indecomposable injective objects of* $\underline{\mathcal{V}}$ *are exactly the simple abelian varieties.*

Proof. Let G be an extension of an abelian variety A by a unipotent group U.

(i) If G is projective in $\underline{\mathcal{V}}$, then $\mathrm{Ext}^1_{\underline{\mathcal{V}}}(G, \mathbb{G}_a) = 0$. In view of Proposition 5.4, it follows that the map $\mathrm{Hom}_{\mathcal{U}}(U, \mathbb{G}_a) \to \mathrm{Hom}_{\mathcal{U}}(U(A), \mathbb{G}_a)$, $u \mapsto u \circ \gamma$ is surjective. Equivalently, γ is injective; hence so is $\delta : E(A) \to G$. Identifying $E(A)$ with a subgroup of G, it follows that $G = U + E(A)$, and $U(A) \subset U$. We may choose a complement $V \subset U$ to the subspace $U(A) \subset U$; then $G \cong V \times E(A)$. Conversely, every such product is projective by Proposition 5.2. This yields the assertion.

(ii) If G is injective in $\underline{\mathcal{V}}$, then $\mathrm{Ext}^1_{\underline{\mathcal{V}}}(B, G) = 0$ for any abelian variety B. Choose B non-zero and non-isogenous to A; then the natural map $\mathrm{Ext}^1_{\underline{\mathcal{V}}}(B, U) \to \mathrm{Ext}^1_{\underline{\mathcal{V}}}(B, G)$ is an isomorphism, since $\mathrm{Hom}_{\underline{\mathcal{V}}}(B, A) = 0 = \mathrm{Ext}^1_{\underline{\mathcal{V}}}(B, A)$. Also, $\mathrm{Ext}^1_{\underline{\mathcal{V}}}(B, U) \cong \mathrm{Hom}_{\mathcal{U}}(U(B), U)$ by Proposition 5.4. It follows that $U = 0$, i.e., G is an abelian variety. Conversely, every abelian variety is injective in $\underline{\mathcal{V}}$ by Proposition 4.7. □

We now describe the structure of $\mathcal{V}$ and $\underline{\mathcal{V}}$ in terms of linear algebra. Let $\mathcal{D}$ be the category with objects the triples (A, U, γ), where A is an abelian variety, U a unipotent group, and $\gamma : U(A) \to U$ a morphism; the $\mathcal{D}$-morphisms from (A_1, U_1, γ_1) to (A_2, U_2, γ_2) are those pairs of $\mathcal{C}$-morphisms $u : U_1 \to U_2$, $v : A_1 \to A_2$ such that the square

$$\begin{array}{ccc} U(A_1) & \xrightarrow{U(v)} & U(A_2) \\ {\scriptstyle\gamma_1}\big\downarrow & & {\scriptstyle\gamma_2}\big\downarrow \\ U_1 & \xrightarrow{u} & U_2 \end{array}$$

commutes. We also introduce the 'isogeny category' $\underline{\mathcal{D}}$, by allowing v to be a $\underline{\mathcal{C}}$-morphism in the above definition (this makes sense in view of Corollary 5.3). Next, define a functor

$$D : \mathcal{V} \longrightarrow \mathcal{D}$$

by assigning to each object G the triple $(A(G), R_u(G), \gamma)$, where $\gamma : U(A(G)) \to R_u(G)$ denotes the classifying map, and to each morphism $f : G_1 \to G_2$, the pair $(A(f), U(f))$. By Corollary 5.3 again, we may define similarly a functor

$$\underline{D} : \underline{\mathcal{V}} \longrightarrow \underline{\mathcal{D}}.$$

Proposition 5.6. *With the above notation, the functors* D *and* $\underline{D}$ *yield equivalences of categories.*

We omit the easy proof.

5.2 Semi-abelian varieties

Recall that a semi-abelian variety is an algebraic group G that lies in an extension

$$\xi : \quad 0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where T is a torus, and A an abelian variety. We now adapt part of the results of §5.1 to this setting, leaving the (easy) verifications to the motivated reader. The groups $T = T(G)$ and $A = A(G)$ are uniquely determined by G, and the extension ξ has no non-trivial automorphisms. Thus, the data of G and of the extension ξ are equivalent. Moreover, recall the natural isomorphism

$$c : \mathrm{Ext}^1_{\mathcal{C}}(A, T) \xrightarrow{\cong} \mathrm{Hom}^{\Gamma}(X(T), \widehat{A}(k_s)),$$

which arises from the Weil-Barsotti isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}_{k_s}}(A_{k_s}, \mathbb{G}_{m,k_s}) \xrightarrow{\cong} \widehat{A}(k_s)$$

together with the pairing $\mathrm{Ext}^1_{\mathcal{C}}(A, T) \times X(T) \to \mathrm{Ext}^1_{\mathcal{C}_{k_s}}(A_{k_s}, \mathbb{G}_{m,k_s})$ given by push-out of extensions via characters of T.

Denote by $\mathcal{S}$ the full subcategory of $\mathcal{C}$ with objects the semi-abelian varieties. Then the analogue of Lemma 5.1 holds in view of [Br15b, §5.4] (but there is no analogue of the universal vector extension in this setting). Thus, the isogeny category of semi-abelian varieties, $\underline{\mathcal{S}}$, is an abelian category. As for vector extensions of abelian varieties, we have natural isomorphisms

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(G_1, G_2) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathcal{S}}}(G_1, G_2), \quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{C}}(G_1, G_2) \xrightarrow{\cong} \mathrm{Ext}^1_{\underline{\mathcal{S}}}(G_1, G_2)$$

for any objects G_1, G_2 of $\underline{\mathcal{S}}$. This yields e.g. natural isomorphisms

$$\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, T) \xrightarrow{\cong} \mathrm{Hom}^{\Gamma}(X(T)_{\mathbb{Q}}, \widehat{A}(k_s)_{\mathbb{Q}}).$$

Note that the assignement $A \mapsto \widehat{A}(k_s)_{\mathbb{Q}}$ yields an exact functor from $\underline{\mathcal{A}}$ to the category of $\mathbb{Q}$-vector spaces equipped with a continuous representation of Γ, as follows e.g. from Remarks 4.8.

Next, we obtain a description of $\underline{\mathcal{S}}$ in terms of linear algebra. Let $\underline{\mathcal{E}}$ be the category with objects the triples (A, M, c), where A is an abelian variety, M a finite-dimensional $\mathbb{Q}$-vector space equipped with a continuous action of Γ, and $c : M \to \widehat{A}(k_s)_{\mathbb{Q}}$ a Γ-equivariant linear map; the $\underline{\mathcal{E}}$-morphisms from (A_1, M_1, c_1) to (A_2, M_2, c_2) are those pairs $(\underline{u}, v)$, where $\underline{u} : A_1 \to A_2$ is a $\underline{\mathcal{A}}$-morphism and $v : M_2 \to M_1$ a Γ-equivariant linear map, such that the square

$$\begin{array}{ccc} M_2 & \xrightarrow{c_2} & \widehat{A}_2(k_s)_{\mathbb{Q}} \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\underline{u}}} \\ M_1 & \xrightarrow{c_1} & \widehat{A}_1(k_s)_{\mathbb{Q}} \end{array}$$

commutes. Then one may check that the assignement $G \mapsto (A(G), X(T(G))_{\mathbb{Q}}, c(G)_{\mathbb{Q}})$ yields an equivalence of categories $\underline{\mathcal{S}} \to \underline{\mathcal{E}}$. Moreover, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\underline{\mathcal{S}}}(G_1, G_2) \xrightarrow{\varphi} \mathrm{Hom}^{\Gamma}(M_2, M_1) \times \mathrm{Hom}_{\underline{\mathcal{A}}}(A_1, A_2)$$

$$\xrightarrow{\psi} \mathrm{Hom}^\Gamma(M_2, \widehat{A}_1(k_s)_\mathbb{Q}) \longrightarrow \mathrm{Ext}^1_{\underline{\mathcal{S}}}(G_1, G_2) \longrightarrow 0$$

turns out to be exact for any semi-abelian varieties G_1, G_2, where $\underline{\mathcal{E}}(G_i) := (A_i, M_i, c_i)$ for $i = 1, 2$, $\varphi(\underline{f}) := (\underline{X}_\mathbb{Q} \circ \underline{T})(\underline{f}), \underline{A}(\underline{f}))$, and $\psi(u, v) := c_1 \circ u - \hat{v} \circ c_2$.

Yet there are important differences between the isogeny categories of vector extensions and semi-abelian varieties. For example, the latter does not have enough projectives in general:

Proposition 5.7. (i) *If k is not locally finite, then the projective objects of $\underline{\mathcal{S}}$ are exactly the tori.*

(ii) *If k is locally finite, then the product functor $\underline{\mathcal{T}} \times \underline{\mathcal{A}} \to \underline{\mathcal{S}}$ yields an equivalence of categories.*

Proof. (i) Let G be a semi-abelian variety, extension of an abelian variety A by a torus T; denote by $c = c(G) : X(T) \to \widehat{A}(k_s)$ the classifying map. If G is projective in $\underline{\mathcal{C}}$, then $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(G, T') = 0$ for any torus T'. Thus, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(A, T') \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(G, T') \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(T, T') \xrightarrow{\partial} \mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, T') \longrightarrow 0.$$

Moreover, the boundary map ∂ may be identified with the map

$$\mathrm{Hom}^\Gamma(X(T')_\mathbb{Q}, X(T)_\mathbb{Q}) \longrightarrow \mathrm{Hom}^\Gamma(X(T')_\mathbb{Q}, \widehat{A}(k_s)_\mathbb{Q}). \quad f \longmapsto c \circ f.$$

Since ∂ is surjective, and $X(T')_\mathbb{Q}$ may be chosen arbitrarily among finite-dimensional $\mathbb{Q}$-vector spaces equipped with a continuous representation of Γ, the map

$$c_\mathbb{Q} : X(T)_\mathbb{Q} \longrightarrow \widehat{A}(k_s)_\mathbb{Q}$$

is surjective as well. In particular, the abelian group $\widehat{A}(k_s)$ has finite rank. In view of [FJ74, Thm. 9.1], this forces A to be zero, i.e., G is a torus.

(ii) This follows readily from Proposition 2.5 and Corollary 2.12. □

5.3 Product decompositions

In this subsection, we first prove the remaining assertions (ii) and (iii) of Theorem 1. Then we describe the isogeny category $\mathcal{C}$ in characteristic 0, and its projective objects in arbitrary characteristics.

Proposition 5.8. (i) *If k is perfect, then the product functor $\mathcal{T} \times \mathcal{U} \to \mathcal{L}$ yields an equivalence of categories.*

(ii) *For any field k, the product functor $\underline{\mathcal{T}} \times \underline{\mathcal{U}} \to \underline{\mathcal{L}}$ yields an equivalence of categories.*

Proof. (i) This follows readily from Theorem 2.4 and Proposition 2.5.

(ii) This is a consequence of (i) in view of Theorem 3.11. (Alternatively, the assertion follows from Theorem 2.11 by arguing as in the proof of our next result). □

Proposition 5.9. *If* $\mathrm{char}(k) > 0$*, then the product functor* $\underline{\mathcal{S}} \times \underline{\mathcal{U}} \to \underline{\mathcal{C}}$ *yields an equivalence of categories.*

Proof. Let G be a smooth connected algebraic group. Recall from Theorem 2.11 that G lies in a unique extension

$$0 \longrightarrow H \longrightarrow G \longrightarrow U \longrightarrow 0,$$

where U is smooth, connected and unipotent, and H is an extension of an abelian variety by a group of multiplicative type; moreover, this extension splits after pull-back under a surjective morphism with finite kernel $V \to U$. Thus, $G \cong H \times U$ in $\underline{\mathcal{C}}$. Also, H/F is smooth and connected for some finite subgroup $F \subset H$ (Lemma 2.2), and hence is a semi-abelian variety. Thus, the product functor Π is essentially surjective.

Next, let S_1, S_2 be semi-abelian varieties, and U_1, U_2 smooth connected unipotent groups. We check that Π induces an isomorphism

$$\mathrm{Hom}_{\underline{\mathcal{C}}}(S_1, S_2) \times \mathrm{Hom}_{\underline{\mathcal{C}}}(U_1, U_2) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(S_1 \times U_1, S_2 \times U_2), \quad (\underline{\varphi}, \underline{\psi}) \longmapsto \underline{\varphi} \times \underline{\psi}.$$

Assume that $\underline{\varphi} \times \underline{\psi} = 0$. Choose representatives $\varphi : S_1 \to S_2/S_2'$, $\psi : U_1 \to U_2/U_2'$, where S_2', U_2' are finite. Then $\varphi \times \psi : S_1 \times U_1 \to (S_2 \times U_2)/(S_2' \times U_2')$ has finite image, and hence is zero by Lemma 2.1. So $\underline{\varphi} = \underline{\psi} = 0$.

Let $\underline{\gamma} \in \mathrm{Hom}_{\underline{\mathcal{C}}}(S_1 \times U_1, S_2 \times U_2)$ be represented by $\gamma : S_1 \times U_1 \to (S_2 \times U_2)/F$, where F is finite. Then $F \subset S_2' \times U_2'$ for some finite subgroups $S_2' \subset S_2$, $U_2' \subset U_2$. Thus, we may assume that $F = S_2' \times U_2'$. Then the composite morphisms

$$S_1 \longrightarrow S_1 \times U_1 \xrightarrow{\gamma} S_2/S_2' \times U_2/U_2' \longrightarrow U_2/U_2',$$

$$U_1 \longrightarrow S_1 \times U_1 \xrightarrow{\gamma} S_2/S_2' \times U_2/U_2' \longrightarrow S_2/S_2'$$

are zero by Lemma 2.1 and Proposition 2.5. Thus, $\gamma = \varphi \times \psi$ for some morphisms $\varphi : S_1 \to S_2/S_2'$, $\psi : U_1 \to U_2/U_2'$. □

Combining Propositions 5.9 and 5.7 (i), we obtain readily:

Corollary 5.10. *If k is locally finite, then the product functor*

$$\underline{\mathcal{T}} \times \underline{\mathcal{A}} \times \underline{\mathcal{U}} \longrightarrow \underline{\mathcal{C}}$$

yields an equivalence of categories.

Remarks 5.11. (i) With the notation of the above corollary, each of the categories $\underline{\mathcal{T}}$, $\underline{\mathcal{A}}$, $\underline{\mathcal{U}}$ admits a description of its own. By Proposition 4.1, $\underline{\mathcal{T}}$ is equivalent to the category of $\mathbb{Q}$-vector spaces equipped with an automorphism of finite order. Also, the isomorphism classes of objects of $\underline{\mathcal{A}}$, i.e., the isogeny classes of abelian varieties over a locally finite field, are classified by the Honda-Tate theorem (see [Ho68, Tat66]); their endomorphism rings are investigated in [MW69, Wa69]. Finally, the structure of $\underline{\mathcal{U}}$ has been described in Remark 3.12.

(ii) Combining Lemma 2.2, Theorem 2.4 and Lemma 3.10, one may show that the product functor $\mathcal{M}/\mathcal{IM} \times \mathcal{U}/\mathcal{IU} \to \mathcal{L}/\mathcal{I}$ yields an equivalence of categories. Here $\mathcal{I}$ denotes the category of infinitesimal algebraic groups, and $\mathcal{IM}$ (resp. $\mathcal{IU}$) the full subcategory of infinitesimal groups of multiplicative type (resp. unipotent).

Next, assume that $\mathrm{char}(k) = 0$. Then every algebraic group is isogenous to a fibered product $E \times_A S$, where E is a vector extension of the abelian variety A, and S is semi-abelian with Albanese variety isomorphic to A (see e.g. Remark 2.10). This motivates the consideration of the fibered product $\underline{\mathcal{V}} \times_{\underline{\mathcal{A}}} \underline{\mathcal{S}}$: this is the category with objects the triples $(E, S, \underline{f})$, where E is a vector extension of an abelian variety, S a semi-abelian variety, and $\underline{f} : A(E) \to A(S)$ an $\underline{\mathcal{A}}$-isomorphism. The morphisms from $(E_1, S_1, \underline{f}_1)$ to $(E_2, S_2, \underline{f}_2)$ are those pairs of $\underline{\mathcal{C}}$-morphisms $\underline{u} : E_1 \to E_2$, $\underline{v} : S_1 \to S_2$ such that the square

$$\begin{array}{ccc} A(E_1) & \xrightarrow{A(\underline{u})} & A(E_2) \\ {\scriptstyle \underline{f}_1}\big\downarrow & & {\scriptstyle \underline{f}_2}\big\downarrow \\ A(S_1) & \xrightarrow{A(\underline{v})} & A(S_2) \end{array}$$

commutes in $\underline{\mathcal{A}}$.

Proposition 5.12. *If* $\mathrm{char}(k) = 0$*, then* $\underline{\mathcal{C}}$ *is equivalent to* $\underline{\mathcal{V}} \times_{\underline{\mathcal{A}}} \underline{\mathcal{S}}$.

The proof is similar to that of Proposition 5.9, and will be omitted. Note that the descriptions of $\underline{\mathcal{V}}$ and $\underline{\mathcal{S}}$ in terms of linear algebra, obtained in §5.1 and §5.2, can also be reformulated in terms of fibered products of categories.

Returning to an arbitrary field k, we obtain:

Theorem 5.13. *The projective objects of* $\underline{\mathcal{C}}$ *are exactly:*

- *the linear algebraic groups, if* $\mathrm{char}(k) = 0$.
- *the semi-abelian varieties, if k is locally finite.*
- *the tori, if* $\mathrm{char}(k) > 0$ *and k is not locally finite.*

Proof. Let G be a smooth connected algebraic group. As a consequence of Theorem 2.9 and Proposition 5.9, we have an exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow U \longrightarrow G \longrightarrow S \longrightarrow 0,$$

where U is smooth, connected, and unipotent, and S is a semi-abelian variety.

If G is projective in $\underline{\mathcal{C}}$, then $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(G, T') = 0$ for any torus T'. Since $\mathrm{Hom}_{\underline{\mathcal{C}}}(U, T') = 0$ (as a consequence of Proposition 2.5) and $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(U, T') = 0$ (by Lemma 4.9), the long exact sequence for Ext groups yields that $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(S, T') = 0$ as well. By arguing as in the proof of Proposition 5.7, this forces either A to be zero, or k to be locally finite.

In the former case, G is linear, and hence $G \cong T \times U$ in $\underline{\mathcal{C}}$ by Proposition 5.8. Moreover, tori are projective in $\underline{\mathcal{C}}$ by Proposition 4.4; thus, we may assume that G is unipotent. If $\mathrm{char}(k) = 0$, then G is projective, as follows e.g. from Lemma 4.9. If $\mathrm{char}(k) > 0$ and $G \neq 0$, then we claim that G is not projective. To check this, we may assume k perfect by Theorem 3.11. Then G is isogenous to a direct sum of groups of the form W_n (the group of Witt vectors of length n), by [DG70, V.3.6.11]. Moreover, the canonical exact sequence

$$0 \longrightarrow \mathbb{G}_a \longrightarrow W_{n+1} \longrightarrow W_n \longrightarrow 0$$

(see e.g. [DG70, V.1.1.6]) is not split in $\underline{\mathcal{C}}$, as follows e.g. from [DG70, V.1.5.2] together with Lemma 3.3. Thus, $\mathrm{Ext}^1_{\underline{\mathcal{U}}}(G, \mathbb{G}_a) \neq 0$; this yields the claim.

In the latter case, $G \cong T \times A \times U$ in $\underline{\mathcal{C}}$ (by Corollary 5.10) and it follows as above that U is zero. Conversely, every semi-abelian variety is projective in $\underline{\mathcal{C}}$, by Corollary 5.10 again. □

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