The dynamics of holomorphic correspondences on compact Riemann surfaces

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Holomorphic correspondences

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with the following properties: for each $\Gamma_j$,

- $\pi_1|_{\Gamma_j}$ & $\pi_2|_{\Gamma_j}$ are surjective $\forall j$; AND
- for $x \in X$, the set $\bigcup_{1 \leq j \leq N} (\pi_1^{-1}\{x\} \cap \Gamma_j)$ is finite.
Why should we care?

In this talk, $X_1 = X_2$ ($= X$, say), both Riemann surfaces. A correspondence can be composed with itself (which we’ll define). Wish to study the dynamical system that arises.
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By the *de Franchis Theorem*, there are only finitely many holomorphic self-maps on a compact hyperbolic Riemann surface. I.e., *no interesting holomorphic dynamics!*

**But** not too hard to show that there are infinitely many holomorphic correspondences $\Gamma$ on a compact hyperbolic Riemann surface; even satisfying

\[
d_{\text{top}}(\Gamma) > d_{\text{top}}(\Gamma^\dagger).
\]
Point of entry

In 2006, Dinh–Sibony proved a result that, paraphrased for holomorphic correspondences, is:

\[
\text{Result.}
\]

Let \( \Gamma \) be a holomorphic correspondence on a \( k \)-dim'l. compact Kähler manifold \((X, \omega)\) and assume that \( d_{\text{top}}(\Gamma) > d_{\text{top}}(\Gamma^\dagger) \). Suppose \( \int_X \omega^k = 1 \). Then, \( \exists \mu_{\Gamma} \) a Borel prob. measure on \( X \) that satisfies 

\[
\Gamma^* (\mu_{\Gamma}) = d_{\text{top}}(\Gamma) \mu_{\Gamma},
\]

and 

\[
\frac{1}{d_{\text{top}}(\Gamma)} n(\Gamma^n)^* (\omega^k) \text{ weak}^* \rightarrow \mu_{\Gamma} \text{ as measures, as } n \rightarrow \infty.
\]

Clarifications:

The pullback \( \Gamma^* (\omega^k) \) is carried out in the sense of currents.

\( d_{\text{top}}(\Gamma) := \sum_{1 \leq j \leq N} m_j \text{degree}(\pi_2) \).
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**Result.** Let $\Gamma$ be a holomorphic correspondence on a $k$-dim’l. compact Kähler manifold $(X, \omega)$ and assume that $d_{\text{top}}(\Gamma) > d_{\text{top}}(\dag \Gamma)$. Suppose $\int_X \omega^k = 1$. Then,
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**Result.** Let $\Gamma$ be a holomorphic correspondence on a $k$-dim’l. compact Kähler manifold $(X, \omega)$ and assume that $d_{top}(\Gamma) > d_{top}(\dagger \Gamma)$. Suppose $\int_X \omega^k = 1$. Then, $\exists \mu_{\Gamma} – a$ Borel prob. measure on $X – that satisfies $\Gamma^*(\mu_{\Gamma}) = d_{top}(\Gamma) \mu_{\Gamma}$, and $d_{top}(\Gamma) \xrightarrow{\text{weak}^*} \mu_{\Gamma}$ as measures, as $n \to \infty$. 

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- When $k = 1$, $X = \mathbb{CP}^1$ and $\Gamma = \text{graph}(f)$, $f$ a rational map, then $\text{supp}(\mu_\Gamma)$ is the Julia set — denoted $\mathcal{J}(f)$ — of $f$. 

Problem 1. Describe the set $\mathcal{F}(\Gamma)$ on which the orbits of $\Gamma$ are insensitive to small perturbations of initial condition.

Problem 2. Describe the complex geometry of the components of $\mathcal{F}(\Gamma)$ in terms analogous to classical complex dynamics. 

In classical complex dynamics, a crucial part of studying geometric structure is the fact that $\mathcal{J}(f) \cup \mathcal{F}(f) = \mathbb{CP}^1$.

$\triangleright$ Theorem A will address Problem 1 above.

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Composing two holomorphic correspondences

Given a holomorphic correspondence $\Gamma$, we denote by

$$|\Gamma| := \bigcup_{j=1}^{N} \Gamma_j$$

the set underlying $\Gamma$. Now, $|\Gamma|$ is a relation on $X$. 

To code the $k$-chain data into the above "composition" we need to do some work...
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Composing two holomorphic correspondences, cont’d.

To begin with, we now use an alternative representation:

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\Gamma^1 = \sum_{1 \leq j \leq L_1} \Gamma^1_{1,j}, \quad \Gamma^2 = \sum_{1 \leq j \leq L_2} \Gamma^2_{2,j},
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primed sums indicating that the irred. subvarieties \(\Gamma^s_{\cdot,j}\), \(s = 1, 2\), are not necessarily distinct and repeated according to the coeffs. \(m^s_{\cdot,j}\).
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We then define

\[ \Gamma^2 \circ \Gamma^1 := \sum_{j=1}^{L_1} \sum_{l=1}^{L_2} \sum_{S \in S(j, l)} \nu_s S, \]

where :

\( S(j, l) := \text{set of distinct irred. components of } \Gamma^2_{2, l} \star \Gamma^1_{1, j}. \)
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To understand the coefficient $\nu_S$, consider the following:

**Example.** Take the correspondences $\Gamma^1 = \Gamma^2 = \text{the completion in } \mathbb{CP}^1 \times \mathbb{CP}^1 \text{ of } \{(z, w) \in \mathbb{C}^2 : w^2 = z^2 + 1\} =: \Gamma$. 

$\nu_S := \text{generic no. of } y \text{'s} - \text{as } (x, z) \text{ varies through } S \text{ for which the memberships given in } (\ast) \text{ hold.}$
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$$z_0 \rightarrow \sqrt{z_0^2 + 1} \rightarrow \sqrt{z_0^2 + 2}$$

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Two distinct occurrences of $y$ indicated by $(\ast)$ associated to the point $(z_0, \sqrt{z_0^2 + 2}) \in \Gamma \ast \Gamma$. 
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Statement of main results

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Recall that when $X = \mathbb{C}P^1$ and $\Gamma$ is the graph of a rational map $f$, $d_{top}(\Gamma) = 1$, $\mu_\Gamma$ exists and $\text{supp}(\mu_\Gamma)$ equals the Julia set $\mathcal{J}(f)$.
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For various reasons (e.g., see the Dinh–Sibony theorem) $\text{supp}(\mu_{\Gamma})$ is a natural analogue of the Julia set for general correspondences.
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The following theorem provides a relationship between $\text{supp}(\mu_\Gamma)$ and the Fatou set $\mathcal{F}(\Gamma)$.
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**Theorem A (B., 2014)**

*Let $X$ be a compact Riemann surface and let $\Gamma$ be a holomorphic correspondence on $X$ such that $d_{\text{top}}(\Gamma) > d_{\text{top}}(\Gamma)$. Let $\mu_\Gamma$ denote the Dinh–Sibony measure associated to $\Gamma$.***
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The Fatou set: preliminary notations/notions

Fix a compact Riemann surface $X$ and a correspondence $\Gamma$ on it.
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- **$N$-path:**
  
  $$(z_0, \ldots, z_N; \alpha_1, \ldots, \alpha_N) \in X^{N+1} \times [1 \ldots L]^N : (z_{j-1}, z_j) \in \Gamma^\bullet_{\alpha_j}, \: j \leq N.$$
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- $\mathcal{P}_N(z_0) =$ set of all $N$-paths starting at $z_0$. 

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- $\mathcal{P}_N(z_0) = \text{set of all } N\text{-paths starting at } z_0.$

- Given a multi-index $\alpha \in [1..L]^j$
  $$\Gamma^\bullet_{\alpha} := \{(x_0, \ldots, x_j) \in X^{j+1} : (x_{i-1}, x_i) \in \Gamma^\bullet_{\alpha_i}, \ 1 \leq i \leq j\}.$$
The Fatou set: preliminary notations/notions

Fix a compact Riemann surface $X$ and a correspondence $\Gamma$ on it.

- **$N$-path:**
  $$(z_0, \ldots, z_N; \alpha_1, \ldots, \alpha_N) \in X^{N+1} \times [1..L]^N : (z_{j-1}, z_j) \in \Gamma_{\alpha_j}, \ j \leq N.$$

- $\mathcal{P}_N(z_0)$ = set of all $N$-paths starting at $z_0$.

- Given a multi-index $\alpha \in [1..L]^j$
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**Basic idea underlying the Fatou set:**
A point $z_0$ belongs to the Fatou set if there exists a nbhd. $U \ni z_0$ such that
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**Basic idea underlying the Fatou set:**

A point $z_0$ belongs to the Fatou set if there exists a nbhd. $U \ni z_0$ such that for every infinite path $(z_0, z_1, z_2, \ldots; \alpha_1, \alpha_2, \alpha_3, \ldots)$, each sequence of analytic germs of $\Gamma^\bullet_{(\alpha_1, \ldots, \alpha_n)}$ at $(z_0, z_1, \ldots, z_n)$, $n = 1, 2, 3, \ldots$, determined by lifting $U$ into these varieties admits
The Fatou set: preliminary notations/notions

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The Fatou set: secondary notations

A useful map:

\[ \pi_j^{(k)} : X^{k+1} \to X, \quad \pi_j^{(k)} : (z_0, z_1, \ldots, z_k) \mapsto z_j, \ 0 \leq j \leq k. \]
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\[ \pi_j^{(k)} : X^{k+1} \longrightarrow X, \quad \pi_j^{(k)} : (z_0, z_1, \ldots, z_k) \longmapsto z_j, \quad 0 \leq j \leq k. \]

Let \( z_0 \in X \) and \( U \ni x_0 \) be an open nbhd. Denote paths in \( P_N(z_0) \) by \( Z \equiv (Z; \alpha) \).
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Let \( z_0 \in X \) and \( U \ni x_0 \) be an open nbhd. Denote paths in \( P_N(z_0) \) by \( Z \equiv (Z ; \alpha) \). Then

\[ Z_{[j]} := (z_0, \ldots, z_j; \alpha_1, \ldots, \alpha_j), \quad 1 \leq j \leq N \]
\[ \text{pre}(Z) := Z_{[N-1]}, \quad N \geq 2. \]
The Fatou set: secondary notations

A useful map:

\[ \pi_k^j : X^{k+1} \to X, \quad \pi_k^j : (z_0, z_1, \ldots, z_k) \mapsto z_j, \ 0 \leq j \leq k. \]

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\[ \mathcal{Z}[j] := (z_0, \ldots, z_j; \alpha_1, \ldots, \alpha_j), \quad 1 \leq j \leq N \]

\[ \text{pre}(\mathcal{Z}) := \mathcal{Z}[N-1], \quad N \geq 2. \]

Next, we define sequences of analytic germs:

\[ \mathcal{I}(U, \mathcal{Z}) := \text{set of irred. components of } \Gamma_\alpha^\bullet \cap (U \times X) \text{ containing } Z, \quad \text{if } \mathcal{Z} \in \mathcal{P}_1(z_0), \]
The Fatou set: secondary notations

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Next, we define **sequences of analytic germs**:

\[ \mathcal{S}(U, \mathcal{Z}) := \text{set of irred. components of } \Gamma_\alpha^* \cap (U \times X) \text{ containing } Z, \quad \text{if } \mathcal{Z} \in \mathcal{P}_1(z_0), \]

\[ \mathcal{S}(U, \mathcal{Z}) := \text{set of irred. components of } \Gamma_\alpha^* \cap \left( U \times \left( X^{N-1} \prod_{k=1}^{N-1} \pi_k^{(k)}(S) \right) \times X \right) \text{ containing } Z, \quad \text{if } \mathcal{Z} \in \mathcal{P}_N(z_0), \; N \geq 2. \]

where
The Fatou set: secondary notations

A useful map:

\[ \pi_j^{(k)} : X^{k+1} \to X, \quad \pi_j^{(k)} : (z_0, z_1, \ldots, z_k) \mapsto z_j, \quad 0 \leq j \leq k. \]

Let \( z_0 \in X \) and \( U \ni x_0 \) be an open nbhd. Denote paths in \( P_N(z_0) \) by \( Z \equiv (Z; \alpha) \). Then

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Z_j := (z_0, \ldots, z_j; \alpha_1, \ldots, \alpha_j), \quad 1 \leq j \leq N
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Next, we define sequences of analytic germs:

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if \( Z \in P_1(z_0) \),

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\mathcal{I}(U, Z) := \text{set of irred. components of } \Gamma_\alpha \cap (U \times (X_{k=1}^{N-1} \pi_k^{(k)}(S)) \times X) \text{ containing } Z,
\]

if \( Z \in P_N(z_0), \quad N \geq 2. \)

Fundamental compositional relation
The Fatou set

Definition

Given a path $\mathcal{Z} \in \mathcal{P}_N(z_0)$, the list $(\mathcal{A}_1, \ldots, \mathcal{A}_N; U)$ is called an analytic branch of $\Gamma$ along $\mathcal{Z}$ if $U$ is a connected open nbhd. of $z_0$ and
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(a) $\mathcal{A}_j \in \mathcal{L}(U, \mathcal{Z}_{[j]}), \ j = 1, \ldots, N$, and each $\mathcal{A}_{j+1}$ is related to $\mathcal{A}_j$, $j = 1, \ldots, N - 1$ by the fundamental compositional relation;
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(b) $[\mathcal{A}_j]_p$ is irreducible $\forall p \in \mathcal{A}_j, \ j = 1, \ldots, N$. 
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Definition (The Fatou set)

A point $z_0$ is said to belong to the Fatou set of $\Gamma$ if there exists a single connected open nbhd. $U$ $\ni$ $z_0$ such that for each $n \in \mathbb{Z}_+$, each $\mathcal{Z} \in \mathcal{P}_n(z_0)$ admits an analytic branch $(\mathcal{A}_1, \ldots, \mathcal{A}_n; U)$ of $\Gamma$ along $\mathcal{Z}$, and
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Definition

Given a path \( \mathcal{Z} \in \mathcal{P}_N(z_0) \), the list \((\mathcal{A}_1, \ldots, \mathcal{A}_N; U)\) is called an analytic branch of \( \Gamma \) along \( \mathcal{Z} \) if \( U \) is a connected open nbhd. of \( z_0 \) and

1. \( \mathcal{A}_j \in \mathcal{I}(U, \mathcal{Z}_{[j]}), j = 1, \ldots, N \), and each \( \mathcal{A}_{j+1} \) is related to \( \mathcal{A}_j \), \( j = 1, \ldots, N - 1 \) by the fundamental compositional relation;

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Definition (The Fatou set)

A point \( z_0 \) is said to belong to the Fatou set of \( \Gamma \) if there exists a single connected open nbhd. \( U \ni z_0 \) such that for each \( n \in \mathbb{Z}_+ \), each \( \mathcal{Z} \in \mathcal{P}_n(z_0) \) admits an analytic branch \((\mathcal{A}_1, \ldots, \mathcal{A}_n; U)\) of \( \Gamma \) along \( \mathcal{Z} \), and such that the set

\[ \mathcal{F}(z_0) := \left\{ \pi_0^{(n)} \times \pi_n^{(n)}(\mathcal{A}_n) : n \in \mathbb{Z}_+, \mathcal{Z} \in \mathcal{P}_n(z_0), \text{ and } (\mathcal{A}_1, \ldots, \mathcal{A}_n; U) \right\}, \]

viewed as a set comprising currents of integration, is relatively compact in the space of \((1,1)\)-currents on \( U \times X \).
The iterative tree

Suppose, for $z_0 \in X$, $\exists U \ni z_0$, a connected nbhd. of $z_0$, such that for each $n \in \mathbb{Z}_+$, each $\mathcal{Z} \in \mathcal{P}_n(z_0)$ admits an analytic branch $(\mathcal{A}_1, \ldots, \mathcal{A}_N; U)$ of $\Gamma$ along $\mathcal{Z}$. We can define an infinite tree $\tau(\Gamma, U)$ as follows.
The iterative tree

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\[
V(\tau(\Gamma, U)) := \bigcup_{n \in \mathbb{Z}_+} \bigcup_{\mathcal{Z} \in \mathcal{P}_n(z_0)} \mathcal{I}(U, \mathcal{Z}),
\]

\( E(\tau(\Gamma, U)) \) is defined by the condition

there is an edge between \( A, B \in V(\tau(\Gamma, U)) \)

\( \iff A \in \mathcal{I}(U, \mathcal{Z}) \) for some \( \mathcal{Z} \in \mathcal{P}_n(z_0), n \geq 2 \), and \( B \in \mathcal{I}(U, \text{pre}(\mathcal{Z})) \).
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Such a tree is called the iterative tree at \( z_0 \).
The Fatou set: very basic properties

Unlike the case with rational maps, \( \mathcal{F}(\Gamma) \) and \( \text{supp}(\mu_\Gamma) \) do not, in general, partition \( X \) under the condition

\[
d_{\text{top}}(\Gamma) > d_{\text{top}}(\dagger \Gamma) \geq 2.
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This follows from certain computer experiments by Shaun Bullett from the 1990s, read together with an entropy estimate of Dinh–Sibony.
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This follows from certain computer experiments by Shaun Bullett from the 1990s, read together with an entropy estimate of Dinh–Sibony. This is the motivation of Theorem B, which we shall see soon.
Preliminaries to proving Theorem A

**Formal** principle behind the pull-back of a current:

For a \( k \)-dim'l. manifold \( X \) and a \((p,p)\)-current \( \Gamma \ast (S) := (\pi_1) \ast (\pi_2 \ast (S) \wedge [\Gamma]) \), whenever the intersection of \( \pi_2 \ast (S) \) with \([\Gamma]\) makes sense.

So, for instance, viewing a smooth \((k,k)\)-form \( \Omega \) as a current, and a test function as a \((0,0)\)-form, \( \langle \Gamma \ast (\Omega), \phi \rangle := \sum_{j=1}^{m} \int_{reg(\Gamma_j)} (\pi_1|\Gamma_j) \ast \phi (\pi_2|\Gamma_j) \ast \Omega. \)

• comes from dualising \((\pi_1) \ast\),
• is the interpretation of "\((\pi_2 \ast (\Omega) \wedge [\Gamma])\)" in this case.
Preliminaries to proving Theorem A

**Formal** principle behind the pull-back of a current:
For a $k$-dim'l. manifold $X$ and a $(p, p)$-current

$$\Gamma^*(S) := (\pi_1)_* (\pi_2^*(S) \wedge [\Gamma]),$$

whenever the intersection of $\pi_2^*(S)$ with $[\Gamma]$ makes sense.
Preliminaries to proving Theorem A

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So, for instance, viewing a smooth $(k, k)$-form $\Omega$ as a current, and a test function as a $(0, 0)$-form,

$$
\langle \Gamma^*(\Omega), \varphi \rangle := \sum_{j=1}^{N} m_j \int_{\text{reg}(\Gamma_j)} (\pi_1|_{\Gamma_j})^* \varphi (\pi_2|_{\Gamma_j})^* \Omega.
$$
Preliminaries to proving Theorem A

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The proof of Theorem A

Assume \( \mathcal{F}(\Gamma) \neq \emptyset \). Nothing to prove otherwise.
The proof of Theorem A

Assume $\mathcal{F}(\Gamma) \neq \emptyset$. Nothing to prove otherwise.

Pick a $z_0$ in $\mathcal{F}(\Gamma)$, and let $U \ni z_0$ be as given by the definition. It suffices to show that for any non-negative function $\varphi \in \mathcal{C}(X; \mathbb{R})$ with $\text{supp}(\varphi) \subset U$, $\int_X \varphi \, d\mu_{\Gamma} = 0$. 

Let $\omega_X$ denote the normalized Kähler form associated to the hyperbolic metric. Call $d_{\text{top}}(\Gamma) =: d_1$ and $d_{\text{top}}(\Gamma^\dagger) =: d_0$. Easy to show that 

$$
\left\langle \left(\Gamma^n\right)^* \left(\omega_X\right), \varphi \right\rangle = \sum_{Z \in P_n(z_0)} \sum_{A \in S(U, Z)} \int_{\text{reg}} \left(\tilde{A} \cdot \pi_1|_{\Gamma^j}\right)^* \varphi \left(\pi_2|_{\Gamma^j}\right)^* \omega_X,
$$

where $\tilde{A} = \pi_n(0) \times \pi_n(A(n))$. Thus:

$$
d_{\text{top}} - n d_1 |\left\langle \left(\Gamma^n\right)^* \left(\omega_X\right), \varphi \right\rangle| \leq d_{\text{top}} - n d_1 \sup |\varphi| \sum_{Z \in P_n(z_0)} \sum_{A \in S(U, Z)} \int_{\text{reg}} \left(\tilde{A} \cdot \pi_1|_{\Gamma^j}\right)^* \varphi \left(\pi_2|_{\Gamma^j}\right)^* \omega_X,
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Let $\omega_X$ denote the normalized Kähler form associated to the hyperbolic metric. Call $d_{top}(\Gamma) =: d_1$ and $d_{top}(\dagger \Gamma) =: d_0$. 
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\langle (\Gamma^n)^* (\omega_X), \varphi \rangle = \sum_{\mathcal{P} \in \mathcal{P}(z_0)} \sum_{\mathcal{A} \in \mathcal{I}(U, \mathcal{P})} \int_{\text{reg}(\mathcal{A})} (\pi_1|_{\Gamma_j})^* \varphi (\pi_2|_{\Gamma_j})^* \omega_X,
$$

where $\mathcal{A} = \pi_0^{(n)} \times \pi^n_{\text{top}}(\mathcal{A})$. 

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Dynamics of holomorphic correspondences

Gautam Bharali
The proof of Theorem A

Assume $\mathcal{F}(\Gamma) \neq \emptyset$. Nothing to prove otherwise.

Pick a $z_0$ in $\mathcal{F}(\Gamma)$, and let $U \ni z_0$ be as given by the definition. It suffices to show that for any non-negative function $\varphi \in C(X; \mathbb{R})$ with $\text{supp}(\varphi) \subset U$, $\int_X \varphi \, d\mu_\Gamma = 0$.

Let $\omega_X$ denote the normalized Kähler form associated to the hyperbolic metric. Call $d_{top}(\Gamma) =: d_1$ and $d_{top}(\hat{\Gamma}) =: d_0$. Easy to show that

$$\langle (\Gamma^n)^*(\omega_X), \varphi \rangle = \sum_{\mathcal{Z} \in \mathcal{P}_n(z_0)} \sum_{\mathcal{A} \in \mathcal{I}(U, \mathcal{Z})} \int_{\text{reg}(\widetilde{\mathcal{A}})} (\pi_1|_{\Gamma_j})^* \varphi \left( \pi_2|_{\Gamma_j} \right)^* \omega_X,$$

where $\widetilde{\mathcal{A}} = \pi_0^{(n)} \times \pi_n^{(n)}(\mathcal{A})$. Thus:

$$d_1^{-n} |\langle (\Gamma^n)^*(\omega_X), \varphi \rangle| \leq d_1^{-n} \sup |\varphi| \sum_{\mathcal{Z} \in \mathcal{P}_n(z_0)} \sum_{\mathcal{A} \in \mathcal{I}(U, \mathcal{Z})} \int_{\text{reg}(\widetilde{\mathcal{A}})} \left( \pi_2|_{\Gamma_j} \right)^* \omega_X,$$
The proof of Theorem A

Hence

\[ d_1^{-n} |\langle (\Gamma^n)^* (\omega_X), \varphi \rangle | \leq C d_1^{-n} \sum_{\mathcal{Z} \in \mathcal{P}_n(z_0)} \sum_{\mathcal{A} \in \mathcal{J}(U, \mathcal{Z})} \text{Vol} (\mathcal{A}). \quad (**) \]
The proof of Theorem A

Hence

\[ d_1^{-n} |\langle (\Gamma^n)^*(\omega_X), \varphi \rangle | \leq C d_1^{-n} \sum_{Z \in \mathcal{P}_n(z_0)} \sum_{\mathcal{A} \in \mathcal{I}(U, Z)} \text{Vol}(\mathcal{A}) \]  

(\ast\ast)

At this stage, we need a new tool:

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The proof of Theorem A

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At this stage, we need a new tool:

**Bishop’s Compactness Theorem (in the style of Harvey–Schiffman).**

Let \((X_1, \omega_1)\) and \((X_2, \omega_2)\) be compact \(k\)-dim’l. Kähler manifolds, and let \(U\) be a relatively compact open subset of \(X_1\). Let \(F\) be a family of reduced, irreducible, analytic subsets of \(U \times X_2\) of pure dimension \(p : 1 \leq p \leq k\).
The proof of Theorem A

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The proof of Theorem A

Hence

\[ d_1^{-n} \left| \langle (\Gamma^n)^* (\omega_X), \varphi \rangle \right| \leq C d_1^{-n} \sum_{\mathcal{Z} \in \mathcal{P}_n(z_0)} \sum_{\mathcal{A} \in \mathcal{F}(U, \mathcal{Z})} \text{Vol}(\mathcal{A}). \quad (**) \]

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Let \((X_1, \omega_1)\) and \((X_2, \omega_2)\) be compact \(k\)-dim’l. Kähler manifolds, and let \(U\) be a relatively compact open subset of \(X_1\). Let \(F\) be a family of reduced, irreducible, analytic subsets of \(U \times X_2\) of pure dimension \(p : 1 \leq p \leq k\). Then, \(F\) is compact in the space of currents of bidimension \((p, p)\) if & only if

(a) The volumes of the sets in \(F\) are uniformly bounded; and
The proof of Theorem A

Hence

\[ d_1^{-n} |\langle (\Gamma^n)^* (\omega_X), \varphi \rangle| \leq C d_1^{-n} \sum_{Z \in \mathcal{P}_n(z_0)} \sum_{\mathcal{A} \in \mathcal{I}(U, Z)} \text{Vol}(\mathcal{A}). \]  

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(a) The volumes of the sets in \(F\) are uniformly bounded; and

(b) Given a compact \(K \subset U\), there \(\exists C_K > 0\) such that, for \(\mathcal{A}, \mathcal{B} \in F\), \(\mathcal{A} \cap (K \times X_2)\) and \(\mathcal{B} \cap (K \times X_2)\) are no farther than \(C_K\) in the Hausdorff metric.
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Thus, from (**) we have that

\[ d_1^{-n} |\langle (\Gamma^n)^*(\omega_X), \varphi \rangle| \leq C \left( \frac{d_0}{d_1} \right)^n, \]

whence

\[ \int_X \varphi d\mu = \lim_{n \to \infty} d_1^{-n} \langle (\Gamma^n)^*(\omega_X), \varphi \rangle = 0. \]

Hence the result. □
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Hence the result. ■
Theorem B: preliminaries

Observe: (**) suggests that one could allow the volumes of branches to grow at a certain exponential rate. This motivates the following:

We say that most analytic branches of $\Gamma$ around $z_0$ converge if there exist a connected nbhd. $U \ni z_0$ that admits an iterative tree $\tau(\Gamma,U)$, and an $\varepsilon \in (0,1)$ such that there is a connected subtree $\tilde{\tau}(\Gamma,U)$, and so that $\text{The n-th generation of } \tilde{\tau}(\Gamma,U) \text{ contains at least } (1 - \varepsilon^n) d^n_0 \text{ vertices from the n-th generation of the iterative tree}$; and the family $F(z_0) := \{ \pi(n) \times \pi(n) : n \in \mathbb{Z}^+, \text{and } A_n \in V_n(\tilde{\tau}(\Gamma,U)) \}$ is relatively compact in the space of $(1,1)$-currents on $U \times X$. 
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$$\mathcal{F}(z_0) := \{ \pi_0^{(n)} \times \pi_n^{(n)}(\mathcal{A}_n) : n \in \mathbb{Z}_+, \text{ and } \mathcal{A}_n \in V_n(\tilde{\tau}(\Gamma, U))\}$$

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Gautam Bharali

Dynamics of holomorphic correspondences
Theorem B

Let $X$ be a compact Riemann surface. Let $\Gamma$ and $\mu_\Gamma$ be as in Theorem A. Suppose the postcritical set of $\Gamma$ is disjoint from $\text{supp}(\mu_\Gamma)$. Define:

$$F(\Gamma) := \text{the largest open subset of } X \text{ consisting of points } z_0 \in X \text{ such that most analytic branches of } \Gamma \text{ around } z_0 \text{ converge.}$$

Then, $F(\Gamma) \supseteq \text{supp}(\mu_\Gamma)$.

The proof of $F(\Gamma) \supset\text{supp}(\mu_\Gamma)$ follows from the fact that there is at most exponential volume-growth of analytic branches of $\Gamma$, and that $F(\Gamma)$ does not depend on the size of $\epsilon$ as long as $0 < \epsilon < 1$. We then apply $(\ast\ast)$. 

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Theorem B (B., 2015)

Let $X$ be a compact Riemann surface. Let $\Gamma$ and $\mu_\Gamma$ be as in Theorem A. Suppose the postcritical set of $\Gamma$ is disjoint from $\text{supp}(\mu_\Gamma)$. Define:

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Then, $\overline{F}(\Gamma)^c = \text{supp}(\mu_\Gamma)$. 

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