# Harish-Chandra philosophy for enhanced Langlands parameters

"Matching colors"

Anne-Marie Aubert

Institut de Mathématiques de Jussieu - Paris Rive Gauche

16 janvier 2016

# Plan of the talk I

# The theory of Harish-Chandra I

## Definition

A $p$-adic reductive group $G$ is the group of the $F$-rational points of a connected reductive algebraic group, defined over $F$, with $F$ a local non archimedean field. $G$ is a is a locally compact topological group.

## Examples

$\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_n(F)$, $\mathrm{U}_n(F)$.

# The theory of Harish-Chandra II

## Definition

A smooth representation of $G$ is a pair $(\pi, V)$ such that $V$ is a $\mathbb{C}$-vector space, $\pi\colon G \to \mathrm{GL}(V)$ is a group morphism, and

$$\mathrm{Stab}_G(v) := \{g \in G \,|\, \pi(g)(v) = v\}$$

is open for each $v \in V$. $(\pi, V)$ is irreducible if there is no proper $G$-stable subspace of $V$. Let $\mathrm{Irr}(G)$ denote the set of equivalence classes of irreducible representations of $G$.

# The theory of Harish-Chandra III

## Parabolic Induction

Let

- $P$ be a parabolic subgroup of $G$, with unipotent radical $U$ and Levi factor $L$
- $(\sigma, V_L)$ be an irreducible smooth representation of $L$
- $(\tilde{\sigma}, V_L)$ its inflation to $P$
- $I_L^G(\sigma) := \mathrm{Ind}_P^G(\tilde{\sigma})$.

## Definition

An irreducible smooth representation $\pi$ of G that does not occur as a subquotient of any $I_L^G(\sigma)$ with $P$ a proper parabolic is called supercuspidal.

# The theory of Harish-Chandra IV

## Theorem (Harish-Chandra)

Every irreducible smooth representation $\pi$ of $G$ is contained in a parabolically induced representation $I_L^G(\sigma)$, with $\sigma$ supercuspidal. The pair $(L, \sigma)$ is unique up-to $G$-conjugation, it is called the *cuspidal support* of $\pi$.

## Notation

Let $\mathfrak{H}(G)$ be the set of $\mathfrak{h} = (L, \sigma)_G$ with $L$ Levi subgroup of $G$ and $\sigma$ supercuspidal irreducible representation of $L$, and let

$$\mathrm{Sc}\colon \mathrm{Irr}(G) \to \mathfrak{H}(G)$$

be the map sending $\pi$ to its cuspidal support.

# The theory of Harish-Chandra V

## Theorem (Harish-Chandra)

We have a partition

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{h} \in \mathfrak{H}(G)} \mathrm{Irr}^{\mathfrak{h}}(G),$$

where $\mathrm{Irr}(G)^{\mathfrak{h}} := \{\pi \in \mathrm{Irr}(G) \,|\, \mathrm{Sc}(\pi) = \mathfrak{h}\}$.

## An intuitive version of Harish-Chandra Theorem

Via "painting" : Think of the $\mathfrak{h}$ as different "colors" : all the elements in $\mathrm{Irr}^{\mathfrak{h}}(G)$ are painted in the color $\mathfrak{h}$.

# The theory of Harish-Chandra VI

"One of Harish-chandras guiding principles was what he later formulated as the Philosophy of cusp forms, which for the Plancherel formula effectively meant that the crucial case was the discrete spectrum, from which the continuous spectrum could be constructed by well-established techniques." (from "Harish-Chandra 1923-1983, A Biographical Memoir" by Roger Howe, published by the National Academy of Sciences).

# The approach of Langlands I

## An intuitive version of the approach of Langlands

Via "packing" : Putting the elements in $\mathrm{Irr}(G)$ into boxes (a box will be called an L-packet).

## Remarks

If $G = \mathrm{GL}_n(F)$, each box contains a unique element. Also the following may happen :

- Elements of the same color are in different boxes (for instance, for $G = \mathrm{GL}_n(F)$, already for $n = 2$).
- A given box contains elements of different colors (for instance for $G = \mathrm{Sp}_4(F)$).

# The approach of Langlands II

## Weil group of $F$

Let $W_F$ be the absolute Weil group of $F$ (a subgroup of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}/F)$. We have an exact sequence

$$1 \to I_F \to \mathrm{Gal}(\overline{\mathbb{F}}/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1$$

where $I_F$ is the inertia subgroup and $\mathbb{F}_q$ the residual field of $F$. Then $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $W_F$ is the inverse image of $\mathbb{Z}$ under $\mathrm{Gal}(\overline{\mathbb{F}}/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Write $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$ (the Weil-Deligne group of $F$).

# The approach of Langlands III

## Langlands dual of $G$

For the simplicity of the exposition, let us assume from now on that $G$ is split. Let $G^\vee$ be the complex algebraic group with root datum dual to that of $G$ :

$$G \leftrightarrow (X^*, R, X_*, R^\vee)$$

$$G^\vee \leftrightarrow (X_*, R^\vee, X^*, R).$$

## Examples

$\mathrm{GL}_n(F)^\vee = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(F)^\vee = \mathrm{PGL}_n(\mathbb{C})$,
$\mathrm{Sp}_{2n}(F)^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{SO}_{2n}(F)^\vee = \mathrm{SO}_{2n}(\mathbb{C})$.

## A more precise version of the approach of Langlands

- Each "box" is "labeled" via (the $G^\vee$-conjugacy class of) an L-parameter, that is, a morphism

$$\phi\colon W_F' \to G^\vee$$

such that $\phi(w)$ is semisimple for each $w \in W_F$, and $\phi|\mathrm{SL}_2(\mathbb{C})$ is a morphism of complex algebraic groups. Let $\Phi(G^\vee, F)$ be the set of $G^\vee$-conjugacy classes of L-parameters.

- It is expected that

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G^\vee, F)/G^\vee} \Pi_\phi(G),$$

where $\Pi_\phi$ is the "box labeled by $\phi$", called the L-packet attached to $\phi$.

# Enhanced L-parameters I

**Definition/Question**

The $\pi \in \mathrm{Irr}(G)$ inside an L-packet are said to be L-indistinguishable. How could we distinguish them each other ?

# Enhanced L-parameters II

## Idea (Lusztig, Vogan, Arthur)

Enhanced $\phi$ by some some additional datum : an irreducible representation $\rho$ of the component group $\mathfrak{R}_\phi$ of $\mathrm{Z}_{G^\vee}(\phi)/Z_{G^\vee}$, where $\mathrm{Z}_{G^\vee}(\phi)$ is the centralizer of $\phi(W'_F)$ in $G^\vee$, and $Z_{G^\vee}$ is the center of $G^\vee$. We call

$$\Phi(G^\vee, F)_{\mathrm{e}} := \{(\phi, \rho) \mid \phi \in \Phi(G^\vee, F),\ \rho \in \mathrm{Irr}(\mathfrak{R}_\phi)\}$$

the set of enhanced L-parameters for $G$.

# Coming back to the intuitive approach I

## Local Langlands Correspondence (LLC)

View $\Phi(G^\vee, F)_{\mathrm{e}}/G^\vee$ as a collection of labels that one expects to be able to put on all the element of $\mathrm{Irr}(G)$ (in a bijective way).

## Next steps

(1) Find a way of painting all the labels.

(2) Conjecture (A-Moussaoui-Solleveld) : Assume that the LLC holds. Then the colors in (1) match with Harish-Chandra's way of coloring, that is, for each element $\pi$ in $\mathrm{Irr}(G)$, its Harish-Chandra-color is the same as the color of its LLC-label $(\phi, \rho)$. Conjecture is true for classical groups if $\mathrm{char}(F) = 0$ (Moussaoui, 2015).

# The generalized Springer correspondence I

## Notation

- Let $\mathcal{G}$ be a complex reductive group, and let $\mathrm{Unip}(\mathcal{G})$ the unipotent variety of $\mathcal{G}$. We set

$$\mathrm{Unip}(\mathcal{G})_{\mathrm{e}} := \{(u, \rho)\ u \in \mathrm{Unip}(\mathcal{G}),\ \rho \in \mathrm{Irr}(\pi_0(\mathrm{Z}_{\mathcal{G}}(u)))\}/\mathcal{G}.$$

  We call $\mathrm{Unip}(\mathcal{G})_{\mathrm{e}}$ the enhanced unipotent variety of $\mathcal{G}$.
- Assume that $\mathcal{G}$ is connected. Let $D^b_{\mathcal{G}}(\mathrm{Unip}(\mathcal{G}))$ be the constructible $\mathcal{G}$-equivariant derived category defined by Bernstein and Lunts, and let $\mathrm{Perv}_{\mathcal{G}}(\mathrm{Unip}(\mathcal{G}))$ its subcategory of $\mathcal{G}$-equivariant perverse sheaves on $\mathrm{Unip}(\mathcal{G})$).

# The generalized Springer correspondence II

## Parabolic induction for perverse sheaves

Let $\mathcal{P} = \mathcal{L}\mathcal{U}$ be a parabolic subgroup of $\mathcal{G}$. Lusztig defined a parabolic induction functor

$$\mathrm{I}_{\mathcal{L}}^{\mathcal{G}} \colon \mathrm{Perv}_{\mathcal{L}}(\mathrm{Unip}(\mathcal{L})) \to \mathrm{Perv}_{\mathcal{G}}(\mathrm{Unip}(\mathcal{G})).$$

## Cuspidal perverse sheaves

A simple object $\mathcal{F}$ in $\mathrm{Perv}_{\mathcal{G}}(\mathrm{Unip}(\mathcal{G}))$ is called cuspidal if for any proper parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$, with Levi factor $\mathcal{L}$, and for any object $\mathcal{F}'$ in $\mathrm{Perv}_{\mathcal{L}}(\mathrm{Unip}(\mathcal{L}))$, $\mathcal{F}$ does occur in the socle of $\mathrm{I}_{\mathcal{L}}^{\mathcal{G}}(\mathcal{F}')$.

# The generalized Springer correspondence III

For $u$ a given unipotent element in $\mathcal{G}$, let $A_{\mathcal{G}}(u)$ denote the component group of its centralizer $\mathrm{Z}_{\mathcal{G}}(u)$. Then $\mathrm{Irr}(A(u))$ is in bijection with the irreducible $\mathcal{G}$-equivariant local systems $\mathcal{E}$ on $\mathcal{O} = (u)_{\mathcal{G}}$ :

$$\rho \mapsto \mathcal{E}_\rho.$$

## Definition

$\rho \in \mathrm{Irr}(A(u))$ is called cuspidal if the perverse sheaf $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ is cuspidal.

# The generalized Springer correspondence IV

**Theorem (Lusztig)**

Let $\mathfrak{L}(\mathcal{G})$ be the set of $\mathcal{G}$-conjugacy classes of triples $\tau := (\mathcal{L}, v, \epsilon)$, where $\mathcal{L}$ is a Levi subgroup of $\mathcal{G}$, $v \in \mathrm{Unip}(\mathcal{L})$, and $\epsilon \in \mathrm{Irr}(A_{\mathcal{L}}(v))$ is cuspidal. Then $W_{\mathcal{L}} := \mathrm{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$ is a Weyl group and

$$\mathrm{Unip}(\mathcal{G})_{\mathrm{e}} = \bigsqcup_{\tau \in \mathfrak{L}(\mathcal{G})} \mathrm{Unip}(\mathcal{G})_{\mathrm{e}}^{\tau},$$

where $\mathrm{Unip}(\mathcal{G})_{\mathrm{e}}^{\tau}$ is in bijection with $\mathrm{Irr}(W_{\mathcal{L}})$.

# The generalized Springer correspondence V

## A generalized Springer correspondence for disconnected groups

Remove the assumption of the connectedness of $\mathcal{G}$. Let $\mathcal{G}^\circ$ be the identity component of $\mathcal{G}$. Let $u \in \mathrm{Unip}(\mathcal{G}^\circ)$. We have $A_{\mathcal{G}^\circ}(u) \subset A_{\mathcal{G}}(u)$. We say that $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$ is *cuspidal* if its restriction to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of cuspidal irreducible representations of $A_{\mathcal{G}^\circ}(u)$. Then we will say that $(u, \rho)$ is a *cuspidal pair* in $\mathcal{G}$. Extend the notation above to the disconnected case.

# The generalized Springer correspondence VI

**Theorem (A, Moussaoui, Solleveld, 2015)**

Let $\tau \in \mathfrak{L}(\mathcal{G})$. Then there exists a 2-cocycle

$$\kappa_\tau \colon W_{\mathcal{L}}/W_{\mathcal{L}^\circ} \times W_{\mathcal{L}}/W_{\mathcal{L}^\circ} \to \mathbb{C}^\times$$

and a twisted algebra $\mathbb{C}[W_{\mathcal{L}}, \kappa_\tau]$ such that

$$\mathrm{Unip}(\mathcal{G})_{\mathrm{e}} = \bigsqcup_{\tau \in \mathfrak{L}(\mathcal{G})} \mathrm{Unip}(\mathcal{G})_{\mathrm{e}}^{\tau},$$

where $\mathrm{Unip}(\mathcal{G})_{\mathrm{e}}^{\tau}$ is in bijection with $\mathrm{Irr}(\mathbb{C}[W_{\mathcal{L}}, \kappa_\tau])$.

# The generalized Springer correspondence VII

**Remark**

The 2-cocycle $\kappa_\tau$ may be non trivial : it happens for instance for $\mathcal{G} = \mathrm{Z}_{\mathrm{SL}_{10}(\mathbb{C})}(Q)$, with $Q$ some subgroup of order 8 of $\mathrm{SL}_2(\mathbb{C})^5$.

# Plugging the GSC into the LLC I

**Attach to each box a complex algebraic group (Moussaoui's thesis, 2015)**

Let $G^\vee_{\mathrm{sc}}$ be the simply connected cover of $G^\vee$. For each $\phi \in \Phi(G^\vee, F)$, let $\mathcal{G}_\phi$ denote the centralizer in $G^\vee_{\mathrm{sc}}$ of $\phi(W_F)$.

**Remark**

We set $u_\phi = \phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$. Then we have $\mathfrak{R}_\phi \simeq A_{\mathcal{G}_\phi}(u_\phi)$.

# Plugging the GSC into the LLC II

## Definition (Moussaoui)

Let $(\phi, \rho) \in \Phi(G^\vee, F)_{\mathrm{e}}$. It is called cuspidal if $\phi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$, and $(u_\phi, \rho)$ is a cuspidal pair in $\mathcal{G}_\phi$.

# Plugging the GSC into the LLC III

## Theorem (A, Moussaoui, Solleveld, 2015)

Let $\mathfrak{L}(G^\vee, F)$ be the set of $G^\vee$-conjugacy classes of triples $\mathfrak{l} = (L^\vee, \phi, \rho)$ such that $(\phi, \rho) \in \Phi(L^\vee, F)_{\mathrm{e}}$ is cuspidal. One can define a *cuspidal support* map

$$\mathrm{Sc}\colon \Phi(G^\vee, F)_{\mathrm{e}} \to \mathfrak{L}(G^\vee, F).$$

Then

$$\Phi(G^\vee, F)_{\mathrm{e}} = \bigsqcup_{\mathfrak{l} \in \mathfrak{L}(G^\vee, F)} \Phi(G^\vee, F)_{\mathrm{e}}^{\mathfrak{l}},$$

where $\Phi(G^\vee, F)_{\mathrm{e}}^{\mathfrak{l}}$ is the subset formed by the elements of $\Phi(G^\vee, F)_{\mathrm{e}}$ with cuspidal support $\mathfrak{l}$.

# Plugging the GSC into the LLC IV

**Intuive version :**

Rephrasing the Theorem above : we have colored all the elements of $\Phi(G^\vee, F)_{\mathrm{e}}^{\mathfrak{l}}$ with the color $\mathfrak{l}$.

# References I

## References

- A, P. Baum, R. Plymen, M. Solleveld, “Conjectures about *p*-adic groups and their noncommutative geometry”, arXiv :1508.02837, 2015.
- A, A. Moussaoui, M. Solleveld, “Generalizations of the Springer correspondence and cuspidal Langlands parameters”, arXiv :1511.05335, 2015.
- A. Moussaoui, "Centre de Bernstein stable et conjecture d’Aubert-Baum-Plymen-Solleveld", Thèse UPMC - 2015 - tel.archives-ouvertes.fr.