A classical Ramsey-type result of Schur

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January 11–24, 2016
Indo-French meeting, Chennai

Prologue

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Theodore Motzkin: "Complete disorder is impossible."

Plan of the talk

- The pigeonhole principle and Ramsey's Theorem

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- Schur's theorem: generalizations and some open questions

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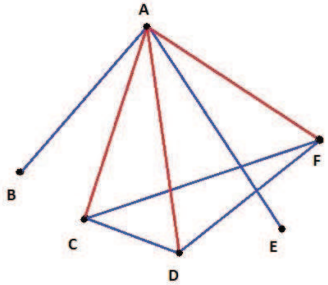
Proof by graphs (joining by red line when two people know each other)



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One observes that writing

$$S = \chi^{-1}(1) \cup \chi^{-1}(2) \cup \cdots \cup \chi^{-1}(r),$$

an r-coloring of a set S is nothing but a partition of S into r parts.

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such that the elements of $\binom{L}{k}$ are of the same color.

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For odd primes e, the congruence

$$x^e + y^e + z^e \equiv 0 \pmod{p}$$

has integral solutions x, y, z, each prime to p, for every $p > E(e)$.

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The equation $x + y = z$ has a monochromatic solution for a finite coloring of $\mathbb{Z}^+$.

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Ramsey implies Schur

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Theorem. Given $k, r \in \mathbb{Z}^+$, there exists $W(k, r) \in \mathbb{Z}^+$ such that for any r-coloring of $\{1, 2, \cdots W(k, r)\}$, there is a monochromatic arithmetic progression (A.P.) of k terms.

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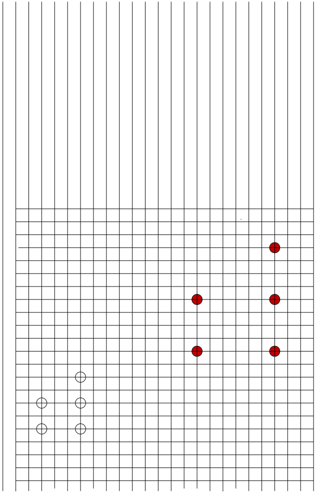
Theorem. Let $d, r \in \mathbb{Z}^+$. Then given any finite set $S \subset (\mathbb{Z}^+)^d$, and an r-coloring of $(\mathbb{Z}^+)^d$, there exists a positive integer 'a' and a point 'v' in $(\mathbb{Z}^+)^d$ such that the set $aS + v$ is monochromatic.

Monochromatic translated homothety: when $d = 1$, putting $S = \{1, \cdots, k\}$, one gets van der Waerden's theorem.



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Remark. Taking $s = 1$ in the above, a monochromatic set $\{a, a + d, \} \cup \{d\}$ already implies Schur's theorem. We shall see later that a much stronger statement follows from the above theorem.

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Theorem. Given any two positive integers r and k, there is a positive integer $n = n(r, k)$ such that if $[n]$ is r colored, there are positive integers $a_1 < a_2 < \cdots < a_k$ satisfying $\sum_{1\le i\le k} a_i \le n$ such the elements $\sum_{i\in I} a_i$ are identically colored as I varies over different non-empty subsets of $\{1, 2, \cdots, k\}$.

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Different proofs were later given by Furstenberg and Weiss (1978) and Glazer.

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it has a monochromatic solution $(x_1, \cdots, x_n)$, where x_i's may not be distinct, in $\mathbb{Z}^+$ with respect to any finite coloring if and only if some non-empty subset of $\{c_1, \cdots, c_n\}$ sums to zero.

From Schur's theorem and a special case of van der Waerden's theorem we see that
for any finite coloring of $\mathbb{Z}^+$, there are monochromatic solutions of the following equations:

$$\begin{aligned} x+y &= z, \\ x+z &= 2y. \end{aligned}$$

Theorem.(Rado, 1933) (abridged version). Given an equation

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Remark. If an equation $c_1x_1+\cdots+c_nx_n=0$ has a monochromatic solution with respect to any finite coloring (respectively, r-coloring), it is called regular (respectively, r-regular).

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S_p is a $(p-1)$-coloring of $\mathbb{Q}^*$ with the property that

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The first nontrivial case of the conjecture has been proved by Fox and Kleitman.

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We have established it (part of it to appear in Math. Comp.) when $(k-1)$ divides c or $k \leq 7$.

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Our equation (1) is a special case of equation (L), namely where

$$\alpha = (1, 1, \ldots, 1, -1) \in \mathbb{Z}^{k+1}.$$

Definitions. Given $n \in \mathbb{Z}^+$, the equation (L) is said to be n-regular if, for every n-coloring of $\mathbb{Z}^+$, there exists a monochromatic solution $x \in (\mathbb{Z}^+)^{k+1}$ to (L).

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If $dor(L) = \infty$, then (L) is regular.

In general one speaks about n-regularity over $A \subseteq \mathbb{Z}$ and defines $dor_A(L)$.
For any $A \subseteq B \subseteq \mathbb{Z}$, clearly

$$1 \leq dor_A(L) \leq dor_B(L).$$

We show that if (L_0) is regular, and the coordinate sum of α is nonzero, then

$$dor_{\mathbb{Z}^+}(L) = dor_{\mathbb{Z}}(L).$$

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Clearly, the chromatic number is monotonous with respect to restriction.

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Its set of vertices is $\mathbb{Z}$, and a subset $E \subseteq \mathbb{Z}$ is a hyperedge in $\mathcal{H}$ if and only if

$$E = U(\delta)$$

for some solution $\delta \in \mathbb{Z}^{k+1}$ to equation (L).

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By a study of the combinatorics of the stable triples, considering equation (L) with parameters $k = 5$, $\alpha = (1, 1, 1, 1, 1, -1) \in \mathbb{Z}^6$ and $c = 2$, that is, the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 - x_6 = -2,$$

and observing that the corresponding homogeneous equation is regular, we show that $\chi(\mathcal{H}|_V) \geq 4$,

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We observed that, if $\mathcal{H} = \mathcal{H}(L)$ is the hypergraph associated to equation (L), and $A \subseteq \mathbb{Z}$, then

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Therefore, by (2), $dor_V(L) \geq 3$ and hence $dor_{\mathbb{Z}}(L) \geq 3$.

We observed that, if $\mathcal{H} = \mathcal{H}(L)$ is the hypergraph associated to equation (L), and $A \subseteq \mathbb{Z}$, then

$$dor_A(L) = \chi(\mathcal{H}|_A) - 1. \tag{2}$$

So, we work with $\chi(\mathcal{H}|_A)$.

By a study of the combinatorics of the stable triples, considering equation (L) with parameters $k = 5$, $\alpha = (1, 1, 1, 1, 1, -1) \in \mathbb{Z}^6$ and $c = 2$, that is, the equation

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Therefore, by (2), $dor_V(L) \geq 3$ and hence $dor_{\mathbb{Z}}(L) \geq 3$.

But, from the 4-coloring of the integers according to the class mod 4, it follows that $dor_{\mathbb{Z}}(L) < 4$.

THANK YOU !!!