Eigenvalue statistics of random operators

M Krishna

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Best Wishes to Kesavan

On completing his tenure at IMSc

and starting a new leaf

Eigenvalue statistics

The problem is to determine how eigenvalues of some random operators are distributed.

We look at the Anderson Model

$$H^\omega = \Delta + V^\omega$$

$$(V^\omega u)(n) = \omega_n u(n), \quad (\Delta u)(n) = \sum_{|j|=1|} u(n+j)$$

where ω_n are i.i.d real random variables distributed according to μ with the operators defined on $\ell^2(\mathbb{Z}^d)$ where u belongs.

When variance of μ is large the spectrum of H^ω has only eigenvalues and the corresponding eigenfunctions are exponentially decreasing sequences.

We also have

$$\sigma(H^\omega) = [-2d, 2d] + \operatorname{supp}(\mu)$$

This shows that the eigenvalues are dense in the spectrum. Our attempt is to find out how they behave in the vicinity of a point in the spectrum. To be able to do this we have to somehow separate the eigenvalues and count them.

It is known that when μ is absolutely continuous, the integrated density of states measure $\mathcal{N}$ given by

$$\mathcal{N} = \mathbb{E}\langle \delta_0, E_{H^\omega}(\cdot)\delta_0 \rangle = \mathbb{E}\langle \delta_n, E_{H^\omega}(\cdot)\delta_n \rangle$$

is also a.c. and we denote its density by the symbol n

Take a box Λ_L centered at the origin with the side length 2L+1 in $\mathbb{Z}^d$

Consider the finite matrices

$$H_L^\omega = \chi_{\Lambda_L} H^\omega \chi_{\Lambda_L}$$

with χ_{Λ_L} denoting the projection onto $\ell^2(\Lambda_L)$. Being finite matrices, they have finitely many eigenvalues.

These eigenvalues come together as L increases, so we have to separate them to understand their local behavior. We separate the eigenvalues by scaling and centering at a point in the spectrum. The point is chosen so there are enough of these around.

namely the point E is chosen such that $n(E) > 0.$ Then we look at

$$H^{\omega}_{L,E} = |\Lambda_L|(H^{\omega}_L - E)$$

and consider the random measures

$$\xi^{\omega}_{L,E}(I) = \#(\sigma(H^{\omega}_{L,E}) \cap I)$$

for any interval $I.$

An alternate expression is

$$\xi^{\omega}_{L,E}(I) = Tr(E_{H^{\omega}_{L,E}}(I)).$$

The question then is what is the limit

of the point random measures above.

Minami [1] gave the first proof that

$$\xi^{\omega}_{L,E}(I) \to X^{\omega}(I)$$

where $X^{\omega}()$ is a Poisson random measure with intensity $\mathbb{E}(X^{\omega}()) = n(E)\mathcal{L}$ with $\mathcal{L}$ the Lebesgue measure.

$$\mathbb{P}\left(X^{\omega}(I) = k\right) = \frac{(n(E)|I|)^k}{k!} e^{-n(E)|I|}$$

A simple way to recognize the nature of the random variable X is to look at the Fourier transform $\mathbb{E}\left(e^{itX}\right)$. X is infinitely divisible iff the Levy-Khintchine formula below

$$\mathbb{E}\left(e^{itX}\right) = e^{ita+bt^2+\int(e^{itx}-itx-\frac{(itx)^2}{2!})d\nu(x)}.$$

holds and ν is called the Levy measure.

For the Anderson model Minami

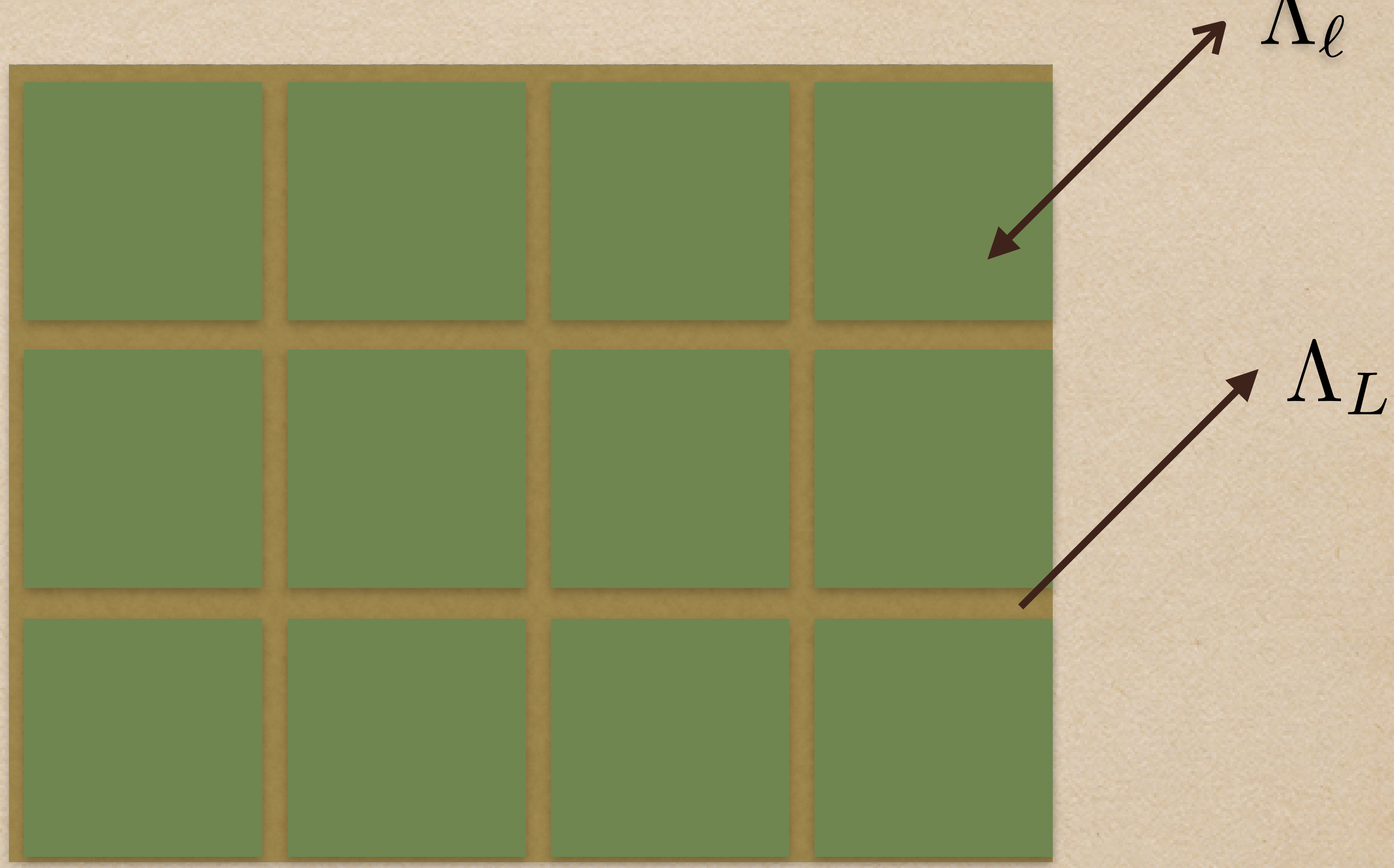
showed that $a, b = 0$

and the measure ν

is atomic supported at the point

$\{1\}$.

Ideas of Proof



$$H^{\omega}_{j,\ell} = \chi_{\Lambda_\ell(j)} H^{\omega} \chi_{\lambda_\ell(j)}$$

Write $H^{\omega}_{L} = \oplus H^{\omega}_{j,\ell} + \mathrm{M}$

and $H^{\omega} = H^{\omega}_{L} \oplus H^{\omega}_{\Lambda^{c}_{L}} + M_1$

M, M_1 are some constant matrices.

Associated with these operators we form three sequences of processes.

$$\eta_L^\omega(I) = Tr\left(E_{|\Lambda_L|(H_{j,\ell}^\omega - E)}(I)\right)$$

$$\xi_L^\omega(I) = Tr\left(E_{|\Lambda_L|(H_L^\omega - E)}(I)\right)$$

$$\zeta_L^\omega(I) = Tr\left(\chi_{\Lambda_L} E_{|\Lambda_L|(H^\omega - E)}(I)\right)$$

and show that as $L \to \infty$,

$$\lim \sum_{j=1}^{N_L} \eta_L^\omega(I) = \lim \xi_L^\omega(I) = \lim \zeta_L^\omega(I).$$

We have to compute the middle limit.

The first limit gives infinite divisibility with a bit more specifies support of the Levy measure and the last limit computes the mass there.

There are two estimates that play a role in the calculations. The Wegner estimate

$$\mathbb{E}\left(\frac{1}{|\Lambda_L|}Tr\left(E_{H_L^\omega(I)}\right)\right) \leq c|I|$$

and the Minami estimate

$$\mathbb{P}\left(\eta_L^\omega(I) \geq 2\right) \to 0, \quad \text{as} \quad L \to \infty.$$

It was not clear if the above result extends to the Schrodinger operators

$$-\Delta + \sum_{n \in \mathcal{Z}^d} \omega_n \chi_{\Lambda(n)}$$

on $L^2(\mathbb{R}^d)$.

The main difference is that the operator coefficients of ω_n are infinite rank here.

Peter Hislop, MK [2] considered

$$\Delta + \sum_{n \in \mathcal{J}} \omega_n P_n$$

on $\ell^2(\mathbb{Z}^d)$ with finite rank P_n

and showed that as random variables

$$\xi^{\omega}_{L,E}(I) \to X^{\omega}(I)$$

with the limit being a compound Poisson random variable for each I.

This means that there are i.i.d random variables Y_i such that

$$X^\omega(I) = \sum_{j=1}^{N^\omega(I)} Y_i$$

with $N^\omega(I)$ being Poisson random variables.

They showed that the associated Levy measure is supported in the set

$$\{1, 2, 3, ..., \mathrm{rank}(P_n)\}.$$

In the random Schrodinger case also they show that it is Compound Poisson and the Levy measure sits on $\mathbb{Z}^+$.

- They showed that Wegner estimate is enough together with exponential decay of eigenfunctions to get Compound Poisson limit

- They generalized Minami estimate to

$$\mathbb{P}\left(\eta_\ell^\omega(I) \geq k\right) \to 0, \quad k = \mathrm{rank}(P_0).$$

In all these cases it is a consequence that the level spacing distribution which is the distribution of the distance $E_{j+1}^{\omega} - E_j^{\omega}$ between eigenvalues of the random operators is exponential.

In a series of papers [3], [4], [5]

Dhriti Ranjan Dolai
Anish Mallick

worked out the statistics.

- Eigenfunction statistics when μ is singular they show that the centers of localization of eigenfunctions are uniformly distributed. Centre of localization is the point where the eigenfunction attains its maximum.

- When ω_n is replaced by $a_n\omega_n$ in the model, where a_n decays or grows as $n \to \infty$. In this model they showed when the dimension is 1 that the limit is the clock process, which is the sum of atomic measures supported on a constant multiple of $\mathbb{N}$. This result is in the region of a.c. spectrum.

French Mathematicians who contributed significantly to eigenvalue statistics for the Anderson type models are :

Jean Michael Combes

Francois Germinet

Frederic Klopp

among others.

References

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