Writing 8 as, $8 = 2 \times 4$, and further using $4 = \sqrt{16}$, we can write 8 as, $8 = 2\sqrt{16}$. Substituting in the above equation, we have,

$$3 = \sqrt{1 + 2\sqrt{16}}\,.$$

You may already have noticed that you can keep repeating this trick: write $16 = 1 + 15 = 1 + (3 \times 5) = 1 + (3 \times \sqrt{25})$, etc., to get, sequentially,

$$\begin{aligned} 3 &= \sqrt{1 + 2\sqrt{1 + 3\sqrt{25}}}\,, \\ 3 &= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{36}}}}\,, \\ 3 &= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \cdots}}}}}}\,, \end{aligned}$$

and so we have the proof!

Infinite nested radicals for larger integers?

Now we have infinite nested square-root expression for 2,3. Obviously we may ask the next question, what about expressions for 4,5,..., or in general any positive integer n? To find the answer, we just need to follow in the foot-steps of Ramanujan's proof for 3. Let us do this now—just square the Ramanujan expression for 3. We have,

$$9 = 1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \cdots}}}}}\,.$$

Take the first number, 1, to the left hand side to get $(9 - 1) = 8$ and then divide the entire equation by 2. We have,

$$4 = \sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + 7\sqrt{1 + \cdots}}}}}}\,.$$

There you have it for 4. Now things become simple, square the expression for 4 and we get,

$$16 = 1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + 7\sqrt{1 + \cdots}}}}}\,.$$